JACO

Volume 329

Grundlehren
der mathematischen
Wissenschaften

Grundlehren der mathematischen Wissenschaften 329

A Series of Comprehensive Studies in Mathematics

Springer-Verlag Berlin Heidelberg GmbH

Jacob Korevaar

Tauberian Theory

A Century of Developments

Springer

Jacob Korevaar
KdV Mathematical Institute
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands
e-mail: korevaar@science.uva.nl

Library of Congress Control Number: 2004104246

Mathematics Subject Classification (2000): Primary 40E05
Secondary 11M45, N05, P82; 26A12; 30B, C, D; 40-XX; 41Axx; 42Axx; 44Axx; 46A45, F05, Jxx; 47A10, 30; 60E15

ISSN 0072-7830

ISBN 978-3-642-05919-3 ISBN 978-3-662-10225-1 (eBook)
DOI 10.1007/978-3-662-10225-1

springeronline.com

Originally published by Springer-Verlag Berlin Heidelberg New York in 2004
Softcover reprint of the hardcover 1st edition 2004

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper 41/3142db- 5 4 3 2 1 0

To Pia

Preface

Summability methods have been used at least since the days of Euler to assign a reasonable sum to an infinite series, whether it is convergent or not. In its simplest form, Tauberian theory deals with the problem of finding conditions under which a summable series is actually convergent. A first condition of this kind, which applies to Abel summability (the power series method), was given by Alfred Tauber in 1897. However, Tauberian theory began in earnest only around 1910 with the work of Hardy and Littlewood. Over a period of thirty years they obtained a large number of refined 'Tauberian theorems', and they gave the subject its euphonious name.

A summability method for a series typically involves an averaging process of the partial sums. The step from summability to convergence requires a reversal of the averaging. For this one generally needs an additional condition on the series, known as a Tauberian condition. There is an endless variety of summability methods, and a corresponding variety of possible Tauberian theorems. However, the subject acquires a certain unity by similarities among optimal Tauberian conditions. One may also note the frequent appearance of the Riemann–Lebesgue lemma.

The aim of the book is to treat the principal Tauberian theorems in various categories and to provide attractive proofs. We sometimes use more than one approach and occasionally generalize the results in the literature.

The arrangement of the material roughly follows the historical development. The first three chapters deal with the basic theory. Chapter I describes the major Tauberian results of Hardy and Littlewood. They involve power series and related transforms. Over the years, many of the difficult original proofs have been simplified considerably. Karamata's surprising approach by polynomial approximation receives ample attention. The famous 'high-indices theorem', which involves lacunary series with 'Hadamard gaps', is treated by a variation on Ingham's peak function method. The proof of some other difficult theorems is postponed till later chapters.

An important impulse for Tauberian theory came from number theory, in particular, the search for relatively simple proofs of the prime number theorem. In this area the Tauberian work of Hardy and Littlewood had not been definitive. The unsatisfactory situation was one of the factors that led Wiener to his comprehensive Tauberian

theory of 1932. Here the central theme is the comparison of different limitation methods. In Chapter II, Wiener's theory is developed on the basis of a convenient testing equation, which is treated both classically and by the author's distributional method. With Wiener's theory in hand we describe several paths to the prime number theorem. Additional proofs of the PNT may be found in other chapters; cf. the Index.

For some of the proofs the natural setting is 'complex Tauberian theory', which is treated in Chapter III. Here conditions in the complex domain play an essential role. The complex theory had two roots in the early 1900's. One was Fatou's theorem, which involves power series, the other was Landau's treatment of the prime number theorem, based on Dirichlet series. A common framework is provided by Laplace transforms. The beautiful extension of Landau's theorem by Wiener and Ikehara is treated both in a classical manner and in an elegant contemporary way. Another attractive approach to the prime number theorem uses the newer complex analysis method due to Newman. Adapted to the Fatou area, this method has recently been applied in operator theory. The effect of conditions in the complex domain is illustrated also by results in Tauberian remainder theory. An earlier version of Chapter III has appeared as a survey under the title 'A century of complex Tauberian theory', Bulletin of the American Mathematical Society (N.S.), vol. 39 (2002), pp 475–531.

After Hardy, Littlewood and Wiener, the principal actor in Tauberian theory was Karamata. Chapter IV deals with his heritage involving 'regular variation', which has become indispensable in asymptotics of all kinds, including probability theory. Besides the standard theory, the chapter contains a variety of Tauberian theorems involving large Laplace transforms. Regular Variation is now a subject in its own right, witness the 1987 book with that title by Bingham, Goldie and Teugels.

Chapter V treats other extensions of the classical theory. The first part deals with the Banach algebra approach to Wiener theory. Going back to Beurling and Gelfand, it has led to important generalizations. Other parts of the chapter serve to reduce 'general' Tauberian theorems to the easier case involving the limitation of bounded functions or sequences. After an important boundedness theorem of Pitt, we discuss the functional-analytic approach initiated by the Polish school, and greatly developed by Zeller and Meyer-König. The chapter concludes with some interesting special theorems.

One of the more difficult Tauberian areas concerns the Borel method of summation, which is best known as a tool for analytic continuation. Although the Tauberian theory for this method was started by Hardy and Littlewood, several basic results are of relatively recent origin. In Chapter VI we present a new unified Tauberian theory for Borel summability and the related 'circle methods', of which Euler summability is the best known representative. The treatment includes a common theory for lacunary series with 'square-root gaps'.

Chapter VII is devoted to (real) Tauberian remainder theory. The basic question is to obtain remainder estimates for convergent series, given the order of approximation provided by a summability process. The chapter starts with the case of power series and Laplace transforms, for which Freud and the author refined the method of polynomial approximation. It continues with the broader approach by Ganelius and others which is based on refinement of Wiener theory. The final part of the chapter treats some

difficult nonlinear problems of the type, first considered by Erdős in connection with the elementary proof of the prime number theorem. The material based on Siegel's unpublished 'fundamental relation' was taken from the author's article 'Tauberian theorem of Erdős revisited', which appeared in Combinatorica, vol. 21 (2001), pp 239–250.

The idea of a 'Tauberian book' came up in the sixties when I lectured at Stanford and Oregon, but the project became dormant soon after I moved back from the U.S. to Amsterdam in the seventies. This notwithstanding a solemn promise, made to Springer series editor Sz.-Nagy (on Tolstoy's grave, of all places) to complete a Tauberian book. Naturally, the present volume has evolved a great deal from the early notes. In the meantime, a few of the topics have been treated very nicely in small books by A.G. Postnikov [1979] and Jan van de Lune [1986]. Prior to these, the only 'general' Tauberian book had been Pitt's monograph [1958].

Although the present book deals with a large variety of results, it does not aspire to completeness. The Tauberian literature is just too extensive – there are so many summability methods! This is clear already from Hardy's book *Divergent Series* of 1949 and the 'Ergebnisse' book by Zeller and Beekmann of 1970. The present Bibliography lists a substantial number of contributions (both old and new), but by no means all. Our emphasis is on Tauberian theorems for the principal summability methods and on results that belong to the area of classical analysis. We do not consider multidimensional theory or absolute summability; the book does not deal with Tauberian theorems for topological groups or generalized functions. There is no systematic treatment of the many applications, but a number of them are scattered through the book; see the Index. For all that is missing, the interested reader is referred to the well-known reference journals and databases.

The various chapters of the book are largely independent of each other. For the newcomer to Tauberian theory, the first ten sections of Chapter I may serve as orientation. Beyond that, every chapter has its own introduction. The Index refers to a few challenging open problems.

ACKNOWLEDGEMENTS. Under the German occupation (1940-45), students in the Netherlands had a difficult time. When university attendance was impossible, one tried to study from notes taken by older students. I am very grateful to my former high-school teacher C. Visser in Dordrecht and my later Ph.D. advisor H.D. Kloosterman at Leiden for help during this period. Visser encouraged me to explore Tauberian theorems that were stated without proof in notes of Kloosterman's introductory analysis course. Kloosterman continued to receive students at his home for examinations and encouraged my early independent work. For many years, my heroes were Hardy and Littlewood, next to Pólya, Szegő and Landau; a list soon extended to Karamata and Wiener.

Shortly after the war, I was fortunate to meet Paul Erdős in Amsterdam, where he lectured on the elementary proof of the prime number theorem. He challenged me with related Tauberian questions, also after I moved to the U.S. Some of my early papers owe a great deal to his suggestions.

Jumping to the last few years, it is a pleasure to mention some of the many friends and colleagues who have helped with the book. After Fred Gehring rekindled my interest, Nick Bingham provided constant support and encouragement. Nick, Harold Diamond, Kenneth Ross, anonymous referees and other experts commented on early drafts of several chapters, and Ronald Kortram read through the whole book. I thank all of them for their useful suggestions. However, even after several rounds of editing, some of my mistakes and omissions are likely to remain, for which my apologies.

On the technical side, notably LaTeX questions, my former student and junior colleague Jan Wiegerinck was always ready to assist with day-to-day problems. He and Jan van de Craats skillfully executed the drawings. At Springer Verlag, Dr. Byrne and her staff kindly met my wishes on the styling of the book.

Special thanks are due to the Mathematical Institute at the University of Amsterdam, which provided a desk and access to its facilities after my retirement. Our library is well supplied with older material, and librarian Sjoerd Lashley was always willing to go after newer items. The library of the CWI (Centrum voor Wiskunde en Informatica) in Amsterdam was very helpful too, as were friends at the University of Wisconsin and elsewhere; Armen Sergeev in Moscow provided me with a (photo)copy of Subhankulov's hard-to-get book on remainder theory.

The interest of the mathematical community in Tauberian theory has been relatively constant over the years. That the book may provide a new impulse!

Amsterdam, January 2004 *Jaap Korevaar*

Contents

I

The Hardy–Littlewood Theorems

1 Introduction

In various contexts – think of Fourier series or analytic continuation – it is important to have a method which sums a given infinite series $\sum_{n=0}^{\infty} a_n$. It may be difficult to determine the sum of a convergent series directly, or one may wish to assign a reasonable sum to a possibly divergent series. The simplest summability method is Cesàro's or the method of arithmetic means. Here one forms the arithmetic means

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} \tag{1.1}$$

of partial sums $s_n = \sum_{k=0}^{n} a_k$, and one looks for a limit of the averages σ_n instead of $\lim s_n$. If $\lim \sigma_n$ exists, it is called the Cesàro sum of $\sum a_n$. More powerful (in the sense that it sums more series) is the so-called Abel method, or power series method, sometimes called the Abel–Poisson method. Assuming that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to a sum function $f(x)$ for $|x| < 1$, one can write $f(x)$ as a weighted average of the partial sums s_n:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{\sum_{n=0}^{\infty} x^n}. \tag{1.2}$$

If $f(x)$ tends to a (finite) limit A as $x \nearrow 1$, then A is called the Abel sum of $\sum a_n$. It follows from (1.2) that the Abel sum of a convergent series is equal to the usual sum, $\lim s_n$. This is Abel's continuity theorem [1826] which gave the summability method its name. Furthermore, the divergent series

$$1 - 1 + 1 - 1 + \cdots, \qquad 1 - 2 + 3 - 4 + \cdots$$

have Abel sums $1/2$ and $1/4$, respectively. The first series has Cesàro sum $1/2$; the second is not summable by the use of (1.1). There is a nice essay on the history of summability theory in chapters 1 and 2 of Hardy's book *Divergent Series* [1949].

For many applications one would like to know under what conditions a summable series is actually convergent; cf. Section 4 for series in number theory and Section 6 for Fourier series. Summability usually involves averaging, hence one would need a condition under which one can 'undo' the averaging. A first converse result was obtained in [1897] by the Austrian mathematician Alfred Tauber (1866–1942). (Tauber later specialized in the mathematics of insurance; cf. the biographical information in Binder [1984]). Tauber's sufficient condition for the convergence of an Abel summable series $\sum a_n$ was '$na_n \to 0$'. In [1910], Hardy asked whether boundedness of the sequence $\{na_n\}$ would be sufficient, and Littlewood [1911] proved that it is. His paper started the lifelong collaboration between Hardy and Littlewood, one of the most successful of all time. Jointly they obtained a large variety of conditional converses of continuity theorems or *Abelian* theorems. Beginning with the brief note [1913c], Hardy and Littlewood called their converses *Tauberian* theorems. This chapter deals with a substantial number of their results, both for series and for integrals.

Littlewood's theorem has fascinated many mathematicians. It was not that his 'Tauberian condition' for the convergence of Abel summable series $\sum a_n$ was spectacular. The condition '$\{na_n\}$ bounded' is simple and, perhaps surprisingly, it is optimal as an order condition on the terms a_n; cf. Section 24. Littlewood's result was impressive because the simple answer seemed to require a very complicated proof. An important aspect of his proof was a condition under which asymptotic relations can be differentiated; cf. Section 17 for fairly refined results. The resulting technique of 'repeated differentiation' was used by Hardy and Littlewood in many of their papers (starting with [1914a]). It serves to give a great deal of weight to a particular term involving s_n, so that the behavior of s_n can be studied.

It came as a big surprise to the mathematical community when Karamata [1930a] found a much simpler proof for Littlewood's theorem and other Tauberian theorems for power series. His technique emphasizes a group of terms in a series with the aid of polynomial approximation; see Sections 11, 12. In Sections 15, 21 we discuss more general integral analogs of the Hardy–Littlewood theorems for power series.

However, some of the most striking Hardy–Littlewood theorems remain difficult until today. Among them are their Tauberians for Borel and Lambert summability (Hardy and Littlewood [1916], [1921]), the latter of great interest in number theory. The theorems are stated in this chapter but will be proved later. A breakthrough in their treatment came with Wiener's general Tauberian theory [1928], [1932], which is based on Fourier transforms (see Chapter II). Another Hardy–Littlewood spectacular [1926] was the so-called *high-indices* theorem, which involves lacunary power series. If the gaps between the powers are large enough, the analog of Littlewood's theorem no longer requires an order condition on the coefficients!! Ingham [1937] found a simpler proof for this result with the aid of 'peak functions'; cf. also his paper [1965]. Our version of the proof is somewhat more direct; see Section 23.

Very often, the first step in a convergence proof for series is to show that the partial sums are bounded. Useful boundedness results may be found in Sections 5, 19 and 20. Optimality of Tauberian conditions is discussed in Section 24.

It is not possible to discuss all the work on summability by Hardy, Littlewood and their contemporaries in a book on Tauberian theory. Fortunately much of that material can be found in Hardy's book *Divergent Series* which was mentioned earlier; other sources are Zeller and Beekmann [1958/70], Baron [1966/77] (in Russian) and Boos [2000]. A few books or booklets are devoted entirely to certain aspects of Tauberian theory: Karamata [1937b], Pitt [1958], Subhankulov [1976] (in Russian), Postnikov [1980], and van de Lune [1986]. Other books contain more or less elaborate Tauberian chapters, notably the books on summability mentioned above. One may also consult Wiener [1933], Widder [1941], Peyerimhoff [1969], the survey Kangro [1974] (in Russian) for the years 1964–1973, and Bingham, Goldie and Teugels [1987]. Volume 6 of Hardy's 'Collected Papers' [1974], with editorial comment, is a valuable source. For a quick look at some of the older theory and interesting commentary, see Gaier's expanded edition of Landau's *'Neuere Ergebnisse der Funktionentheorie'* (Landau and Gaier [1986]).

Many results in this chapter hold only for *real* series and functions. This is rarely stated explicitly, but should be clear from the form. For possible extensions to the complex case one may consider real and imaginary parts.

2 Examples of Summability Methods. Abelian Theorems and Tauberian Question

In each example below we compare two summability methods, P and Q, for infinite series $\sum_{n=0}^{\infty} a_n$ (often written $\sum_0^{\infty} a_n$). Here method Q will always be 'stronger' than P in the following sense. All P-summable series are Q-summable, to the same finite (generalized) sum, briefly, Q is *consistent* with P. However, some Q-summable series would fail to be P-summable. A method which sums all convergent series to the usual sum is called *regular*.

If the series $\sum_0^{\infty} a_n$ is P-summable to A, or has 'P sum' A, the partial sums $s_n = \sum_{k=0}^{n} a_k$ are said to be P*-limitable* to A.

Example 2.1. (Cauchy [1821]):

P	Q
Ordinary convergence:	*Cesàro summability* (of order 1) : the arithmetic mean
$s_n = a_0 + a_1 + \cdots + a_n \to A$	$\sigma_n = (s_0 + s_1 + \cdots + s_{n-1})/n$ tends to a limit A' as $n \to \infty$, the 'Cesàro sum' of $\sum a_n$

If $s_n \to A$ then also $\sigma_n \to A$.

The divergent series $1 - 1 + 1 - 1 + \cdots$ has Cesàro sum $1/2$.

Remarks. There is extensive literature on a whole hierarchy of Cesàro methods. For these (C, k) methods, see for example Hardy [1949], Baron [1966/77]. In Cesàro's

work [1890] these methods played an important role in the multiplication of series; cf. Section 3. For us Cesàro summability will mean the $(C, 1)$ summability of the present example, except in Section 18.

Conceptually simpler (but not analytically!) are the Hölder methods (H, k), which (for integral $k \geq 2$) are obtained by repeated formation of arithmetic means. It was shown by Knopp and Schnee that the (C, k) and (H, k) method are equivalent: they sum the same series, to the same sum; see for example Hardy (loc. cit.).

Example 2.2. (Abel [1826]):

P	Q
Ordinary convergence:	*Abel summability:*
$s_n = \sum_{k=0}^{n} a_k \to A$	the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $\lvert x \rvert < 1$, and $f(x) = \sum a_n x^n$ tends to a limit A' as $x \nearrow 1$, the 'Abel sum' of $\sum a_n$

If $s_n \to A$ then $f(x)$ is well-defined for $|x| < 1$ and $f(x) \to A$ as $x \nearrow 1$ ('Abel's continuity theorem'):

$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{s_0 + s_1 x + s_2 x^2 + \cdots}{1 + x + x^2 + \cdots} \to A. \tag{2.1}$$

The series $1 - 2 + 3 - 4 + \cdots$ has Abel sum $1/4$. It is not Cesàro summable.

Example 2.3. (Frobenius [1880]):

P	Q
Cesàro summability:	*Abel summability:*
$\sigma_n = (s_0 + s_1 + \cdots + s_{n-1})/n \to A$	$f(x) = \sum_{n=0}^{\infty} a_n x^n \to A'$ as $x \nearrow 1$

If $\sigma_n \to A$ then $f(x)$ is well-defined for $|x| < 1$ and $f(x) \to A$ as $x \nearrow 1$:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} (s_n - s_{n-1}) x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=0}^{\infty} s_n x^{n+1} \\
&= (1-x) \sum_{n=0}^{\infty} s_n x^n = (1-x)^2 \sum_{n=0}^{\infty} (s_0 + \cdots + s_n) x^n \\
&= \frac{\sum_{n=0}^{\infty} (n+1) \sigma_{n+1} x^n}{\sum_{n=0}^{\infty} (n+1) x^n} = A + \frac{\sum_{n=0}^{\infty} (n+1)(\sigma_{n+1} - A) x^n}{\sum_{n=0}^{\infty} (n+1) x^n} \to A.
\end{aligned} \tag{2.2}$$

The higher-order Cesàro methods described in Section 18 are stronger than the first order method, but not as powerful as Abel summability.

ABELIAN THEOREMS. Because of Examples 2.2 and 2.3, continuity theorems of the form

$$\sum a_n \text{ is } P\text{-summable to } A \quad \Rightarrow \quad \sum a_n \text{ is } Q\text{-summable to } A$$

(so that Q is 'stronger' than P) are called Abelian theorems.

Example 2.4. (Borel [1899], [1901/28]):

P	Q
Ordinary convergence:	*Borel summability:*
	$\sum_{n=0}^{\infty} s_n x^n/n!$ converges for all x
$s_n \to A$	and $F(x) = e^{-x} \sum s_n x^n/n!$
	tends to a limit A' as $x \to \infty$,
	the 'Borel sum' of $\sum a_n$

If $s_n \to A$ then $F(x)$ is well-defined for all $x > 0$ and $F(x) \to A$ as $x \to \infty$:

$$F(x) = \frac{\sum\limits_{n=0}^{\infty} s_n x^n/n!}{\sum\limits_{n=0}^{\infty} x^n/n!} \to A. \tag{2.3}$$

Example 2.5. (Hardy [1914a], Ananda-Rau [1921]):

P	Q
Ordinary convergence:	*Lambert summability:*
	the 'Lambert series' $\sum_{n=0}^{\infty} a_n L(x^n)$,
$s_n \to A$	where $L(x) = x\{\log(1/x)\}/(1-x)$
	(Lambert kernel; $L(1) = 1$)
	converges for $0 < x < 1$,
	and $g(x) = \sum a_n L(x^n)$
	tends to a limit A' as $x \nearrow 1$,
	the 'Lambert sum' of $\sum a_n$

If $s_n \to A$ then $g(x)$ is well-defined for $|x| < 1$ and $g(x) \to A$ as $x \nearrow 1$. This may be derived from the observation that for $0 < x < 1$,

$$\sum_{n=0}^{\infty} a_n L(x^n) = \sum_{n=0}^{\infty} s_n \{L(x^n) - L(x^{n+1})\} \;\text{ with }\; \{\cdots\} \geq 0. \tag{2.4}$$

The Lambert kernel looks more tractable if one substitutes $x = e^{-t}$:

$$L(x) = x\frac{\log(1/x)}{1-x} \quad \text{gives} \quad L(e^{-t}) = \frac{t}{e^t - 1}. \tag{2.5}$$

We now come to the basic question of this chapter:

Question 2.6. (Tauberian Problem) *Which series $\sum_0^\infty a_n$, summable by the 'stronger' method Q, are also summed by the 'weaker' method P ?*

Is there some (nontrivial) condition on the terms a_n of the series, under which its Q-summability implies its P-summability? Such a condition $T\{a_n\}$ is called a TAUBERIAN CONDITION, the resulting theorem is a TAUBERIAN THEOREM, occasionally called a 'Tauberian'.

Classical Form 2.7 of Tauberian theorems for series:

$$\sum a_n \text{ is } Q\text{-summable } \& \ T\{a_n\} \ \Rightarrow \sum a_n \text{ is } P\text{-summable.} \qquad (2.6)$$

Although they may not precisely conform to this pattern, the early Tauberian theorems were almost always conditional converses of continuity theorems; see Sections 5–10. However, there are classical Tauberian-type theorems (2.6) for cases where not all P-summable series are Q-summable. Examples of such 'nonstandard' theorems may be found in Section 25.

3 Simple Applications of Cesàro, Abel and Borel Summability

CESÀRO SUMMABILITY. Cesàro [1890] used his method to discuss multiplication of series. He proved that for convergent series $\sum_0^\infty a_n = A$ and $\sum_0^\infty b_n = B$, the Cauchy product $\sum_0^\infty c_n$ (given by $c_n = \sum_{k=0}^n a_k b_{n-k}$) is Cesàro summable to $C = AB$. This is a special case of his results for (C, k) summable series; cf. Hardy [1949] (chapter 10).

A more important application involves *Fourier series*. Fejér [1904] proved that the Fourier series of an arbitrary continuous function f of period 2π is *Cesàro summable* to $f(t)$ at every point t. (The series need not converge at every point.) To verify Fejér's theorem, one may first write the partial sum

$$s_k(f,t) = \frac{1}{2}a_0 + \sum_{j=1}^{k}(a_j \cos jt + b_j \sin jt) = \sum_{j=-k}^{k} c_j e^{ijt}$$

of the Fourier series in integral form:

$$s_k(f,t) = \sum_{j=-k}^{k} c_j e^{ijt} = \sum_{j=-k}^{k} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iju}du \cdot e^{ijt}$$
$$= \int_{-\pi}^{\pi} f(u)\frac{\sin(k+\frac{1}{2})(t-u)}{2\pi \sin\frac{1}{2}(t-u)}du = \int_{-\pi}^{\pi} f(t-u)\frac{\sin(k+\frac{1}{2})u}{2\pi \sin\frac{1}{2}u}du.$$

The average $\sigma_n(f,t)$ of the partial sums $s_0, s_1, \cdots, s_{n-1}$ may now be expressed with the aid of the Fejér kernel F_n:

$$\sigma_n(f,t) = \int_{-\pi}^{\pi} f(t-u)F_n(u)du, \qquad (3.1)$$

where

$$F_n(u) = \frac{1}{n}\sum_{k=0}^{n-1} \frac{\sin(k+\frac{1}{2})u}{2\pi \sin\frac{1}{2}u} = \frac{\sin^2 \frac{1}{2}nu}{2\pi n \sin^2 \frac{1}{2}u}. \tag{3.2}$$

Observing that $\sigma_n(1,t) \equiv 1$ and considering small $\delta > 0$, one derives that

$$\begin{aligned}\sigma_n(f,t) - f(t) &= \int_{-\pi}^{\pi} \{f(t-u) - f(t)\} F_n(u) du \\ &= \int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi} \;\to\; 0 \quad \text{as } n \to \infty.\end{aligned}$$

For periodic continuous (hence uniformly continuous) functions f, the Fourier series is uniformly Cesàro summable to f.

Since the Fejér kernel F_n is even, one may also write

$$\sigma_n(f,t) = \int_{-\pi}^{\pi} \frac{1}{2}\{f(t-u) + f(t+u)\} F_n(u) du. \tag{3.3}$$

This formula shows that for a function f which is of bounded variation over a period, the Fourier series is Cesàro summable to the value $\frac{1}{2}\{f(t-) + f(t+)\}$ at the point t.

Abel Summability. By Example 2.3, the Fourier series of a continuous function f of period 2π will also be Abel summable. An independent proof may be derived from Poisson's integral formula,

$$A_r(f,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) r^n = \int_{-\pi}^{\pi} f(t-u) P_r(u) du, \tag{3.4}$$

where $P_r(u)$ is the Poisson kernel,

$$P_r(u) = \frac{1-r^2}{2\pi(1 - 2r\cos u + r^2)} \qquad (0 \le r < 1). \tag{3.5}$$

Poisson [1820], [1823] made this formula the basis for his use of 'Fourier series'. Abel summability is sometimes called Poisson summability!

Borel Summability. Finally we mention an application of Borel summability. For the series $\sum_{n=0}^{\infty} z^n$ one has $s_n = (1 - z^{n+1})/(1-z)$, so that with the notation of Example 2.4,

$$\begin{aligned}F(x) = e^{-x}\sum_{n=0}^{\infty} s_n \frac{x^n}{n!} &= e^{-x}\sum_{n=0}^{\infty} \frac{1-z^{n+1}}{1-z}\frac{x^n}{n!} = \frac{1}{1-z} - \frac{ze^{(z-1)x}}{1-z} \\ &\to \frac{1}{1-z} \quad \text{as } x \to \infty \ \text{ whenever } \operatorname{Re} z < 1.\end{aligned}$$

The Borel sum gives an *analytic continuation* of the ordinary sum of the series $\sum_{n=0}^{\infty} z^n$ for $|z| < 1$. This is a very special case of a general property of Borel summability (Borel [1899], [1901/28]). There is a second Borel method which is even more convenient for analytic continuation; see Chapter VI.

4 Lambert Summability in Number Theory

We begin with a few remarks about the so-called Riemann ZETA FUNCTION of analytic number theory. This function has played an important role in the development of Tauberian theory. For $\operatorname{Re} z > 1$ it is given by the sum of a 'classical Dirichlet series' $\sum_{n=1}^{\infty} a_n/n^z$:

$$\zeta(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{4.1}$$

The series converges absolutely and uniformly for $\operatorname{Re} z \geq 1 + \delta > 1$. Since every term is analytic in z, the sum $\zeta(z)$ is analytic for $\operatorname{Re} z > 1$. We will verify in Section 26 that the difference

$$F(z) = \zeta(z) - \frac{1}{z-1}$$

can be continued analytically to the whole complex z-plane. For complex $z \neq 1$, the function $\zeta(\cdot)$ is defined by this continuation.

For $\operatorname{Re} z > 1$ one can write $\zeta(z)$ as an infinite product involving the prime numbers, the *Euler product*. One knows that every positive integer n has a unique representation as a product of prime powers, $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_1 < \cdots < p_r$ and $\alpha_j > 0$. Using this fact, one may readily verify that for $\operatorname{Re} z > 1$,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots\right) = \prod_{p \text{ prime}} \frac{1}{1 - 1/p^z}. \tag{4.2}$$

In particular $\zeta(z)$ has no zeros for $\operatorname{Re} z > 1$.

Example 4.1. The *Möbius function* $\mu(\cdot)$ on $\mathbb{N}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ different primes}, \\ 0 & \text{if } n \text{ has one or more multiple prime factors.} \end{cases} \tag{4.3}$$

Using the definition in Example 2.5 we will show that

$$\textit{the series } \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \textit{ is Lambert summable to } 0. \tag{4.4}$$

Proof. We take $\operatorname{Re} z > 1$. By (4.3) and (4.2)

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right) = \frac{1}{\zeta(z)}. \tag{4.5}$$

An analytic function in a half-plane can have at most one representation by a Dirichlet series; cf. Titchmarsh [1939] (section 9.6). Thus the identity

$$1=\zeta(z)\frac{1}{\zeta(z)}=\sum_{k=1}^{\infty}\frac{1}{k^z}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^z}=\sum_{k,m}\frac{\mu(m)}{(km)^z}=\sum_{n=1}^{\infty}\frac{1}{n^z}\sum_{m|n}\mu(m)$$

implies that the sum $\sum_{m|n}\mu(m)$ over all divisors m of n is equal to 1 for $n=1$ and equal to 0 for $n>1$. Going to a Lambert series, it follows that

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{\mu(m)}{m}L(x^m)&=\sum_{m=1}^{\infty}\frac{\mu(m)}{m}\frac{x^m\log(1/x^m)}{1-x^m}=\log\frac{1}{x}\sum_{m=1}^{\infty}\mu(m)\frac{x^m}{1-x^m}\\&=\log\frac{1}{x}\sum_{m=1}^{\infty}\mu(m)\sum_{k=1}^{\infty}x^{km}=\log\frac{1}{x}\sum_{n=1}^{\infty}x^n\sum_{m|n}\mu(m)\\&=x\log\frac{1}{x}\qquad(0<x<1).\end{aligned}$$

Hence the first member indeed has limit 0 as $x\nearrow 1$; the series $\sum_{m=1}^{\infty}\mu(m)/m$ has Lambert sum 0.

One may use Tauberian theory to show that the series in (4.4) is convergent; see for example Sections 10 and III.6.

Example 4.2. *Von Mangoldt's function* $\Lambda(\cdot)$ on $\mathbb{N}$ is defined by

$$\Lambda(n)=\begin{cases}0 & \text{if } n=1,\\ \log p & \text{if } n=p^\alpha \text{ with } p \text{ prime and } \alpha\geq 1,\\ 0 & \text{if } n \text{ has at least two different prime factors.}\end{cases}\tag{4.6}$$

It will be shown that

$$\textit{the series}\ \sum_{n=1}^{\infty}\frac{\Lambda(n)-1}{n}\ \textit{is Lambert summable to}\ -2\gamma,\tag{4.7}$$

where γ is Euler's constant, $\lim_{n\to\infty}\{1+(1/2)+\cdots+(1/n)-\log n\}$.

Proof. Again taking $\operatorname{Re} z>1$ one derives from the Euler product (4.2) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}&=\sum_{p\text{ prime}}\left(\frac{1}{p^z}+\frac{1}{p^{2z}}+\cdots\right)\log p\\&=\sum_{p\text{ prime}}\frac{\log p}{p^z-1}=-\frac{d}{dz}\log\zeta(z)=-\frac{\zeta'(z)}{\zeta(z)}.\end{aligned}\tag{4.8}$$

Proceeding as above, one finds that the identity

$$\zeta(z)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}=\sum_{k=1}^{\infty}\frac{1}{k^z}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^z}=-\zeta'(z)=\sum_{n=1}^{\infty}\frac{\log n}{n^z}$$

implies that $\sum_{m|n}\Lambda(m)=\log n$. Writing $d(n)$ for the number of divisors of n, one obtains the following result for a Lambert series:

$$\sum_{m=1}^{\infty}\{\Lambda(m)-1\}\frac{x^m}{1-x^m}=\sum_{n=1}^{\infty}x^n\sum_{m|n}\{\Lambda(m)-1\}=\sum_{n=1}^{\infty}\{\log n-d(n)\}x^n$$
$$=(1-x)\sum_{n=1}^{\infty}\{\log 1+\cdots+\log n-[d(1)+\cdots+d(n)]\}x^n \qquad (0<x<1).$$

Now by Stirling's formula and by a result of Dirichlet for the divisor function, cf. Pólya and Szegő [1925/78] (chapter 2, problem 46),

$$\begin{aligned}\log 1+\cdots+\log n&=n\log n-n+\mathcal{O}(\log n),\\ d(1)+\cdots+d(n)&=n\log n+(2\gamma-1)n+\mathcal{O}(n^{1/2}).\end{aligned} \tag{4.9}$$

As a result,

$$\sum_{m=1}^{\infty}\frac{\Lambda(m)-1}{m}L(x^m)=\log(1/x)\cdot(1-x)\sum_{n=1}^{\infty}\{-2\gamma n+o(n)\}x^n. \tag{4.10}$$

Since $\sum_{n=1}^{\infty}nx^n=x/(1-x)^2$, one concludes that the left-hand side of (4.10) indeed has limit -2γ as $x\nearrow 1$. Thus the series in (4.7) has Lambert sum -2γ.

One may again use Tauberian theory to show that the series in (4.7) is convergent. The convergence implies the so-called *prime number theorem*; see Section 10. (A similar remark applies to $\sum_{n=1}^{\infty}\mu(n)/n$; cf. Section III.6.)

Background material in number theory may be found in many books; classics are Hardy and Wright [1979], Landau [1909].

5 Tauber's Theorems for Abel Summability

Tauberian theory began with the following theorem of Tauber [1897] (cf. Chatterji [1984] for historical remarks).

Theorem 5.1. *The following implication holds:*

$$\sum a_n \text{ is Abel summable } \;\&\; na_n\to 0 \;\Rightarrow\; \sum a_n \text{ converges.}$$

This is not difficult to prove. In fact, for $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $0<x<1$,

$$\begin{aligned}\left|\sum_{n=0}^{N}a_n-f(x)\right|&=\left|\sum_{n=1}^{N}a_n(1-x^n)-\sum_{n=N+1}^{\infty}a_nx^n\right|\\ &\le\sum_{n=1}^{N}n(1-x)|a_n|+\frac{1}{N}\sum_{n=N+1}^{\infty}|na_n|x^n\\ &\le(1-x)\sum_{n=1}^{N}|na_n|+\frac{1}{N(1-x)}\sup_{n>N}|na_n|.\end{aligned} \tag{5.1}$$

Setting $\sum_{n=0}^{N} a_n = s_N$ and $x = 1 - 1/N$, one finds that

$$|s_N - f(1 - 1/N)| \le \frac{1}{N} \sum_{n=1}^{N} |na_n| + \sup_{n>N} |na_n|. \tag{5.2}$$

Suppose now that $na_n \to 0$. Then the right-hand side of (5.2) tends to 0 as $N \to \infty$. Hence if $f(x) \to A$ as $x \nearrow 1$, then

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} f(1 - 1/N) = A. \tag{5.3}$$

The Abel summability of $\sum a_n$ implies that the limit A is finite. However, the conclusion $s_N \to A$ remains valid if $f(x)$ is real and tends to $A = \pm\infty$ as $x \nearrow 1$, even under the weaker condition '$\{na_n\}$ bounded'.

Inequality (5.2) also implies a simple boundedness theorem:

Corollary 5.2. *Boundedness of the sequence $\{na_n\}$ and of the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $[0, 1)$ implies boundedness of the sequence $\{s_N\}$.*

TAUBERIAN CONSTANTS. If $na_n \to 0$, it follows from (5.2) that the difference $s_N - f(1-1/N)$ tends to 0 even if $\sum a_n$ is not Abel summable. Assuming that the sequence $\{na_n\}$ is bounded, one may ask for the best constant K such that for appropriate sequences $x_N \nearrow 1$,

$$\limsup_{N\to\infty} |s_N - f(x_N)| \le K \limsup_{N\to\infty} |na_n|.$$

For extensive discussion of this and related questions, also for other summability methods, see Agnew [1954], Zeller and Beekmann [1958/70] (section 50). Later contributions are Meir [1963] and Rajagopal [1974].

Tauber used Theorem 5.1 to derive his 'second theorem' which is more general. It gives a condition which is both necessary and sufficient for the step from Abel summability to convergence:

Theorem 5.3. *An Abel summable series $\sum_0^\infty a_n$ is convergent if and only if*

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} = s_n - \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} \to 0 \quad \textit{as } n \to \infty. \tag{5.4}$$

A proof may be obtained from Section 14 which contains an extension to integrals. For real a_n condition (5.4) is sufficient for the step from Abel 'summability' to 'convergence' also if $f(x) \to A = \pm\infty$.

SERIES SIMILAR TO POWER SERIES. The arguments above readily extend to series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n g(x^n) \tag{5.5}$$

such as Lambert series (Example 2.5). A suitable condition on $g(\cdot)$ would be

$$|g(x)| \le C_1 x^\delta, \quad |1 - g(x)| \le C_2(1 - x), \quad 0 \le x \le 1 \tag{5.6}$$

with positive constants δ and C_j.

Theorem 5.4. *Let $g(\cdot)$ satisfy the inequalities* (5.6). *If $|na_n| \le C$, boundedness of $f(x)$ in* (5.5) *on* $[0, 1)$ *implies boundedness of the sequence $\{s_N\}$; if $na_n \to 0$, convergence $f(x) \to A$ as $x \nearrow 1$ implies convergence $s_N \to A$.*

Remarks 5.5. Under some additional conditions on g, for example, that it is increasing and that $h(x) = g(x) - g(x^2)$ is increasing on some interval $[0, b)$, one can also obtain *boundedness* of $\{s_N\}$ under the *one-sided* condition $na_n \ge -C$. Of course one has to suppose then that the series in (5.5) *converges* on $[0, 1)$, to a bounded function f. Cf. Wiener [1933] (pp 108–110) for the case of power series, $g(x) = x$.

In connection with conformal maps $w = f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the unit disc, Fejér [1914] considered the 'area condition'

$$\sum_{n=1}^{\infty} n|a_n|^2 < \infty, \tag{5.7}$$

which expresses finiteness of the area of the image. Condition (5.7) implies (5.4): just write

$$\sum_{k=1}^{n} ka_k = \sum_{k\le m} ka_k + \sum_{m<k\le n} ka_k$$

with large m and $n > m$, and apply Cauchy–Schwarz to the final sum. Hence (5.7) is a Tauberian condition for Abel summability. Using a similar splitting, Fejér proved directly that condition (5.7) implies $s_N - f(1 - 1/N) \to 0$ as $N \to \infty$. The result is sometimes called *Fejér's Tauberian theorem*; cf. Landau and Gaier [1986] (section 13 and comments); we give an application in Section V.25. Hölder's inequality would similarly show that the condition $\sum_{n=1}^{\infty} n^{p-1}|a_n|^p < \infty$ is a Tauberian condition for Abel summability whenever $p > 1$.

6 Tauberian Theorem for Cesàro Summability

Since Cesàro summability implies Abel summability (Example 2.3), it follows from Tauber's Theorem 5.1 that the condition '$na_n \to 0$' is *a fortiori* sufficient for the implication 'Cesàro summability $\Rightarrow$ convergence'. In [1910] Hardy published a converse theorem for Cesàro summability in which he weakened the 'little o'-condition '$na_n \to 0$' to the 'big $\mathcal{O}$'-condition '$\{na_n\}$ bounded'. The latter condition is optimal as an order condition; cf. Section 24. Hardy also asked whether the little o-condition in Tauber's theorem could be relaxed to a big $\mathcal{O}$-condition.

Landau [1910] extended Hardy's Cesàro theorem to the case of one-sided boundedness.

Theorem 6.1. *One has the following implications:*

$$\sum a_n \ \textit{is Cesàro summable} \ \ \& \ \ \text{either} \ \ |na_n| \le C \ \ \text{(Hardy)}$$
$$\text{or} \ \ na_n \ge -C \ \ \text{(Landau)} \ \Rightarrow \ \sum a_n \ \ \textit{converges.}$$

This result has interesting applications and extensions.

Proof. (Kloosterman's method [1940a]) We consider real a_n and set

$$s_n = a_0 + \cdots + a_n = a_n^{(-1)}, \quad a_0^{(-1)} + \cdots + a_n^{(-1)} = a_n^{(-2)}. \tag{6.1}$$

Then one has the following *discrete* analog of *Taylor's formula*: for integers $h > 0$,

$$a_{n+h}^{(-2)} = a_n^{(-2)} + ha_n^{(-1)} + \frac{1}{2}h(h+1)a_\xi^*, \tag{6.2}$$

where a_ξ^* is a number $\geq \min a_k$ but $\leq \max a_k$ for $n < k \leq n+h$. Indeed,

$$\begin{aligned} a_{n+h}^{(-2)} &= a_n^{(-2)} + \{a_{n+1}^{(-1)} + \cdots + a_{n+h}^{(-1)}\} \\ &= a_n^{(-2)} + ha_n^{(-1)} + \{ha_{n+1} + (h-1)a_{n+2} + \cdots + a_{n+h}\}; \end{aligned} \tag{6.3}$$

the final sum does not exceed $(1/2)h(h+1)\max a_k$, where $n+1 \leq k \leq n+h$, etc.

We may assume that $\sum a_n$ is Cesàro summable to 0. (If the Cesàro sum is A, one may replace a_0 by $a_0 - A$; this will decrease every partial sum s_k by A.) Then $a_n^{(-2)}/n \to 0$ as $n \to \infty$. We now solve equation (6.2) for $a_n^{(-1)} = s_n$. The inequalities $na_n \geq -C$ with $C > 0$ and $|a_n^{(-2)}| \leq n\varepsilon$ with small $\varepsilon > 0$ and $n \geq$ large n_0 imply that for $h \approx 2n\sqrt{\varepsilon/C}$,

$$\begin{aligned} s_n &= \frac{a_{n+h}^{(-2)} - a_n^{(-2)}}{h} - \frac{1}{2}(h+1)a_\xi^* \\ &\leq \frac{2n+h}{h}\varepsilon + C\frac{h+1}{2n} < 3\sqrt{C\varepsilon}. \end{aligned} \tag{6.4}$$

For an estimate in the other direction one may take $h \approx -2n\sqrt{\varepsilon/C}$; for $h < 0$ the discrete Taylor formula needs some adjustment. The conclusion is that $s_n \to 0$ as had to be proved.

Application 6.2. Hardy (loc. cit.) used Theorem 6.1 to deduce or verify the following results for Fourier series; cf. Section 3.

(i) Let f be a continuous function of period 2π whose Fourier coefficients $a_n(f)$ and $b_n(f)$ are $\mathcal{O}(1/n)$. Then the Fourier series of f at the point t converges to $f(t)$. Indeed, by Fejér's theorem the series is Cesàro summable to $f(t)$.

(ii) Let g be a periodic function of bounded variation (over a period). Using integration by parts or some other method, one may show that the Fourier coefficients $a_n(g)$ and $b_n(g)$ are $\mathcal{O}(1/n)$. Thus the Fourier series of g converges at every point t to the value $\frac{1}{2}\{g(t-) + g(t+)\}$.

Remarks 6.3. The proof of Theorem 6.1 may be adapted to show that Cesàro summability also implies convergence under the weaker Tauberian condition

$$-w(\rho) = \liminf_{n\to\infty} \inf_{n\leq p\leq \rho n} (s_p - s_n) \to 0 \quad \text{as } \rho \searrow 1. \tag{6.5}$$

Indeed, by (6.3), the first line in formula (6.4) may be replaced by

$$s_n = \frac{a_{n+h}^{(-2)} - a_n^{(-2)}}{h} - \frac{(s_{n+1} - s_n) + \cdots + (s_{n+h} - s_n)}{h},$$

etc. Condition (6.5) is equivalent to so-called 'slow decrease' of the sequence $\{s_n\}$; cf. formula (7.1) and Section 16.

The proof readily gives a Tauberian *remainder theorem* for Cesàro summability. More generally the method may be used to obtain a *convexity* theorem. Define $a_n^{(-k)}$ as $a_0^{(-k+1)} + \cdots + a_n^{(-k+1)}$. If $|a_n| \le \phi(n)$, or only $a_n \ge -\phi(n)$, and $|a_n^{(-k)}| \le \psi(n)$, with ϕ and ψ 'of regular growth', then

$$|a_n^{(-j)}| \le C_j \phi(n)^{1-j/k} \psi(n)^{j/k}, \quad 0 < j < k; \tag{6.6}$$

cf. Section 17. Such an inequality may be used to prove Hardy's analog [1910] of Theorem 6.1 for higher-order Cesàro summability; cf. Section 18 where we obtain a more general result. Knopp [1954] used induction to prove the extension of Theorem 6.1 to (C, k) summability.

There are other applications of difference formulas to Cesàro summability. In that way Kloosterman [1940b] obtained simple proofs of gap Tauberians of Meyer-König [1939] for (C, k) summability; cf. Section V.21 for a special case involving $k = 1$. A more recent application is in Marić and Tomić [1984].

7 Hardy–Littlewood Tauberians for Abel Summability

Littlewood [1911] answered Hardy's question whether the condition '$na_n \to 0$' in Tauber's theorem could be relaxed to boundedness of the sequence $\{na_n\}$.

Theorem 7.1. (Littlewood)

$$\sum a_n \textit{ is Abel summable } \& \ |na_n| \le C \quad \Rightarrow \sum a_n \textit{ converges.}$$

This 'big $\mathcal{O}$-theorem' for Abel summability is more difficult than the earlier results. The theorems in this section have attracted much interest and invited many alternative proofs, frequently with Theorem 7.3 as the first step towards Theorem 7.1. Littlewood's original proof was rather complicated; his key tool was repeated differentiation; cf. Section 17. For comments on Littlewood's fundamental article of 1911, see his Collected Papers [1982]; for the history of his discovery, see Littlewood [1953]. A first simple proof for the Theorem was found by Karamata [1930a]; see Section 11 for his method. A related more direct proof by Wielandt [1952] will be described in Section 12.

Like Littlewood's and Karamata's, many proofs depend on a construction which makes one partial sum stand out; cf. Section 23 and Izumi [1954], Zygmund [1959] (chapter 3, (1.3.8)). A general discussion of the use of *peak functions* in Tauberian theory may be found in Ingham [1965].

Another popular method depends on a selection principle, usually followed by a uniqueness theorem; see Sections II.3, II.6, IV.7 and cf. Delange [1949], König

[1960], Feller [1963], [1966/71]. Proofs involving complex analysis were given by Delange [1952] and Jurkat [1956a], [1957]. In the 1957 paper, Jurkat found it convenient to use the (more general) average-type Tauberian condition (7.2) below, with $p = 2$; cf. Landau and Gaier [1986] (appendix 2, section 1C). Still other proofs have been given by Reid [1954], Rubel [1960], and Tietz and Zeller [1998a], [1998b]. It was observed by Northcott [1947] that Littlewood's theorem and the proof by Karamata carry over to series $\sum_0^\infty a_n$ in a Banach space; cf. Maddox [1980].

The condition $|a_n| \leq C/n$ in Theorem 7.1 is *optimal* as an *order condition.* Littlewood [1911] has shown that for every positive function $\phi(n) \nearrow \infty$, there is an Abel summable series $\sum a_n$ with $|na_n| \leq \phi(n)$ which fails to converge. In Section 24 we give a simple construction which verifies this fact and the corresponding result for Cesàro summability.

Nevertheless weaker order conditions on the terms a_n may suffice for convergence of $\sum a_n$ if one has more information on $f(x) = \sum a_n x^n$ than just the existence of a limit as $x \nearrow 1$. This is illustrated by Fatou's theorem in Chapter III for the case where f is analytic at the point 1, and by an example in Section VII.2 for the case where $f(x)$ approaches its limit rather rapidly.

If one knows only that the limit of $f(x)$ as $x \nearrow 1$ exists, the condition $|na_n| \leq C$ can be relaxed in other ways. Hardy and Littlewood [1914a] showed that a one-sided condition suffices:

Theorem 7.2. (Hardy and Littlewood) *The following implication is valid:*

$$\sum a_n \textit{ is Abel summable } \ \& \ na_n \geq -C \ \Rightarrow \sum a_n \textit{ converges.}$$

In terms of the partial sums $s_n = \sum_{k=0}^n a_k$, Schmidt [1925a] could relax the Tauberian condition further to

$$\liminf\,(s_m - s_n) \geq 0 \quad \text{for } n \to \infty \text{ and } 1 < m/n \to 1. \tag{7.1}$$

A sequence $\{s_n\}$ as in (7.1) is said to be *slowly decreasing*; cf. Section 16 and for a proof of Schmidt's result, see Remarks 19.4 or Theorem 21.1. Earlier, Landau [1913] had used a corresponding two-sided condition of *slow oscillation* (Section 16). Notice that condition (7.1) is also *necessary* for convergence!

Several alternative Tauberian conditions have been introduced by Szász [1928], [1935], [1948], the simplest being the avarage-type condition

$$\frac{1}{n}\sum_{k=1}^{n} k^p |a_k|^p = \mathcal{O}(1), \quad \text{with } p > 1. \tag{7.2}$$

For $p = 1$ the condition has to be modified; see Rényi [1946], Szász [1951], Jakimovski [1954a].

Hardy and Littlewood jointly obtained a large number of conditional converses of continuity theorems, both for power series and for other kinds of series. Although these theorems were much more difficult than Tauber's original result, they called all of them Tauberian theorems.

The next two results are also in Hardy and Littlewood [1914a].

Theorem 7.3. (Hardy and Littlewood) *One has the following implication:*

$$\sum a_n \text{ is Abel summable } \& \ s_n \geq -C \ \Rightarrow \ \sum a_n \text{ is Cesàro summable.}$$

In the absence of more information on $f(x) = \sum a_n x^n$, the Tauberian condition cannot be relaxed to a condition $s_n \geq -\phi(n)$ with $\phi(n) \nearrow \infty$; see Section 24. Nevertheless the condition $s_n \geq -C$ may be replaced by a weaker condition, which involves boundedness from below in some average sense; see Bingham [1985] and Remarks 24.4.

Observe that the limit condition of Abel summability may be rewritten as

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n \to A;$$

cf. (2.1). Cesàro summability with Cesàro sum A is equivalent to the relation

$$\frac{s_0 + s_1 + \cdots + s_n}{n} \to A.$$

Considering the old s_n as new a_n, so that the old sum $s_0 + s_1 + \cdots + s_n$ becomes the new s_n, we see that Theorem 7.3 is the special case $\alpha = 1$ of the following Tauberian theorem of more general character. A proof may be derived from Section 15 where we consider an extension to integrals.

Theorem 7.4. (Hardy and Littlewood) *Let* $\sum_{n=0}^{\infty} a_n x^n$ *converge for* $|x| < 1$. *Suppose that for some number* $\alpha \geq 0$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim \frac{A}{(1-x)^{\alpha}} \quad \text{as } x \nearrow 1 \tag{7.3}$$

(in the sense that $(1-x)^{\alpha} f(x) \to A$*), while*

$$n a_n \geq -C n^{\alpha}, \quad \forall n \geq 1. \tag{7.4}$$

Then

$$s_n \sim A' n^{\alpha} \quad \text{as } n \to \infty, \text{ where } A' = \frac{A}{\Gamma(\alpha+1)}. \tag{7.5}$$

Remarks 7.5. The corresponding 'Abelian' result may be proved by manipulation with 'infinite Riemann sums': as $t \searrow 0$,

$$t^{\alpha+1} \sum_{n=1}^{\infty} n^{\alpha} e^{-nt} = \sum_{n=1}^{\infty} (nt)^{\alpha} e^{-nt} t \to \int_0^{\infty} v^{\alpha} e^{-v} dv = \Gamma(\alpha+1).$$

Hardy and Littlewood (loc. cit.) actually allowed more general comparison functions in Theorem 7.4, but with roughly the same asymptotic behavior; see Remarks 15.4.

It is not so well known that Theorem 7.4 has a companion which corresponds to *negative* α. Let us replace α by $-\beta$ with $\beta > 0$. If

$$f(x) = o\{(1-x)^\beta\} \text{ as } x \nearrow 1 \quad \text{and} \quad na_n \geq -Cn^{-\beta},$$

then

$$s_n = o(n^{-\beta}) \quad \text{as } n \to \infty.$$

This may be derived from a general *remainder theorem* in SectionVII.3; cf. Korevaar [1954b] and for the present result, de Bruijn [1958/81] (section 7.5).

Theorem 7.3 has analogs involving Cesàro summability of order k. We restrict ourselves to integral k here. If $\sum a_n$ is Abel summable to A and in the notation of Section 6,

$$a_n^{(-k)} \geq -Cn^{k-1}, \quad \text{then } s_n^{(-k)} = a_n^{(-k-1)} \sim A'n^k,$$

which is equivalent to the (C, k) summability of $\sum a_n$ (cf. Remarks 18.3). The relation for $s_n^{(-k)}$ follows from Theorem 7.4 applied to $f(x)/(1-x)^k = \sum_{n=0}^\infty a_n^{(-k)} x^n$. For the case $|a_n^{(-k)}| \leq Cn^{k-1}$ the result goes back to Littlewood [1911]. More general results, also involving nonintegral k, may be found in Hardy and Littlewood [1931], Kogbetliantz [1931] and Rajagopal [1958]; see also the comments on the Hardy–Littlewood article in Hardy's Collected Papers [1974].

8 Tauberians Involving Dirichlet Series

Hardy and Littlewood did not restrict themselves to theorems involving power series. From the beginning, they also considered general Dirichlet series

$$\sum_{n=0}^\infty a_n e^{-\lambda_n t}, \quad \text{where } 0 = \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_n \to \infty. \tag{8.1}$$

The following result is in Hardy and Littlewood [1914b]:

Theorem 8.1. *Let the series in* (8.1) *converge to the sum* $f(t)$ *for* $t > 0$ *and let* $f(t) \to A$ *as* $t \searrow 0$. *Suppose that* $\lambda_{n+1}/\lambda_n \to 1$ *and that*

$$a_n \geq -C\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}, \quad n \geq 1. \tag{8.2}$$

Then $s_N = \sum_{n=0}^N a_n \to A$ *as* $N \to \infty$.

Power series correspond to $\lambda_n = n$, while classical Dirichlet series $\sum_{n=1}^\infty a_n/n^t$ correspond to $\lambda_n = \log n$ ($n \geq 1$). For the latter series, the Tauberian condition becomes

$$a_n \geq -C/(n \log n), \quad n \geq 2.$$

We now turn to Dirichlet series (8.1) in which the positive indices λ_n increase at least geometrically (one speaks of *high indices*):

$$\lambda_{n+1} \geq \rho\lambda_n \quad (n \geq 1) \quad \text{for some} \quad \rho > 1. \tag{8.3}$$

For that case Hardy and Littlewood [1926] proved the *surprising result* that *no order condition* is needed on the sequence $\{a_n\}$:

Theorem 8.2. (High-indices theorem) *Let* $\sum_{n=0}^{\infty} a_n e^{-\lambda_n t}$ *converge to* $f(t)$ *for* $t > 0$. *Then*

$$f(t) \to A \quad \textit{as} \quad t \searrow 0 \quad \& \quad \textit{high-indices condition } (8.3) \quad \Rightarrow \quad \sum a_n = A.$$

For proofs and additional results, see Sections 22, 23.

9 Tauberians for Borel Summability

The following Tauberian theorem was obtained by Hardy and Littlewood [1916]; cf. Hardy and Littlewood [1943] for another proof.

Theorem 9.1. *Suppose that* $\sum_{n=0}^{\infty} a_n$ *is Borel summable to A (Example 2.4) and that*

$$|\sqrt{n}a_n| \leq C, \quad \forall\, n. \tag{9.1}$$

Then $s_n = \sum_{k=0}^{n} a_k \to A$ *as* $n \to \infty$.

Although optimal as an order condition (see Section VI.18), the Tauberian condition may be relaxed to $\sqrt{n}a_n \geq -C$, or even

$$\liminf\,(s_m - s_n) \geq 0 \quad \text{for} \quad n \to \infty \quad \text{and} \quad 0 < \sqrt{m} - \sqrt{n} \to 0. \tag{9.2}$$

This Tauberian condition for Borel summability was found by Schmidt [1925b]; see Section VI.12. More recent Tauberian theorems for Borel summability include a result for the case of positive s_n, due to Tenenbaum [1980], and a (difficult) 'high-indices theorem', due to Gaier [1965] and Mel'nik [1965]; see Sections VI.13 and VI.15–VI.17.

10 Lambert Tauberian and Prime Number Theorem

The most exciting of the Hardy–Littlewood theorems was the so-called Lambert Tauberian theorem [1921].

Theorem 10.1. *Suppose that* $\sum_{n=0}^{\infty} a_n$ *is Lambert summable to A:*

$$\sum_{n=0}^{\infty} a_n \frac{nt}{e^{nt} - 1} \quad \textit{converges for} \quad t > 0 \quad \text{and} \quad \textit{the sum function tends to} \quad A$$

as $t \searrow 0$ *(cf. Example 2.5), while*

$$|na_n| \leq C \quad \text{or} \quad na_n \geq -C.$$

Then $\sum_0^{\infty} a_n$ *converges to A.*

Wiener [1932] has shown that the Tauberian condition here may be relaxed to '$\{s_n\}$ slowly decreasing' as in (7.1); see Section II.12.

Theorem 10.1 is particularly interesting since it implies the famous PRIME NUMBER THEOREM (PNT) of Hadamard [1896] and de la Vallée Poussin [1896]:

Theorem 10.2. *The number $\pi(x)$ of primes $\le x$ satisfies the asymptotic relation*

$$\pi(x) \sim \frac{x}{\log x} \quad \textit{as } x \to \infty. \tag{10.1}$$

Derivation of Theorem 10.2 from Theorem 10.1. It was shown in Example 4.2 that the series $\sum_{n=1}^{\infty}\{\Lambda(n)-1\}/n$ is Lambert summable (to -2γ), hence by the Lambert Tauberian theorem it is convergent (to -2γ). Now if a series $\sum_1^{\infty} a_n$ converges, then

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} = s_n - \frac{s_1 + \cdots + s_{n-1}}{n} \to 0 \quad \text{as } n \to \infty.$$

Thus in our case

$$\frac{\Lambda(1) - 1 + \Lambda(2) - 1 + \cdots + \Lambda(n) - 1}{n} \to 0.$$

In terms of *Chebyshev's function* ψ, this may be written as

$$\psi(x) \stackrel{\text{def}}{=} \sum_{p^\alpha \le x} \log p = \sum_{k \le x} \Lambda(k) \sim x \quad \text{as } x \to \infty. \tag{10.2}$$

We will show that (10.2) implies (10.1). Indeed, $\psi(x)$ is a sum of terms $\log p$; it counts $\log p$ for every prime power $p^\alpha \le x$. Now first observe that the powers of p higher than the first may be ignored in the summation: if $p^2 \le x$ then $p \le x^{1/2}$, etc.; if in (10.2) we omit the sum over the powers $p^2, p^3, \cdots \le x$, the error is bounded by

$$(x^{1/2} + \cdots + x^{1/m}) \log x \le x^{1/2} \log x + m x^{1/3} \log x.$$

Here $m \le 2 \log x$ since one must have $2^m \le x$. Thus by (10.2)

$$\sum_{p \le x} \log p \sim x. \tag{10.3}$$

Next note that for nearly all $p \le x$ one has $\log p \sim \log x$, so that, heuristically,

$$\sum_{p \le x} \log p \sim \sum_{p \le x} \log x = \pi(x) \log x. \tag{10.4}$$

By (10.3) this gives (10.1). To make (10.4) rigorous one may observe that for $p > x^{1-\varepsilon}$ one has

$$\log p > (1-\varepsilon) \log x, \quad \text{while} \quad \sum_{p \le x^{1-\varepsilon}} \log p \le x^{1-\varepsilon} \log x, \quad \text{etc.}$$

Conversely, the prime number theorem (10.1) implies (10.2).

Remarks 10.3. At this point one should mention that the proof of Theorem 10.1 by Hardy and Littlewood [1921] was not independent of number theory. In fact, they used a number-theoretic estimate which is somewhat stronger than the prime number theorem itself! However, their method proved more than Theorem 10.1. It showed that the Lambert summability of a series *always* implies its Abel summability. For the opposite direction one can give Tauberian conditions; see Hardy and Littlewood [1936] and cf. Hardy [1949] (appendix 4).

The unsatisfactory situation around the Lambert Tauberian was resolved by Wiener. In his paper [1928], the beginning of his general Tauberian theory, Wiener succeeded in giving an independent proof for Theorem 10.1; see Chapter II. By his work, the Lambert Tauberian 10.1 is equivalent to the fact that the zeta function has no zeros on the line $\{\mathrm{Re}\, z = 1\}$. Thus Wiener showed that the *nonvanishing* of $\zeta(z)$ on the line $\{\mathrm{Re}\, z = 1\}$ implies the *prime number theorem.* The converse is also true; see Section III.3.

11 Karamata's Method for Power Series

Karamata [1930a] obtained a simple proof for the 'Abel to Cesàro' Theorem 7.3 and (indirectly) for Littlewood's Theorem 7.1.

Theorem 11.1. *Let* $\sum_{n=0}^{\infty} s_n x^n$ *converge for* $|x| < 1$ *and suppose that*

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n \to A \quad as\ x \nearrow 1. \tag{11.1}$$

Then the Tauberian condition

$$s_n \geq -C, \quad \forall n \tag{11.2}$$

implies that

$$\frac{1}{N} s_N^{(-1)} = \frac{1}{N}\sum_{n\leq N} s_n \to A \quad as\ N \to \infty. \tag{11.3}$$

Proof. It may be assumed that $s_n \geq 0$; if this is not the case, one may first add C to s_n, f and A; the conclusion $\sum_{n\leq N}(s_n + C) \sim (A+C)N$ will imply (11.3).

It follows from (11.1) that for every number $k > 0$,

$$(1-x)\sum_{n=0}^{\infty} s_n x^{kn} = \frac{1-x}{1-x^k} f(x^k) \to \frac{A}{k} = A\int_0^1 t^k \frac{dt}{t}$$

as $x \nearrow 1$. Hence if $P(t) = \sum_{k=1}^{m} b_k t^k$, then

$$(1-x)\sum_{n=0}^{\infty} s_n P(x^n) \to A\sum_{k=1}^{m} \frac{b_k}{k} = A\int_0^1 P(t)\frac{dt}{t} \quad \text{as } x \nearrow 1. \tag{11.4}$$

Can one write $s_N^{(-1)}$ in the form $\sum_{n=0}^{\infty} s_n P(x^n)$? Not with a linear combination or polynomial P, but one may use the characteristic function of an interval such as $[1/e, 1]$:

$$g(t) = \begin{cases} 0 \text{ for } 0 \leq t < 1/e, \\ 1 \text{ for } 1/e \leq t \leq 1. \end{cases} \tag{11.5}$$

If we now take $y = y_N = e^{-1/N}$, so that $y^n = e^{-n/N}$ and $g(y^n) = 1$ for $n \leq N$ while $g(y^n) = 0$ for $n > N$, then

$$s_N^{(-1)} = \sum_{n \leq N} s_n = \sum_{n=0}^{\infty} s_n g(y^n). \tag{11.6}$$

Below we will use approximation to show that in the limit relation (11.4), P may be replaced by g:

$$\lim_{x \nearrow 1} (1-x) \sum_{n=0}^{\infty} s_n g(x^n) = A \int_0^1 g(t) \frac{dt}{t} = A. \tag{11.7}$$

From (11.7) and (11.6) one will readily obtain the desired result (11.3) by taking $x = y = e^{-1/N}$:

$$\lim_{N \to \infty} \frac{1}{N} s_N^{(-1)} = \lim_{N \to \infty} (1 - e^{-1/N}) \sum_{n=0}^{\infty} s_n g(e^{-n/N}) = A.$$

To prove (11.7), it will be enough to show that the corresponding lim sup is less than or equal to A; the proof that the lim inf is at least equal to A is similar. For any given number $\varepsilon > 0$, one may construct a polynomial P without constant term such that

$$P \geq g \quad \text{and} \quad \int_0^1 \{P(t) - g(t)\} \frac{dt}{t} < 8\varepsilon, \tag{11.8}$$

see below. This will suffice: since $s_n \geq 0$, (11.8) and (11.4) give

$$\begin{aligned} \limsup_{x \nearrow 1} (1-x) \sum_{n=0}^{\infty} s_n g(x^n) &\leq \lim_{x \nearrow 1} (1-x) \sum_{n=0}^{\infty} s_n P(x^n) \\ = A \int_0^1 P(t) \frac{dt}{t} &\leq A \left\{ \int_0^1 g(t) \frac{dt}{t} + 8\varepsilon \right\} = A + 8A\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the lim sup on the left does not exceed A.

For the construction of a good majorizing polynomial P one may start with the continuous majorant h of g which is equal to g outside the interval $(1/e) - \varepsilon \leq t \leq 1/e$ and linear on that interval; we take $\varepsilon \leq 1/(2e)$. By Weierstrass's approximation theorem one can next determine a polynomial $P(t)/t$ which approximates the continuous function $\{h(t)/t\} + \varepsilon$ with error $\leq \varepsilon$:

$$\left| \frac{h(t)}{t} + \varepsilon - \frac{P(t)}{t} \right| \leq \varepsilon \quad \text{for } 0 \leq t \leq 1.$$

Then $P \geq h \geq g$ and P will also satisfy the integral inequality in (11.8):

$$\int_0^1 \frac{P(t) - h(t)}{t} dt \leq 2\varepsilon, \qquad \int_{(1/e)-\varepsilon}^{1/e} \frac{h(t) - g(t)}{t} dt < \frac{\varepsilon}{(1/e) - \varepsilon} < 6\varepsilon.$$

Remarks 11.2. A standard proof of Littlewood's Theorem 7.1 now goes as follows. The Abel summability of $\sum_{n=0}^{\infty} a_n$ to A implies that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} s_n x^n \to A \quad \text{as } x \nearrow 1,$$

where $s_n = \sum_{k=0}^{n} a_k$. Next, the boundedness of $\{na_n\}$ implies the boundedness of $\{s_n\}$; see Corollary 5.2. Thus by (11.3), $\sum_0^{\infty} a_n$ is Cesàro summable to A: $\sigma_N = s_{N-1}^{(-1)}/N \to A$. Finally, the convergence $\sum_0^{\infty} a_n = A$ follows from Hardy's Theorem 6.1.

The method above can also be used to verify the Hardy–Littlewood Theorem 7.4; cf. Karamata's proof for the extension of Theorem 7.4 to Laplace transforms in Section 15.

The conditions (11.1) and (11.2) imply that

$$(1 - x) \sum_{n=0}^{\infty} s_n \phi(x^n) \to A \int_0^1 \phi(t) \frac{dt}{t} \quad \text{as } x \nearrow 1 \tag{11.9}$$

whenever $\phi(t)/t$ is Riemannn integrable over [0, 1]. Indeed, such a function can be approximated from above and below by step functions (piecewise constant functions) in the appropriate manner; the step functions can be approximated by continuous functions $h(t)/t$ and the latter by polynomials $P(t)/t$.

12 Wielandt's Variation on the Method

We now present Wielandt's variation [1952] on Karamata's method to prove Littlewood's Theorem 7.1 and the Hardy–Littlewood Theorem 7.2. This proof avoids the detour via Cesàro summability. Such a direct proof is important when one wants to obtain sharp remainder estimates, as in Chapter VII.

Theorem 12.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ converge for $|x| < 1$ and suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \to A \quad \textit{as } x \nearrow 1. \tag{12.1}$$

Then each of the Tauberian conditions

$$|na_n| \leq C \quad \text{or} \quad na_n \geq -C, \quad \forall n \tag{12.2}$$

implies that

$$s_N = \sum_{n=0}^{N} a_n \to A \quad \textit{as } N \to \infty. \tag{12.3}$$

Proof. For every polynomial $P(x) = \sum_{k=1}^{m} b_k x^k$,

$$\sum_{n=0}^{\infty} a_n P(x^n) = \sum_{k=1}^{m} b_k \sum_{n=0}^{\infty} a_n x^{kn} \to \sum_{k=1}^{m} b_k \cdot A = P(1)A \tag{12.4}$$

as $x \nearrow 1$. Let g be the characteristic function of $[1/e, 1]$ as in (11.5), so that for $y = e^{-1/N}$,

$$s_N = \sum_{n \le N} a_n = \sum_{n=0}^{\infty} a_n g(y^n); \tag{12.5}$$

cf. (11.6). To complete the proof that $s_N \to A$, we will show below that the conclusion in (12.4) is also valid with g instead of P:

$$\lim_{x \nearrow 1} \sum a_n g(x^n) = g(1)A = A.$$

It will be enough to consider the one-sided Tauberian condition in (12.2) and to show that $\limsup_{x \nearrow 1} \sum a_n g(x^n) \le A$ (cf. Section 11).

To that end, we look for a suitable polynomial majorant

$$P \ge g \quad \text{for which} \quad P(0) = g(0) = 0, \quad P(1) = g(1) = 1.$$

Equivalently,

$$\frac{P(t) - t}{t(1-t)} \text{ must be a polynomial } Q(t) \ge h(t) = \frac{g(t) - t}{t(1-t)}. \tag{12.6}$$

Observe that h is piecewise continuous on $[0, 1]$; cf. Figure I.12.

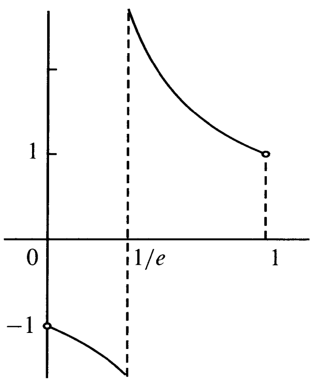

Fig. I.12. The graph of h

For $0 < x < 1$ it follows from (12.6) and the condition $na_n \geq -C$ that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n g(x^n) - \sum_{n=0}^{\infty} a_n P(x^n) = -\sum_{n=1}^{\infty} a_n\{P(x^n) - g(x^n)\} \\
&\leq C\sum_{n=1}^{\infty} \frac{1}{n}\{P(x^n) - g(x^n)\} \leq C\sum_{n=1}^{\infty} \frac{1-x}{1-x^n}\{P(x^n) - g(x^n)\} \\
&= C(1-x)\sum_{n=1}^{\infty} \phi(x^n), \quad \text{where } \phi(t) = \frac{P(t) - g(t)}{1-t}. \qquad (12.7)
\end{aligned}$$

By (12.4), the second term in the first member of (12.7) has limit A as $x \nearrow 1$. It may be derived from (11.9) that the final member in (12.7) has limit

$$C\int_0^1 \phi(t)\frac{dt}{t} = C\int_0^1 \frac{P(t) - t - \{g(t) - t\}}{t(1-t)}dt = C\int_0^1 \{Q(t) - h(t)\}dt. \quad (12.8)$$

For this one needs (11.9) only in the simple case $s_n \equiv 1$ so that $A = 1$; cf. (11.1). This case can be handled very simply with the aid of Riemann sums.

Now for given $\varepsilon > 0$, Weierstrass's theorem makes it possible to construct a polynomial $Q \geq h$ such that $\int_0^1 (Q - h) \leq \varepsilon$; cf. Section 11. For such a Q and the corresponding P determined by (12.6), it follows from (12.7), (12.4) and (12.8) that

$$\begin{aligned}
\limsup_{x\nearrow 1} \sum_{n=0}^{\infty} a_n g(x^n) &\leq \lim_{x\nearrow 1} \sum_{n=0}^{\infty} a_n P(x^n) + \lim_{x\nearrow 1} C(1-x)\sum_{n=1}^{\infty} \phi(x^n) \\
&= P(1)A + C\int_0^1 \{\phi(t)/t\}dt \leq A + C\varepsilon.
\end{aligned}$$

Since ε was arbitrary, our lim sup does not exceed A. □

Remarks 12.2. The essential tool in the proofs by Karamata and Wielandt is (one-sided, weighted) L^1 approximation by linear combinations $\sum_{k=1}^{m} b_k x^k$ (or $\sum b_\mu x^\mu$ with positive exponents μ). For the Lambert Tauberian Theorem 10.1 one would need weighted L^1 approximation by sums $\sum b_\mu L(x^\mu)$, where L is the Lambert kernel (Example 2.5). That kind of approximation is discussed in Wiener theory (Chapter II).

There is a modification of the Karamata–Wielandt proofs which avoids integrals; see Tietz and Zeller [1998b] and cf. Boos [2000] (section 4.4).

13 Transition from Series to Integrals

The Tauberian theorems for series have analogs for integrals which are often easier to treat; cf. Section 15. To obtain integral results which contain those for series one has to use *Stieltjes integrals* as in Sections 14, 21. Some preparatory remarks may be in order.

In principle, the (definite) *integrals* in this book are Lebesgue or Lebesgue–Stieltjes integrals, hence in particular absolutely convergent. The analogs of convergent infinite series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} a_n k_n$ or $\sum_{n=0}^{\infty} k_n a_n$, which may or may not be absolutely convergent, are *improper integrals*, for which we use notations such as

$$\int_0^{\infty-} a(v)dv = \lim_{B\to\infty} \int_0^B a(v)dv, \quad \int_{0-}^{\infty-} k(v)ds(v) = \lim_{B\to\infty} \int_{0-}^B k(v)ds(v). \tag{13.1}$$

[With $k(tv)$ or $k(v/u)$ instead of $k(v)$, the second formula defines a 'general-kernel transform'.] Here and in most subsequent sections,

$$\int_{0-}^{B} k(v)ds(v) = \lim_{\varepsilon\searrow 0} \int_{-\varepsilon}^{B} k(v)ds(v)$$

will be an ordinary Stieltjes integral, involving a function $s(v)$ which vanishes for $v < 0$ and is of bounded variation on every finite interval. Instead of $\int_{0-}^{B} k(v)ds(v)$ we could also write $\int_{-1}^{B} k(v)ds(v)$, but our notation serves as a reminder that $s(v) = 0$ for $v < 0$. It is convenient to normalize $s(\cdot)$ so that it becomes *continuous from the right.* Then $\int_a^b k(v)ds(v)$ (with $b > a$) can also be considered as a Lebesgue–Stieltjes integral over $[a, b]$ or $(a, b]$ with respect to the measure $ds(v)$, given by $(ds)[\alpha, \beta] = s(\beta) - s(\alpha)$. This integral satisfies the usual rule $\int_a^b + \int_b^c = \int_a^c$.

Absolute convergence is important for manipulations with integrals: inversion of the order of integration in repeated integrals (Section 21), or simple estimations such as

$$\left| \int_a^b k(v)ds(v) \right| \le \int_a^b |k(v)||ds(v)|.$$

Here $|ds(v)|$ stands for the variation measure associated with $ds(v)$. Denoting the total variation of $s(\cdot)$ over $[-1, v]$ by $\omega(v)$, one has

$$\int_a^b |k(v)||ds(v)| = \int_a^b |k(v)|d\omega(v).$$

For the transition from series $\sum_{n=0}^{\infty} a_n k_n$, involving coefficients a_n, to integrals it is convenient to define

$$s(v) = \sum_{n\le v} a_n, \quad -\infty < v < \infty. \tag{13.2}$$

Observe that the piecewise constant function $s(v)$ has jumps a_n at the points $v = n$, $n = 0, 1, 2, \dots$ and that $s(v) = 0$ for $v < 0$. The function $s(\cdot)$ is continuous from the right and $s(n)$ is equal to the usual partial sum s_n of the series $\sum a_n$.

We will describe how to obtain integral forms for Cesàro, Abel and Lambert summability. The Cesàro mean

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{1}{n} \sum_{k<n} a_k(n-k), \quad n \ge 1$$

is generalized by

$$\sigma(u) \stackrel{\text{def}}{=} \frac{1}{u}\int_{0-}^{u}(u-v)ds(v) = \frac{1}{u}\int_{0}^{u} s(v)dv = \int_{0}^{1} s(uv)dv, \quad 0 < u < \infty. \tag{13.3}$$

Observe that in the case of series, $\sigma(n) = \sigma_n$ when $n \geq 1$.

One says that a (formal) integral $\int_{0-}^{\infty} ds(\cdot)$ is *Cesàro summable* to A, and that a function $s(\cdot)$ is Cesàro limitable to A, if $\sigma(u) \to A$ as $u \to \infty$. More precisely we should speak here of $(C, 1)$ summability; the case of (C, k) summability will be considered in Section 18.

In the case of power series $\sum_{n=0}^{\infty} a_n x^n$ one can go to *Laplace transforms* by setting $x = e^{-t}$. Limits for $x \nearrow 1$ correspond to limits for $t \searrow 0$. If the series $\sum_{n=0}^{\infty} a_n e^{-nt}$ converges for all $t > 0$, the coefficients a_n are $o(e^{nt})$ for every $t > 0$, and hence $s(v)$ in (13.2) is $\mathcal{O}(e^{\varepsilon v})$ for every $\varepsilon > 0$. Thus one may integrate by parts to obtain

$$\sum_{n=0}^{\infty} a_n e^{-nt} = \int_{0-}^{\infty-} e^{-tv} ds(v) = t\int_{0}^{\infty} s(v)e^{-tv}dv, \quad 0 < t < \infty. \tag{13.4}$$

[Here the first integral is of course also absolutely convergent.] In the following we consider general Laplace–Stieltjes transforms

$$f(t) = \int_{0-}^{\infty-} e^{-tv} ds(v) = \lim_{B\to\infty}\int_{0-}^{B} e^{-tv} ds(v) \tag{13.5}$$

which converge for every $t > 0$, but need not converge absolutely. In this case one can still integrate by parts as in (13.4) to obtain a new integral which is absolutely convergent; see Corollary 13.2 below. The absolute convergence is important for Wiener theory (Chapter II). We sometimes refer to f as the Laplace transform of the measure ds: $f = \mathcal{L}ds$.

A (formal) integral $\int_{0-}^{\infty} ds(\cdot)$ will be called *Abel summable* to A, and the function $s(\cdot)$ Abel limitable, if the improper integral in (13.5) exists for every $t > 0$ and

$$\int_{0-}^{\infty-} e^{-tv} ds(v) \to A \quad \text{as } t \searrow 0.$$

By partial integration this relation can be rewritten as

$$t\int_{0}^{\infty} s(v)e^{-tv}dv \to A \quad \text{as } t \searrow 0. \tag{13.6}$$

The situation in the Lambert case is similar. In the case of convergent Lambert series one obtains

$$\sum_{n=0}^{\infty} a_n \frac{nt}{e^{nt}-1} = \int_{0-}^{\infty-} \frac{tv}{e^{tv}-1} ds(v) = \int_{0}^{\infty} s(v)\frac{d}{dv}\frac{-tv}{e^{tv}-1}dv. \tag{13.7}$$

A (formal) integral $\int_{0-}^{\infty} ds(\cdot)$ is called *Lambert summable* to A if the improper integral (‘Lambert transform’)

$$\tilde{f}(t) = \int_{0-}^{\infty-} \frac{tv}{e^{tv} - 1} ds(v) \tag{13.8}$$

exists for $t > 0$ and has limit A as $t \searrow 0$. One can again integrate by parts to obtain an absolutely convergent integral; see Corollary 13.2.

The old Tauberian conditions for series can be extended to conditions on $s(\cdot)$; see Section 16, where other suitable conditions will be introduced.

We now discuss the integration by parts, for which we consider general monotonic kernels instead of just $k(v) = e^{-tv}$ and $k(v) = tv/(e^{tv} - 1)$.

Proposition 13.1. *Let $k(\cdot)$ on $[-1, \infty)$ be positive, continuous and nonincreasing with $k(\infty-) = 0$. Let $s(v)$ vanish for $v < 0$, be of bounded variation on every finite interval, continuous from the right and such that the improper integral*

$$\int_{0-}^{\infty-} k(v) ds(v) = \lim_{B \to \infty} \int_{0-}^{B} k(v) ds(v) \tag{13.9}$$

exists. Then $s(v) = o\{1/k(v)\}$ as $v \to \infty$ and integration by parts gives

$$\int_{0-}^{\infty-} k(v) ds(v) = \int_{0}^{\infty-} s(v) d\{-k(v)\}. \tag{13.10}$$

Proof. Set

$$\phi(w) = -\int_{w}^{\infty-} k(v) ds(v) = -\int_{0-}^{\infty-} k(v) ds(v) + \int_{0-}^{w} k(v) ds(v). \tag{13.11}$$

Then $\phi(\cdot)$ is locally of bounded variation and continuous from the right. One has $\phi(w) = o(1)$ as $w \to \infty$ and in accordance with (13.11), the measure $d\phi$ is given by $d\phi(w) = k(w) ds(w)$. Integration by parts now shows that

$$\begin{aligned} s(v) &= \int_{0-}^{v} ds(w) = \int_{0-}^{v} \frac{1}{k(w)} d\phi(w) \\ &= \frac{\phi(v)}{k(v)} - \frac{\phi(0-)}{k(0)} - \int_{0}^{v} \phi(w) d\frac{1}{k(w)}. \end{aligned}$$

It will follow that $s(v) = o\{1/k(v)\}$ as $v \to \infty$. Indeed, the integrated terms are $o\{1/k(v)\}$ and so is $\int_0^b \phi(w) d\{1/k(w)\}$ for fixed $b > 0$. Furthermore, for given $\varepsilon > 0$, sufficiently large $b = b(\varepsilon)$ and $v > b$,

$$\left| \int_{b}^{v} \phi(w) d\frac{1}{k(w)} \right| < \varepsilon \int_{b}^{v} d\frac{1}{k(w)} < \frac{\varepsilon}{k(v)}.$$

We turn to (13.10). Integration by parts shows that

$$\int_{0-}^{B} k(v) ds(v) = k(B) s(B) + \int_{0}^{B} s(v) d\{-k(v)\}.$$

For $B \to \infty$ the integrated term goes to 0, and formula (13.10) follows.

Corollary 13.2. *If the Laplace transform of ds in* (13.5) *exists for every number $t > 0$, integration by parts gives an absolutely convergent integral. The same holds for the Lambert transform in* (13.8).

Indeed, in the Laplace case $k(v) = e^{-tv}$, so that by the Proposition, $s(v) = o(e^{tv})$ for every $t > 0$. Equivalently, $s(v) = \mathcal{O}(e^{\varepsilon v})$ for every $\varepsilon > 0$, etc.

14 Extension of Tauber's Theorems to Laplace–Stieltjes Transforms

Tauber's theorems for power series (Section 5) can be extended to Laplace–Stieltjes transforms (13.5). The extension is useful for general Dirichlet series; cf. Section 22. We present a common integral form for Tauber's two theorems.

Theorem 14.1. *Let $s(v) = 0$ for $v < 0$, be of bounded variation on every finite interval, continuous from the right and such that the improper integral*

$$f(t) = \int_{0-}^{\infty-} e^{-tv} ds(v) = \int_{-1}^{\infty-} e^{-tv} ds(v) \text{ exists for } t > 0 \text{ and tends to } A \text{ as } t \searrow 0. \tag{14.1}$$

(i) Suppose that

$$\int_u^{u+1} v|ds(v)| \to 0 \quad \text{as } u \to \infty. \tag{14.2}$$

Then

$$s(u) = \int_{0-}^{u} ds(v) \to A \quad \text{as } u \to \infty. \tag{14.3}$$

(ii) Necessary and sufficient for the convergence in (14.3) *is that*

$$\psi(u) \overset{\text{def}}{=} \int_{0-}^{u} v ds(v) = \int_0^u v ds(v) = o(u) \quad \text{as } u \to \infty. \tag{14.4}$$

A sufficient condition for (14.2) would be: $s(v) = \int_0^v a(w)dw$, with locally integrable $a(\cdot)$ such that $va(v) \to 0$ as $v \to \infty$. In this case $|ds(v)| = |a(v)|dv$. We normally suppose that limits, such as A in (14.1), are finite. However, conclusion (14.3) holds also if $f(t)$ is real and tends to $A = \pm\infty$ as $t \searrow 0$.

Proof of the Theorem. (i) Let (14.2) hold. Then (14.3) follows from the steps below which imply that $s(u) - f(1/u) \to 0$ as $u \to \infty$:

$$s(u) - f(1/u) = \int_0^u (1 - e^{-v/u}) ds(v) - \int_u^{\infty-} e^{-v/u} ds(v);$$

$$\sum_{n\le u}\int_n^{n+1}(1-e^{-v/u})|ds(v)|\le\frac{1}{u}\sum_{n\le u}\int_n^{n+1}v|ds(v)|=o(1),$$

$$\sum_{n>u}\int_n^{n+1}e^{-v/u}|ds(v)|\le\frac{1}{u}\sum_{n>u}e^{-n/u}\int_n^{n+1}v|ds(v)|=o(1).$$

(ii) Condition (14.4) is necessary for (14.3): if $s(u)\to A$, then

$$\frac{1}{u}\psi(u)=s(u)-\frac{1}{u}\int_0^u s(v)dv\to A-A=0.$$

For the proof of sufficiency it is convenient to make $s(v)=0$ for $v<1$. Apart from an inessential change of the limit A in (14.1), the effect is to make $\psi(v)=0$ for $v<1$ so that in the following we can integrate from 0 on. Suppose now that (14.4) is satisfied. It is consistent with (14.4) to write $d\psi(v)=vds(v)$. Then for $t>0$,

$$\begin{aligned}\int_0^{\infty-}e^{-tv}ds(v)&=\int_0^{\infty-}\frac{e^{-tv}}{v}d\psi(v)\\&=t\int_0^{\infty}\frac{\psi(v)}{v}e^{-tv}dv+\int_0^{\infty}\frac{\psi(v)}{v^2}e^{-tv}dv.\end{aligned}\tag{14.5}$$

By (14.1) the left-hand side tends to A as $t\searrow 0$. Since $\psi(v)/v\to 0$ as $v\to\infty$, the first term on the right tends to 0. Thus (14.5) shows that

$$\int_0^{\infty}\frac{\psi(v)}{v^2}e^{-tv}dv\to A\quad\text{as } t\searrow 0.$$

Setting $\psi(v)/v^2=a_1(v)$ one has $va_1(v)\to 0$ as $v\to\infty$, hence by part (i) applied to $ds(v)=a_1(v)dv$,

$$\int_0^u\frac{\psi(v)}{v^2}dv=\int_0^u a_1(v)dv\to A\quad\text{as } u\to\infty.\tag{14.6}$$

Integration by parts will complete the proof that $s(u)\to A$:

$$\int_0^u ds(v)=\int_0^u\frac{d\psi(v)}{v}=\frac{\psi(u)}{u}+\int_0^u\frac{\psi(v)}{v^2}dv\to A\quad\text{as } u\to\infty.$$

The proof above is similar to one given by Widder [1941] (chapter 5, section 3).

15 Hardy–Littlewood Type Theorems Involving Laplace Transforms

In this section some of the Tauberian theorems for power series are extended to Laplace transforms.

Theorem 15.1. *Let $a(\cdot)$ be locally integrable and such that the improper integral $F(t) = \int_0^{\infty-} a(v)e^{-tv}dv$ exists for $t > 0$. Suppose that for some constant $\alpha \geq 0$,*

$$F(t) \sim A/t^\alpha \quad as \quad t \searrow 0, \tag{15.1}$$

and that for certain constants $b \geq 0$ and C

$$a(v) \geq -Cv^{\alpha-1}, \quad b < v < \infty. \tag{15.2}$$

Then

$$s(u) = \int_0^u a(v)dv \sim \frac{A}{\Gamma(\alpha+1)}u^\alpha \quad as \quad u \to \infty. \tag{15.3}$$

The notation $F(t) \sim A/t^\alpha$ stands for $t^\alpha F(t) \to A$, also when $A = 0$.

For the proof one may take $b = 0$: if $b > 0$, one can first prove the Theorem with $a(v) = 0$ on $(0, b)$. In the case $\alpha = 0$, where $\int_0^\infty a(v)dv$ is Abel summable to A, one can give a proof similar to the proof for series in Section 12. We leave this to the reader; a more general result will be obtained in Section 21. The proof for $\alpha > 0$ will be derived from Theorem 15.3 below, but we first illustrate the case $\alpha = 0$.

Example 15.2. Consider the Laplace transform

$$\int_0^\infty \frac{\sin v}{v}e^{-tv}dv = \frac{1}{2}\pi - \arctan t, \quad t > 0. \tag{15.4}$$

[This formula may be obtained by differentiating under the integral sign and integrating the Laplace transform of $\sin v$ from t to ∞.] Since the integral in (15.4) has limit $\pi/2$ as $t \searrow 0$, Theorem 15.1 shows that the improper integral

$$\int_0^{\infty-} \frac{\sin v}{v}dv \quad \text{exists and equals} \quad \pi/2.$$

Theorem 15.3. *Let $s(v)$ vanish for $v < 0$, be nondecreasing, continuous from the right and such that*

$$F(t) = \int_{0-}^{\infty-} e^{-tv}ds(v) = \int_{0-}^{\infty} e^{-tv}ds(v) \quad exists\ for \quad t > 0. \tag{15.5}$$

Suppose that for some constant $\alpha > 0$,

$$F(t) \sim A/t^\alpha \quad as \quad t \searrow 0 \quad [or\ as \quad t \to \infty]. \tag{15.6}$$

Then

$$s(u) \sim \frac{A}{\Gamma(\alpha+1)}u^\alpha \quad as \quad u \to \infty \quad [or\ as \quad u \searrow 0, \quad respectively]. \tag{15.7}$$

Proof. We use Karamata's method [1931]; cf. Section 11. By (15.5) and (15.6)

$$\int_{0-}^{\infty} e^{-ktv} ds(v) \sim Ak^{-\alpha} t^{-\alpha} = \frac{At^{-\alpha}}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-kv} dv^{\alpha} \qquad (k > 0).$$

Hence if $P(x) = \sum_{k=1}^{m} b_k x^k$ and $A' = A/\Gamma(\alpha+1)$,

$$\int_{0-}^{\infty} P(e^{-tv}) ds(v) \sim A' t^{-\alpha} \int_0^{\infty} P(e^{-v}) dv^{\alpha}. \tag{15.8}$$

Let $g(\cdot)$ be as in (11.5) so that $g(e^{-v}) = 1$ for $0 \le v \le 1$ and $= 0$ for $v > 1$. For given $\varepsilon > 0$ we now determine a polynomial P such that

$$P(x) \ge g(x), \quad 0 \le x \le 1 \quad \text{and} \quad \int_0^1 \{P(x) - g(x)\}\alpha \left(\log \frac{1}{x}\right)^{\alpha-1} \frac{dx}{x} \le \varepsilon; \tag{15.9}$$

cf. the construction in Section 11. Then

$$\int_{0-}^{\infty} P(e^{-tv}) ds(v) \ge \int_{0-}^{\infty} g(e^{-tv}) ds(v) = s(1/t), \tag{15.10}$$

provided $1/t$ is a point of continuity for $s(\cdot)$ so that the last integral is well-defined (the function $g(e^{-tv})$ has a jump at $v = 1/t$). Also by (15.9)

$$\int_0^{\infty} P(e^{-v}) dv^{\alpha} \le \int_0^{\infty} g(e^{-v}) dv^{\alpha} + \varepsilon = 1 + \varepsilon. \tag{15.11}$$

By (15.8) we can choose $\delta > 0$ so small that for $0 < t \le \delta$ [or for $t \ge 1/\delta$],

$$\int_{0-}^{\infty} P(e^{-tv}) ds(v) \le (A' + \varepsilon) t^{-\alpha} \int_0^{\infty} P(e^{-v}) dv^{\alpha}. \tag{15.12}$$

Combining (15.10)–(15.12) one finds

$$s(1/t) \le (A' + \varepsilon)(1 + \varepsilon) t^{-\alpha} \quad \text{for } 0 < t \le \delta \quad [\text{or for } t \ge 1/\delta].$$

This holds also at points $1/t$ where $s(\cdot)$ is discontinuous because $s(\cdot)$ is nondecreasing. Approximating g from below one similarly obtains $s(1/t) \ge (A' - \varepsilon)(1 - \varepsilon) t^{-\alpha}$ for $0 < t \le \delta'$ [or $t \ge 1/\delta'$], so that (15.7) follows.

Proof of Theorem 15.1. (Case $\alpha > 0$) The function $a^*(v) = a(v) + Cv^{\alpha-1}$ is nonnegative and has Laplace transform $F^*(t) \sim \{A + C\Gamma(\alpha)\}/t^{\alpha}$, while $s^*(u) = \int_0^u a^*(v) dv$ is nondecreasing. Applying Theorem 15.3 to s^* and F^* one concludes that

$$s^*(u) \sim \frac{A + C\Gamma(\alpha)}{\Gamma(\alpha+1)} u^{\alpha}, \quad s(u) \sim \frac{A}{\Gamma(\alpha+1)} u^{\alpha}.$$

Remarks 15.4. Theorem 15.1 for the case $a(v) \geq 0$ is due to Doetsch [1920]. Results of the type of Theorem 15.3 were obtained by Szász [1929], Hardy and Littlewood [1929], Doetsch [1930] and Karamata [1931]. Cf. also expositions in Doetsch [1937], [1950] and Widder [1941]. Instead of requiring that $s(\cdot)$ be nondecreasing it is enough to have $s(v) + Cv^\alpha$ nondecreasing: since $\alpha > 0$ one can replace $s(v)$ by $s(v) + Cv^\alpha$.

The comparison function A/t^α in (15.6) may be replaced by $L(1/t)/t^\alpha$ (if $t \searrow 0$) or by $L(t)/t^\alpha$ (if $t \to \infty$), where

$$L(t) = (\log t)^{\alpha_1}(\log\log t)^{\alpha_2}\cdots \quad \text{with } \alpha_j \in \mathbb{R};$$

cf. Hardy and Littlewood [1914a] for the case of power series. More generally, L may be any *slowly varying function* in the sense of Karamata [1930b]. This is a positive (locally integrable) function such that

$$\frac{L(\lambda t)}{L(t)} \to 1 \quad \text{as } t \to \infty \text{ for every number } \lambda > 0.$$

Cf. Section IV.2; an example different from the earlier functions is given by $L(t) = \exp\{(\log t)^\alpha\}$ with $0 < \alpha < 1$.

In conclusion (15.7), the constant A would now be replaced by $L(u)$ (if $u \to \infty$) or by $L(1/u)$ (if $u \searrow 0$). The general result is due to Karamata [1931]; see Section IV.8 and cf. Hardy [1949] (section 7.11).

One can prove related results for $\alpha < 0$; cf. Remarks 7.5 and Section VII.3. For monotonic $s(\cdot)$ with $s(v) = 0$ for $v < 0$ and $\beta = -\alpha > 0$,

$$F(t) = o(t^\beta) \text{ as } t \searrow 0 \;\Rightarrow\; s(u) = o(u^{-\beta}) \text{ as } u \to \infty. \tag{15.13}$$

In [1991], Baran used Karamata's method to obtain partial-sum estimates for Dirichlet series with positive coefficients, which he applied to prime number theory.

16 Other Tauberian Conditions: Slowly Decreasing Functions

The Tauberian condition $|na_n| \leq C$ of Sections 6, 7, 10 can be expressed in terms of the partial sum function $s(v) = \sum_{n \leq v} a_n$: taking $0 < v < w$ one finds

$$|s(w) - s(v)| = \left|\sum_{v<n\leq w} a_n\right| \leq C \sum_{v<n\leq w} \frac{1}{n} \leq C\frac{w-v+1}{v}.$$

In many Tauberian theorems the conclusion $s(v) \to A$ as $v \to \infty$ will remain valid for arbitrary (bounded) slowly oscillating functions $s(\cdot)$; cf. Landau [1913], Schmidt [1925a]:

Definition 16.1. One says that a function $s(\cdot)$ is *slowly oscillating* on $(0, \infty)$ if

$$s(w) - s(v) \to 0 \text{ as } v \to \infty \text{ and } w/v \to 1.$$

If $s(\cdot)$ is differentiable and $s'(v) = \mathcal{O}(1/v)$ then $s(\cdot)$ is slowly oscillating. An example would be $s(v) = v^{i\alpha}$ with real α.

Here we also mention a related condition of piecewise constancy. It comes from the case $s(v) = \sum_{n \le v} a_n$ with $a_n \ne 0$ only if n belongs to a 'Hadamard sequence' $\{\lambda_k\}$, that is, $\lambda_{k+1}/\lambda_k \ge \rho > 1$. In this case $s(v)$ is constant for $\lambda_k \le v < \lambda_{k+1}$, an interval of logarithmic length $\log(\lambda_{k+1}/\lambda_k) \ge \log \rho$.

Definition 16.2. We say that $s(\cdot)$ satisfies a *step function condition* on $(0, \infty)$ if $s(\cdot)$ is piecewise constant, with the intervals of constancy having logarithmic length $\ge c > 0$.

The one-sided Tauberian condition $na_n \ge -C$ may be written as

$$s(w) - s(v) \ge -C \sum_{v<n\le w} \frac{1}{n} \ge -C\frac{w-v+1}{v} \qquad (w > v > 0).$$

Generalizing, one is led to consider arbitrary so-called slowly decreasing functions $s(\cdot)$; cf. Schmidt [1925a]. Only the decrease of such functions is restricted, not their increase:

Definition 16.3. One says that a function $s(\cdot)$ is *slowly decreasing* on $(0, \infty)$ if

$$\liminf \{s(\rho v) - s(v)\} \ge 0 \quad \text{for } v \to \infty \text{ and } 1 < \rho \to 1. \tag{16.1}$$

The terminology is standard but misleading: every increasing function is slowly decreasing! If $s(\cdot)$ is differentiable and $s'(v) \ge -C/v$ then $s(\cdot)$ is slowly decreasing.

Given $\varepsilon > 0$, (16.1) implies that there exist B and $0 < \delta \le 1$ such that

$$s(w) - s(v) \ge -\varepsilon \quad \text{for } v \ge B \text{ and } 0 \le \log(w/v) \le \delta. \tag{16.2}$$

Thus $s(w) - s(v) \ge -(n+1)\varepsilon$ if $n\delta \le \log(w/v) \le (n+1)\delta$, so that

$$s(w) - s(v) \ge -(\varepsilon/\delta)\log(w/v) - \varepsilon \quad \text{for } w \ge v \ge B. \tag{16.3}$$

Hence $s(v) + (\varepsilon/\delta)\log v$ is essentially nondecreasing. In most applications $s(\cdot)$ is bounded on every finite interval $(0, B)$, so that finally

$$s(w) - s(v) \ge -c_1 \log(w/v) - c_2 \quad \text{for } w \ge v > 0. \tag{16.4}$$

More generally we will later consider functions $s(\cdot)$ for which

$$s(w) - s(v) \ge -M(w/v) \quad \text{for } w \ge v > 0, \tag{16.5}$$

where $M(\cdot)$ is nonnegative and nondecreasing. There are also conditions in which $s(v)$ is compared to a power v^α; one might have $s(v)/v^\alpha$ slowly decreasing.

17 Asymptotics for Derivatives

Under suitable conditions asymptotic relations can be differentiated. We start with a very simple but useful case.

Lemma 17.1. *Suppose that for some real number* α

$$f(x) \sim Ax^{\alpha} \quad as \;\; x \to \infty,$$

where f *is differentiable and* f' *is nondecreasing. Then*

$$f'(x) \sim A\alpha x^{\alpha-1} \quad as \;\; x \to \infty.$$

Indeed, by the mean-value theorem $f(x+h) - f(x) \geq hf'(x)$ when $h > 0$. Thus for given $\varepsilon > 0$, when x is large and $0 < h < x$, say,

$$\begin{aligned} f'(x) &\leq \frac{A\{(x+h)^{\alpha} - x^{\alpha}\} + \varepsilon x^{\alpha}}{h} \\ &\leq A\alpha x^{\alpha-1} + Cx^{\alpha-2}h + \varepsilon x^{\alpha}/h. \end{aligned}$$

Taking $h = \sqrt{\varepsilon}x$ one finds that

$$f'(x) \leq A\alpha x^{\alpha-1} + (C+1)\sqrt{\varepsilon}x^{\alpha-1}.$$

It follows that $\limsup f'(x)/x^{\alpha-1} \leq A\alpha$. For an inequality in the other direction one would start with $f(x) - f(x-h) \leq hf'(x)$.

There are similar results for a nonincreasing derivative and for a variable going to 0. As an illustration from early Tauberian work, suppose that

$$F(t) = \sum_{n=0}^{\infty} s_n e^{-nt} \text{ converges for } t > 0 \quad \text{and} \quad F(t) \sim At^{-\alpha} \text{ as } t \searrow 0,$$

where $\alpha > 0$. Then the condition $s_n \geq 0$ is enough to ensure that

$$-F'(t) = \sum_{n=1}^{\infty} s_n n e^{-nt} \sim A\alpha t^{-\alpha-1} \quad \text{as } t \searrow 0.$$

By repetition one finds that

$$(-1)^k F^{(k)}(t) = \sum_{n=1}^{\infty} s_n n^k e^{-nt} \sim A\alpha(\alpha+1)\cdots(\alpha+k-1)t^{-\alpha-k};$$

cf. Littlewood [1911], Hardy and Littlewood [1914a], Landau and Gaier [1986]. Littlewood and Hardy used repeated differentiation of $\sum s_n e^{-nt}$ to increase the weight of a specific partial sum s_N, so that it could be estimated.

In the following refinement we go back to variable x instead of t and we let x go to ∞, but there are corresponding results for $x \searrow 0$.

Theorem 17.2. *Let f be defined on an interval (a, ∞) and such that for real numbers α and A,*

$$f(x) \sim Ax^\alpha \quad (\text{that is, } x^{-\alpha} f(x) \to A) \text{ as } x \to \infty. \tag{17.1}$$

(i) Suppose that f is $(k+1)$ times differentiable and

$$f^{(k+1)}(x) \geq -Cx^{\alpha-k-1}. \tag{17.2}$$

Then as $x \to \infty$

$$f^{(j)}(x) \sim A\alpha(\alpha-1)\cdots(\alpha-j+1)x^{\alpha-j} \quad \text{for } 1 \leq j \leq k. \tag{17.3}$$

(ii) The same conclusion holds if f is just k times differentiable and

$$\liminf \{f^{(k)}(y) - f^{(k)}(x)\}/x^{\alpha-k} \geq 0 \quad \text{for } x \to \infty \text{ and } 1 < y/x \to 1. \tag{17.4}$$

For $\alpha = k$ condition (17.4) means that $f^{(k)}(\cdot)$ is slowly decreasing.

Corollary 17.3. *If f is k times differentiable and $f(x) \sim Ax^k$ for $x \to \infty$ while $f^{(k)}$ is slowly decreasing, then $f^{(k)}(x) \to k!A$.*

For the proof of the Theorem we use an auxiliary result on finite differences

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k-j)h);$$

cf. Markov [1896], Kloosterman [1940a], [1950] and Korevaar [1954d].

Proposition 17.4. *(i) Let f be $(k+1)$ times differentiable on (a, b). Then for $h \neq 0$ and x, $x + kh$ in (a, b),*

$$h^{-k}\Delta_h^k f(x) = f^{(k)}(x) + \frac{1}{2}khf^{(k+1)}(\xi), \tag{17.5}$$

with $x < \xi < x + kh$ or $x + kh < \xi < x$ (depending on the sign of h).

(ii) If f is just k times differentiable, the product $hf^{(k+1)}(\xi)$ in (17.5) may be replaced by a suitable difference of $f^{(k)}$:

$$h^{-k}\Delta_h^k f(x) = f^{(k)}(x) + \frac{1}{2}(k+1)\{f^{(k)}(x_2) - f^{(k)}(x_1)\}, \tag{17.6}$$

where $x \leq x_1 < x_2 \leq x_1 + h \leq x + kh$ or $x + kh \leq x_2 < x_1 \leq x_2 + h \leq x$.

Proof. Let us take $h > 0$ for convenience and start with *case (i)*. One can use induction with respect to k. For $k = 1$ relation (17.5) is a special case of Taylor's formula. Next suppose that (17.5) is true for a given k and that f is $(k+2)$ times differentiable. Applying (17.5) to $\Delta_h f(x) = f(x+h) - f(x)$, and later using the case $k = 1$ and the mean-value theorem, one obtains

$$\begin{aligned}
h^{-k}\Delta_h^{k+1} f(x) &= f^{(k)}(x+h) - f^{(k)}(x) + \frac{1}{2}kh\{f^{(k+1)}(\xi+h) - f^{(k+1)}(\xi)\} \\
&= hf^{(k+1)}(x) + \frac{1}{2}h^2 f^{(k+2)}(\xi_1) + \frac{1}{2}kh^2 f^{(k+2)}(\xi_2) \\
&= hf^{(k+1)}(x) + \frac{1}{2}h^2\{f^{(k+2)}(\xi_1) + kf^{(k+2)}(\xi_2)\}. \qquad (17.7)
\end{aligned}$$

Here ξ_1 and ξ_2 belong to the interval $(x, x+(k+1)h)$. Now by the intermediate value property of derivatives (Darboux property), a derivative takes on all values between its infimum and its supremum; cf. Rudin [1953/76] (section 5.12). Hence

$$f^{(k+2)}(\xi_1) + kf^{(k+2)}(\xi_2) = (k+1)f^{(k+2)}(\xi_3) \quad \text{with } \xi_3 \in (x, x+(k+1)h).$$

This gives (17.5) with $k+1$ instead of k.

The case $k = 1$ of *part (ii)* is straightforward:

$$\{f(x+h) - f(x)\}/h = f'(\xi) = f'(x) + \{f'(\xi) - f'(x)\},$$

where $x < \xi < x+h$. For $k > 1$ one may apply formula (17.5), with $k-1$ instead of k, to the function $\Delta_h f(x)$. This gives

$$h^{-(k-1)}\Delta_h^{k-1}\Delta_h f(x) = \Delta_h f^{(k-1)}(x) + \frac{1}{2}(k-1)h\Delta_h f^{(k)}(\xi),$$

where $x < \xi < x+(k-1)h$. Dividing by h and using the case $k = 1$, one obtains

$$h^{-k}\Delta_h^k f(x) - f^{(k)}(x) = \{f^{(k)}(\xi') - f^{(k)}(x)\} + \frac{1}{2}(k-1)\{f^{(k)}(\xi+h) - f^{(k)}(\xi)\}$$

with $x < \xi' < x+h$. The right-hand side is at most equal to

$$\sup \frac{1}{2}(k+1)\{f^{(k)}(x_4) - f^{(k)}(x_3)\} \quad \text{for } x \le x_3 < x_4 \le x_3 + h \le x + kh,$$

and at least equal to the corresponding infimum. Formula (17.6) may now be obtained from the Darboux property. □

Proof of Theorem 17.2. It is sufficient to consider the case $j = k$: for smaller j the argument may be repeated. We only deal with part (ii). Subtracting Ax^α from $f(x)$ one may assume $A = 0$; this step does not spoil (17.4). Let $\varepsilon > 0$ be given. By (17.4) there are constants $\delta \in (0, 1]$ and $B > 0$ such that

$$f^{(k)}(y) - f^{(k)}(x) \ge -\varepsilon x^{\alpha-k} \quad \text{for } x > B \text{ and } 1 < y/x \le 1+\delta. \qquad (17.8)$$

We also take B so large that $|f(x)| \le \varepsilon x^\alpha$ for $x > B$ and now use (17.6) with $a \le B < x < x+kh \le x+\delta x < b$. Then $x \le x_1 < x_2 \le (1+\delta)x$, so that by (17.8)

$$f^{(k)}(x_2) - f^{(k)}(x_1) \ge -\varepsilon x_1^{\alpha-k} \ge -M\varepsilon x^{\alpha-k},$$

where $M = \max\{2^{\alpha-k}, 1\}$. Hence

$$
\begin{aligned}
f^{(k)}(x) &= h^{-k}\Delta_h^k f(x) - \frac{1}{2}(k+1)\{f^{(k)}(x_2) - f^{(k)}(x_1)\} \\
&\le h^{-k}2^k M'\varepsilon x^\alpha + \frac{1}{2}(k+1)M\varepsilon x^{\alpha-k},
\end{aligned}
$$

where $M' = \max\{2^\alpha, 1\}$. Choosing $h = (\delta/k)x$ one concludes that there is a constant $C = C(\delta, k)$ such that

$$
f^{(k)}(x) \le C\varepsilon x^{\alpha-k} \quad \text{for } x > B.
$$

For an estimate in the other direction one would work with $h < 0$. □

Remarks 17.5. The first part of Proposition 17.4 may be used to prove a precise *convexity theorem* for derivatives such as the following (Kloosterman [1940a]). If f is $(k+1)$ times differentiable for $x > 0$ and

$$
|f(x)| \le \phi(x), \quad |f^{(k+1)}(x)| \le \psi(x),
$$

where ϕ and ψ are positive nonincreasing, then

$$
|f^{(k)}(x)| \le 2k\{\phi(x)\}^{1/(k+1)}\{\psi(x)\}^{k/(k+1)}.
$$

There are similar results for other derivatives and other intervals, and for functions of a discrete (integral) variable.

Convexity results for derivatives go back to Hardy and Littlewood [1913b] and Riesz [1923]. See also Bosanquet [1943], Kloosterman [1950] and the book by Chandrasekharan and Minakshisundaram [1952]. A related area is that of convexity results for supremum norms of successive derivatives; cf. Gorny [1939], Cartan [1940].

The three-body problem of celestial mechanics has given rise to some unusual differentiation results. Initial asymptotic formulas by Boas [1939] and Karamata [1939a] were strengthened by Pollard and Saari [1970], and Saari [1969], [1974], to big $\mathcal{O}$-results of the following type. Under certain nonlinear conditions coming from celestial mechanics, the relation $f(x) = \mathcal{O}(x^\alpha)$ implies $f'(x) = \mathcal{O}(x^{\alpha-1})$ as $x \to \infty$ or $x \searrow 0$.

18 Integral Tauberians for Cesàro Summability

There is extensive literature on Cesàro summability; cf. Hardy [1949]. We limit ourselves to the case of integrals and a corresponding Tauberian result. Let $a(\cdot)$ be integrable over every finite interval $(0, B)$ and set

$$
s^{(-k)}(u) = \int_0^u dv_1 \int_0^{v_1} dv_2 \cdots \int_0^{v_k} a(v)dv = \frac{1}{\Gamma(k+1)}\int_0^u (u-v)^k a(v)dv. \tag{18.1}
$$

The Cesàro mean of $a(\cdot)$ of order k is defined by

$$
\sigma_k(u) = \frac{s^{(-k)}(u)}{u^k/\Gamma(k+1)} = \int_0^u \left(1 - \frac{v}{u}\right)^k a(v)dv. \tag{18.2}
$$

One says that the (formal) integral $\int_0^\infty a(v)dv$ is (C, k) *summable* to the value A if

$$\sigma_k(u) \to A \quad \text{or} \quad s^{(-k)}(u) \sim \frac{A}{\Gamma(k+1)} u^k \quad \text{as } u \to \infty; \tag{18.3}$$

cf. Hardy (section 5.14). The final expressions in (18.1) and (18.2) may be used as definitions for nonintegral values $\lambda > 0$ of k, but we will consider only integral values. In fact, a simple operation with a repeated integral shows that (C, λ) summability implies (C, k) summability for any $k > \lambda$. For integral k it is not hard to prove that (C, k) summability implies Abel summability.

For functions $s(\cdot)$ with $s(v) = 0$ for $v < 0$ which are locally of bounded variation and continuous from the right, one may consider formula (18.1) with $ds(v)$ instead of $a(v)dv$. Integration by parts then leads to a definition of (C, k) *limitability* of $s(\cdot)$ through the relation

$$\sigma_k(u) = \int_{0-}^{u} \left(1 - \frac{v}{u}\right)^k ds(v) = \frac{k}{u} \int_0^u \left(1 - \frac{v}{u}\right)^{k-1} s(v)dv \to A. \tag{18.4}$$

Before we derive a Tauberian theorem for (C, k) summability from Section 17 we give an independent proof for the important case of $(C, 1)$ limitability.

Theorem 18.1. *Let $s(\cdot)$ be locally integrable and Cesàro limitable to A:*

$$\sigma(u) = \frac{s^{(-1)}(u)}{u} = \frac{1}{u} \int_0^u s(v)dv = \int_0^1 s(uv)dv \to A \quad \textit{as } u \to \infty. \tag{18.5}$$

In addition, let $s(\cdot)$ be slowly decreasing (Definition 16.3) or satisfy a step function condition as in Definition 16.2. Then

$$s(u) \to A \quad \textit{as } u \to \infty. \tag{18.6}$$

Proof. Replacing $s(v)$ by $s(v) - A$ if necessary one may assume that $A = 0$. Suppose $s(\cdot)$ slowly decreasing. Then for given $\varepsilon > 0$ there exist $B \geq 0$ and $0 < \delta < 1$ such that $s(\rho v) - s(v) \geq -\varepsilon$ for $v \geq B$ and $1 < \rho \leq 1 + \delta$. It follows that

$$\int_u^{(1+\delta)u} s(v)dv \geq \delta u s(u) - \varepsilon\delta u \quad \text{for } u \geq B. \tag{18.7}$$

On the other hand, given $\eta > 0$, relation (18.5) (with $A = 0$) shows that there exists B' such that $|s^{(-1)}(u)| = |\int_0^u s(v)dv| \leq \eta u$ for $u \geq B'$. Hence

$$\int_u^{(1+\delta)u} s(v)dv = \int_0^{(1+\delta)u} - \int_0^u \leq (2+\delta)\eta u \quad \text{for } u \geq B'. \tag{18.8}$$

Combining (18.7) and (18.8) one concludes that for $u \geq \max\{B, B'\}$,

$$s(u) \leq \varepsilon + \{(2+\delta)/\delta\}\eta < 2\varepsilon \quad \text{if one takes } \eta = \delta\varepsilon/3.$$

One similarly shows that $s(u) > -2\varepsilon$ for all large u by integrating $s(\cdot)$ over the interval $(u/(1+\delta), u)$. Conclusion: $s(u) \to 0$ as $u \to \infty$.

In the case of the step function condition, $s(\cdot)$ is constant on intervals $[u, (1+\delta)u)$ with fixed $\delta > 0$ inside every interval of constancy. Here (18.8) shows immediately that $\delta u s(u) \le (2+\delta)\eta u$, etc.

Higher-Order Case. Theorem 17.2 leads to the following form of the Tauberian theorem of Hardy [1910] and Landau [1910] for (C, k) summability; cf. also Korevaar [1954d].

Theorem 18.2. *Suppose that $\int_0^\infty a(v)dv$ is (C, k) summable to A and set $s(v) = \int_0^v a(w)dw$. Then the condition '$s(\cdot)$ slowly decreasing' implies that $s(u) \to A$ as $u \to \infty$.*

Proof. Define $f(u) = s^{(-k)}(u)$. Then f is k times differentiable on $(0, \infty)$, $f(u) \sim Au^k/k!$ as $u \to \infty$ and $f^{(k)}(\cdot) = s(\cdot)$ is slowly decreasing. Hence by Corollary 17.3, $s(u) \to A$.

Remarks 18.3. For infinite series $\sum_0^\infty a_n$, Cesàro summability of (integral) order $k \ge 0$ is defined by convergence of the (C, k) means

$$C_n^{(k)} = s_n^{(-k)} \bigg/ \binom{n+k}{k}; \quad \binom{n+k}{k} \sim \frac{n^k}{k!} \quad \text{as } n \to \infty. \tag{18.9}$$

Here $s_n^{(-k)} = s_0^{(-k+1)} + \cdots + s_n^{(-k+1)}$, $s_n^{(0)} = s_n$. Warning: if one takes $s(v) = \sum_{n \le v} a_n$ in formula (18.4) one obtains different means:

$$\sigma_k(u) = \sum_{n \le u} \left(1 - \frac{n}{u}\right)^k a_n. \tag{18.10}$$

Nevertheless the corresponding summability (with continuous variable u) is equivalent to (C, k) summability. See Hardy [1949] (for integral k) and Ingham [1968b] (for arbitrary $k \ge 0$). The means in (18.10) are the special case $\lambda_n = n$ of the *Riesz means* or *typical means*

$$R_\lambda^{(k)}(u) = \sum_{\lambda_n \le u} \left(1 - \frac{\lambda_n}{u}\right)^k a_n. \tag{18.11}$$

Such means have been used extensively in connection with Dirichlet series. Cf. Hardy (sections 4.16, 5.16) and books by Hardy and Riesz [1915] and Chandrasekharan and Minakshisundaram [1952]; see also Boos [2000]. A special case will be encountered in Section VI.13.

19 The Method of the Monotone Minorant

In the proof of Tauberian theorems for series $\sum a_n$ it is often necessary to begin by establishing boundedness of the sequence of partial sums $s_n = \sum_{k \le n} a_k$. Depending

on the Tauberian condition this may be easy or difficult. As an illustration, suppose one has a power series or Lambert series on $\{0 < x < 1\}$ with bounded sum function. Under the Tauberian condition $|na_n| \le C$ it is easy to deduce boundedness of the sequence $\{s_n\}$; see Theorem 5.4. Somewhat more complicated but elementary arguments also work under the one-sided condition $na_n \ge -C$; cf. Remarks 5.5. However, if one knows only that the partial sums s_n satisfy a more general Schmidt condition, the boundedness is more difficult to prove.

Vijayaraghavan [1926], [1928] found a good way to do this for the case of Abel and Borel summability. His technique may be described as the 'method of the monotone minorant'; cf. the proof below and the (more technical) treatment of the general-kernel transform in Section 20.

Comprehensive boundedness theorems for series which are based on the method may be found in Hardy's book [1949] (sections 12.13, 12.14) and in Tietz and Zeller [2001]. Since those theorems and their proofs are quite complicated, we explain the method here for the simple case of power series; cf. Section VI.8 for the Borel case.

Theorem 19.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ converge for $0 \le x < 1$ to a bounded sum function $f(x)$. Suppose that any decrease of the partial-sum function $s(w) = \sum_{n \le w} a_n$ is moderate in the following sense. One has*

$$s(vu) - s(u) \ge -M(v) \quad \text{for } u \ge 0,\ v \ge 1, \tag{19.1}$$

where $M(\cdot)$ is nonnegative, nondecreasing for $v \ge 1$ and such that

$$\int_{\lambda}^{\infty} M(v/\lambda) e^{-v} dv < \infty, \quad \int_0^{\lambda} M(\lambda/v) e^{-v} dv < \infty, \quad \forall \lambda > 0. \tag{19.2}$$

Then the function $s(\cdot)$ is bounded on $(0, \infty)$.

The conditions (19.1), (19.2) are amply satisfied if $s(\cdot)$ is slowly decreasing: in that case one may take $M(v) = c_1 \log v + c_2$; see (16.4).

Preparations for the Proof. Taking $u > 0$, it is convenient to set

$$x = e^{-1/u}, \quad x^n - x^{n+1} = (1 - e^{-1/u}) e^{-n/u} = K_n(u). \tag{19.3}$$

Observe that $\sum_{n=0}^{\infty} K_n(u) = 1$,

$$H(\lambda u) \overset{\text{def}}{=} \sum_{n<\lambda u} K_n(u) = 1 - e^{-N/u} \qquad (\lambda > 0,\ \lambda u \le N < \lambda u + 1). \tag{19.4}$$

Lemma 19.2. *Let $M(\cdot)$ be as in Theorem 19.1. Then for fixed $\lambda > 0$ and for u running over $[1, \infty)$,*

$$\sum_{n \ge \lambda u} M\left(\frac{n}{\lambda u}\right) K_n(u) = \mathcal{O}(1), \qquad \sum_{1 \le n < \lambda u} M\left(\frac{\lambda u}{n}\right) K_n(u) = \mathcal{O}(1). \tag{19.5}$$

Proof. It will be enough to consider the first sum, $S_1(u)$. By monotonicity,

$$\begin{aligned} S_1(u) &\le \sum_{n\ge\lambda u} \int_n^{n+1} M\left(\frac{t}{\lambda u}\right)(1-e^{-1/u})e^{-(t-1)/u}dt \\ &\le (e^{1/u}-1)\int_{\lambda u}^{\infty} M\left(\frac{t}{\lambda u}\right)e^{-t/u}dt \\ &= u(e^{1/u}-1)\int_{\lambda}^{\infty} M\left(\frac{v}{\lambda}\right)e^{-v}dv = \mathcal{O}(1) \end{aligned}$$

for $u \ge 1$; see (19.2).

We now introduce the negative of the *monotone minorant*,

$$\sigma(u) \stackrel{\text{def}}{=} -\inf_{v\le u} s(v) = \sup_{v\le u}\{-s(v)\} \tag{19.6}$$

(which vanishes for $u < 0$).

Lemma 19.3. *With $s(\cdot)$ and $M(\cdot)$ as in Theorem 19.1, the function $\sigma(\cdot)$ is nonnegative and nondecreasing, and*

$$\sigma(vu) \le \sigma(u) + M(v) \quad \textit{for} \;\; u \ge 0,\;\; v \ge 1. \tag{19.7}$$

Proof. For $w \ge 1$ inequality (19.1) shows that

$$-s(wu) \le -s(u) + M(w) \le \sigma(u) + M(w).$$

If $\sigma(vu) = \sigma(u)$ there is nothing to prove. Suppose now that $\sigma(vu) > \sigma(u)$. For any given small $\eta > 0$ there is then a number $w \in (1, v]$ for which one has $-s(wu) > \sigma(vu) - \eta$. Hence

$$\sigma(vu) < -s(wu) + \eta \le \sigma(u) + M(w) + \eta \le \sigma(u) + M(v) + \eta.$$

For $\eta \searrow 0$ this implies (19.7).

Proof of Theorem 19.1. Writing $s(n) = s_n$, $x = e^{-1/u}$ and $f(x) = F(u)$, we obtain from (19.3) that

$$F(u) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} s_n(x^n - x^{n+1}) = \sum_{n=0}^{\infty} s(n)K_n(u). \tag{19.8}$$

By the hypotheses $F(u)$ is bounded on $\{0 < u < \infty\}$.

(i) For fixed $\lambda > 0$ and u running over $[1, \infty)$, by (19.1), (19.4) and (19.5),

$$\begin{aligned} \sum_{n\ge\lambda u} s(n)K_n(u) &\ge s(\lambda u)\sum_{n\ge\lambda u} K_n(u) - \sum_{n\ge\lambda u} M\left(\frac{n}{\lambda u}\right)K_n(u) \\ &= s(\lambda u)\{1 - H(\lambda u)\} - \mathcal{O}(1). \end{aligned}$$

On the other hand, by (19.6)

$$\sum_{n \ge \lambda u} s(n)K_n(u) = F(u) - \sum_{n<\lambda u} s(n)K_n(u)$$
$$\le \mathcal{O}(1) + \sigma(\lambda u)H(\lambda u).$$

Observe that $H(\lambda u) \approx 1 - e^{-\lambda}$ when u is large by (19.4). Combining the above inequalities one obtains the estimate

$$s(\lambda u) \le \frac{H(\lambda u)}{1 - H(\lambda u)}\sigma(\lambda u) + \mathcal{O}(1) \quad \text{for } u \ge B. \tag{19.9}$$

We will use (19.9) with $\lambda = a$, where $a > 0$ is chosen such that $1 - e^{-a}$ is less than $1/2$. Then $H(au) < 1/2$ for all large u. Thus for $\lambda = a$ it follows from (19.9) that

$$[-\sigma(v) \le]\ s(v) \le \sigma(v) + \mathcal{O}(1), \tag{19.10}$$

first for $v \ge v_0$ and finally, for $0 < v < \infty$. Hence if $\sigma(\cdot)$ is bounded, so is $s(\cdot)$.

(ii) With the aid of (19.10) one can prove an inequality for $s(\cdot)$ which goes in the other direction. By (19.1), (19.4) and (19.5),

$$- \sum_{1 \le n < \lambda u} s(n)K_n(u) \ge -s(\lambda u) \sum_{1 \le n < \lambda u} K_n(u) - \sum_{1 \le n < \lambda u} M\left(\frac{\lambda u}{n}\right) K_n(u)$$
$$= -s(\lambda u)\{H(\lambda u) - K_0(u)\} - \mathcal{O}(1).$$

On the other hand, by (19.8), (19.10), (19.7) and (19.5),

$$- \sum_{1 \le n < \lambda u} s(n)K_n(u) = -F(u) + s(0)K_0(u) + \sum_{n \ge \lambda u} s(n)K_n(u)$$
$$\le \mathcal{O}(1) + \sum_{n \ge \lambda u} \sigma(n)K_n(u)$$
$$\le \mathcal{O}(1) + \sigma(\lambda u) \sum_{n \ge \lambda u} K_n(u) + \sum_{n \ge \lambda u} M\left(\frac{n}{\lambda u}\right) K_n(u)$$
$$= \sigma(\lambda u)\{1 - H(\lambda u)\} + \mathcal{O}(1).$$

Since $H(\lambda u) \to 1 - e^{-\lambda}$ and $K_0(u) \to 0$ as $u \to \infty$, one may combine the above inequalities to obtain

$$-s(\lambda u) \le \frac{1 - H(\lambda u)}{H(\lambda u) - K_0(u)}\sigma(\lambda u) + C(\lambda) \quad \text{for all large } u. \tag{19.11}$$

We will use (19.11) with $\lambda = b$, where b is chosen such that $1 - e^{-b} > 2/3$. As a result $H(bu) - K_0(u) > 2/3$ and $1 - H(bu) < 1/3$ for all large u. It follows that

$$-s(w) \le \frac{1}{2}\sigma(w) + \mathcal{O}(1) \quad \text{as } w \to \infty. \tag{19.12}$$

Suppose now that $\sigma(\cdot)$ is unbounded. Then by (19.12), $-s(w) < (2/3)\sigma(w)$ for all large w, but this would contradict the definition $\sigma(w) = \sup_{v \le w}\{-s(v)\}$. Hence $\sigma(\cdot)$ and $s(\cdot)$ are bounded.

Remarks 19.4. One can use Theorem 19.1 to prove Schmidt's extension of Littlewood's Theorem 7.1 and the Hardy–Littlewood Theorem 7.2. Indeed, if $\sum a_n$ is Abel summable to A and the sequence of partial sums $\{s_n\}$ is slowly decreasing, Theorem 19.1 shows that $\{s_n\}$ is bounded. It then follows from the Hardy–Littlewood Theorem 7.3 that $\sum a_n$ is Cesàro summable to A. The convergence of $\sum a_n$ now follows as in Section 6; see Remarks 6.3.

20 Boundedness Theorem Involving a General-Kernel Transform

The theorem below is formulated for Stieltjes integrals involving a general kernel. It is an extension of Karamata's form ([1932], [1933a]) of Vijayaraghavan's theorem; cf. also Delange [1950]. The theorem will be used for Laplace–Stieltjes integrals in Section 21 and is especially important for the application of Wiener theory to the Lambert transform in Section II.12. Other very general boundedness theorems may be found in Pitt's book [1958] (chapter 2).

Theorem 20.1. (Boundedness Theorem) *Let $k(\cdot)$ on $[-1,\infty)$ be positive, continuous and nonincreasing with $k(\infty-)=0$. Let $s(v)$ vanish for $v<0$, be of bounded variation on every finite interval, continuous from the right and such that the general-kernel transform*

$$F(u)=\int_{0-}^{\infty-}k(v/u)ds(v)=\int_{0-}^{\infty-}k(v)d_v s(vu) \tag{20.1}$$

exists and represents a bounded function on a half-line $\{u>u_0\}$ with $u_0\ge 0$. Finally suppose that

$$s(vu)-s(u)\ge -M(v)\quad \textit{for}\ \ u\ge 0,\ v\ge 1, \tag{20.2}$$

where $M(v)$ is nonnegative, nondecreasing for $v\ge 1$ and such that

$$\int_\lambda^\infty M(v/\lambda)d\{-k(v)\}<\infty,\qquad \int_0^\lambda M(\lambda/v)d\{-k(v)\}<\infty,\quad \forall\lambda>0. \tag{20.3}$$

Then $s(\cdot)$ is bounded.

The notation $d_v s(vu)$ in (20.1) serves to emphasize that v is the variable of integration.

Remark 20.2. If $s(\cdot)$ is *slowly decreasing*, condition (20.3) reduces to

$$\int_0^\infty |\log v| d\{-k(v)\}<\infty; \tag{20.4}$$

cf. (16.4).

Preparations for the Proof. We introduce the negative of the *monotone minorant* of $s(\cdot)$ as in Section 19:

$$\sigma(u) = -\inf_{v \le u} s(v) = \sup_{v \le u}\{-s(v)\}. \tag{20.5}$$

Recall that the function $\sigma(\cdot)$ is nonnegative and nondecreasing, and that

$$\sigma(vu) \le \sigma(u) + M(v) \quad \text{whenever } u \ge 0,\ v \ge 1. \tag{20.6}$$

For the proof of Theorem 20.1 it is convenient to take $k(0) = 1$. By Proposition 13.1 we may apply integration by parts to the final integral in (20.1) to obtain the representation

$$F(u) = \int_0^{\infty -} s(vu)d\{-k(v)\}, \quad u_0 < u < \infty. \tag{20.7}$$

We summarize the next steps in

Proposition 20.3. *There are positive constants C_1 and a, and for any $\lambda > 0$ there are numbers $C_2(\lambda)$, $C_3(\lambda)$, such that*

$$-\sigma(w) \le s(w) \le \sigma(w) + C_1, \quad au_0 < w < \infty, \tag{20.8}$$

$$\int_\lambda^{\infty -} s(vu)d\{-k(v)\} \ge s(\lambda u)k(\lambda) - C_2(\lambda), \quad 0 < u < \infty, \tag{20.9}$$

$$\int_\lambda^{\infty -} s(vu)d\{-k(v)\} \le \sigma(\lambda u)k(\lambda) + C_3(\lambda), \quad au_0 < \lambda u < \infty. \tag{20.10}$$

Proof. (i) We begin with (20.9). By (20.2),

$$\begin{aligned}\int_\lambda^{\infty -} s(vu)d\{-k(v)\} - s(\lambda u)k(\lambda) &= \int_\lambda^{\infty -} \{s(vu) - s(\lambda u)\}d\{-k(v)\} \\ &\ge -\int_\lambda^{\infty} M(v/\lambda)d\{-k(v)\} = -C_2(\lambda).\end{aligned} \tag{20.11}$$

(ii) We turn to (20.8). The first inequality follows directly from (20.5). Also by (20.5)

$$\int_0^\lambda s(vu)d\{-k(v)\} \ge -\sigma(\lambda u)\int_0^\lambda d\{-k(v)\} = -\sigma(\lambda u)\{1 - k(\lambda)\}.$$

Hence by (20.7), (20.9) and the boundedness condition on F,

$$\begin{aligned}C_4 \ge F(u) &= \int_0^{\infty -} s(vu)d\{-k(v)\} = \left(\int_0^\lambda + \int_\lambda^{\infty -}\right)\cdots \\ &\ge s(\lambda u)k(\lambda) - \sigma(\lambda u)\{1 - k(\lambda)\} - C_2(\lambda) \quad \text{for } u > u_0.\end{aligned}$$

It follows that

$$s(\lambda u) \le \frac{1 - k(\lambda)}{k(\lambda)}\sigma(\lambda u) + C_5(\lambda), \quad u_0 < u < \infty.$$

Setting $\lambda = a$, where a is such that $k(a) = 1/2$, we obtain the second inequality in (20.8) for $w = au > au_0$.

(iii) Taking $\lambda u > au_0$, we now estimate $\int_\lambda^{\infty-} s(vu)d\{-k(v)\}$ from above to establish (20.10). Using (20.8), (20.6) and (20.3) on the way, one finds

$$\begin{aligned}&\int_\lambda^{\infty-} s(vu)d\{-k(v)\} \le \int_\lambda^{\infty-} \{\sigma(vu) + C_1\}d\{-k(v)\}\\ &\le \int_\lambda^{\infty} \{\sigma(\lambda u) + M(v/\lambda) + C_1\}d\{-k(v)\} = \sigma(\lambda u)k(\lambda) + C_3(\lambda).\end{aligned}$$

Completion of the Proof of Theorem 20.1. We also have to estimate the integral $\int_0^\lambda s(vu)d\{-k(v)\}$ from above. By (20.2) and (20.3)

$$\begin{aligned}&\int_0^\lambda s(vu)d\{-k(v)\} - s(\lambda u)\{1 - k(\lambda)\} = \int_0^\lambda \{s(vu) - s(\lambda u)\}d\{-k(v)\}\\ &\le \int_0^\lambda M(\lambda/v)d\{-k(v)\} = C_6(\lambda).\end{aligned}$$

Taking $u > u_0$, $\lambda u > au_0$ and combining this inequality with (20.9), one obtains

$$\begin{aligned}-C_7 \le F(u) &= \int_0^{\infty-} s(vu)d\{-k(v)\}\\ &\le s(\lambda u)\{1 - k(\lambda)\} + \sigma(\lambda u)k(\lambda) + C_8(\lambda).\end{aligned}$$

It follows that for $\lambda > a$

$$-s(\lambda u) \le \frac{k(\lambda)}{1 - k(\lambda)}\sigma(\lambda u) + C_9(\lambda), \quad u_0 < u < \infty.$$

We now take $\lambda = b$ where $b > a$ is such that $k(b) = 1/3$. This gives

$$-s(w) \le \frac{1}{2}\sigma(w) + C_{10}, \quad bu_0 < w < \infty.$$

If $\sigma(\cdot)$ would tend to ∞, the final inequality would contradict the fact that $\sigma(w) = \sup_{v\le w}\{-s(v)\}$. Hence $\sigma(\cdot)$ is bounded and by (20.8) the same is true for $s(\cdot)$.

Corollary 20.4. *Under the conditions of Theorem 20.1, the integral for $F(u)$ in* (20.7) *is absolutely convergent.*

21 Laplace–Stieltjes and Stieltjes Transform

We begin with a generalization of the Hardy–Littlewood Theorems 7.3 and 7.2 for power series.

Theorem 21.1. *Let $s(v)$ vanish for $v < 0$, be of bounded variation on every finite interval and continuous from the right. Suppose that $\int_{0-}^{\infty} ds(\cdot)$ is Abel summable to A, that is, the Laplace–Stieltjes transform*

$$f(t) = \mathcal{L}ds(t) = \int_{0-}^{\infty-} e^{-tv} ds(v) \tag{21.1}$$

exists for $t > 0$ and

$$f(t) \to A \quad as \quad t \searrow 0. \tag{21.2}$$

Conclusions:

(i) if $s(\cdot) \geq -C$, then $s(\cdot)$ is Cesàro limitable to A (Section 13);
(ii) if $s(\cdot)$ is slowly oscillating or slowly decreasing, then $s(u) \to A$ as $u \to \infty$.

Related results are in Szász [1929] and Karamata [1931]. Theorem 21.1 includes Schmidt's extension [1925a] of Theorems 7.1, 7.2 to the case of slowly decreasing sequences $\{s_n\}$.

Proof of the Theorem. By Proposition 13.1 and its Corollary we may integrate by parts to obtain

$$f(t) = \int_{0-}^{\infty-} e^{-tv} ds(v) = t \int_{0}^{\infty} s(v) e^{-tv} dv \quad \text{for } t > 0. \tag{21.3}$$

Case (i). Having (21.3) we can apply Theorem 15.1 with $\alpha = 1$, $a(v) = s(v) \geq -C$ and

$$F(t) = \int_{0}^{\infty} s(v) e^{-tv} dv = f(t)/t \sim A/t \quad \text{as } t \searrow 0.$$

The conclusion is that

$$\int_{0}^{u} s(v) dv \sim Au, \qquad \text{so that } \sigma(u) = \frac{1}{u} \int_{0}^{u} s(v) dv \to A \quad \text{as } u \to \infty.$$

Thus $s(\cdot)$ is Cesàro limitable to A.

In case (ii) we set $t = 1/u$ and rewrite $f(1/u)$ as

$$f(1/u) = \int_{0-}^{\infty-} e^{-v} d_v s(vu).$$

The kernel $k(v) = e^{-v}$ and the slowly decreasing function $s(\cdot)$ satisfy the conditions of Boundedness Theorem 20.1 [with $f(1/u)$ in place of $F(u)$]; cf. Remark 20.2. It follows that $s(\cdot)$ is bounded, so that $s(\cdot)$ is Cesàro limitable to A just as in case (i). The Cesàro summability Theorem 18.1 now shows that $s(u) \to A$ as $u \to \infty$.

Stieltjes Transform. The (first order) Stieltjes transform of the measure $ds(\cdot)$ on $[0, \infty)$ is formally given by

$$g(x) = \int_{0-}^{\infty-} \frac{ds(v)}{x+v}. \tag{21.4}$$

Let us assume that $s(v)$ vanishes for $v < 0$, is locally of bounded variation and continuous from the right. If $s(v) \sim Cv^\alpha$ as $v \to \infty$ for some $\alpha \in [0, 1)$, integration by parts shows that the transform $g(x)$ exists for $x > 0$ and that

$$g(x) = \int_0^\infty \frac{s(v)}{(x+v)^2} dv \sim C \int_0^\infty \frac{v^\alpha}{(x+v)^2} dv = C' x^{\alpha-1} \quad \text{as } x \to \infty. \tag{21.5}$$

Taking $x = 1$, the substitution $v = y/(1-y)$ shows that $C' = C$ if $\alpha = 0$ and

$$C' = C \int_0^1 y^\alpha (1-y)^{-\alpha} dy = C\Gamma(1+\alpha)\Gamma(1-\alpha) = C \frac{\alpha\pi}{\sin \alpha\pi}$$

if $0 < \alpha < 1$; cf. Whittaker and Watson [1927/96] (section 12.4).

Tauberian theorems for Stieltjes transforms may be obtained by repeated application of theorems for the Laplace transform. We restrict ourselves to the first-order case here; higher-order results will be discussed in Sections IV.9 and VII.18.

Theorem 21.2. *Let $s(v)$ vanish for $v < 0$, be nondecreasing, continuous from the right and such that the Stieltjes transform $g(x)$ in* (21.4) *exists for $x > 0$. Suppose that for some number $\alpha \in [0, 1)$,*

$$g(x) \sim Ax^{\alpha-1} \quad \textit{as } x \to \infty. \tag{21.6}$$

Then

$$\begin{aligned} &s(u) \sim A'u^\alpha \quad \textit{as } u \to \infty, \quad \textit{where} \\ &A' = \frac{A}{\Gamma(1+\alpha)\Gamma(1-\alpha)} = A\frac{\sin \alpha\pi}{\alpha\pi}. \end{aligned} \tag{21.7}$$

Proof. Substituting

$$\frac{1}{x+v} = \int_0^\infty e^{-(x+v)t} dt$$

in the formula for $g(x)$ and inverting the order of integration (which is justified by absolute convergence), one finds that

$$g(x) = \int_0^\infty e^{-xt} f(t) dt, \quad \text{with } f(t) = \int_{0-}^\infty e^{-tv} ds(v). \tag{21.8}$$

Since $f \geq 0$, its integral $f^{(-1)}(t) = \int_0^t f(w)dw$ is nondecreasing. Hence by Theorem 15.3, relation (21.6) for the Laplace transform $g = \mathcal{L}df^{(-1)}$ implies that

$$f^{(-1)}(t) = \int_0^t f(w)dw \sim \frac{A}{\Gamma(2-\alpha)} t^{1-\alpha} \quad \text{as } t \searrow 0. \tag{21.9}$$

Now f is monotonic, so that relation (21.9) may be differentiated to give

$$f(t) \sim \frac{A}{\Gamma(1-\alpha)} t^{-\alpha} \quad \text{as } t \searrow 0; \tag{21.10}$$

cf. Section 17. Having (21.10) for $f = \mathcal{L}ds$, one may use Theorem 15.3 (if $\alpha > 0$) or Theorem 21.1 (if $\alpha = 0$) to establish the desired result (21.7).

Remarks 21.3. In his book [1949] (section 7.10), Hardy showed how part (ii) of Theorem 21.1 can be derived from Theorem 15.1. In that way one can avoid the 'heavy' Boundedness Theorem 20.1.

Theorem 21.2 is due to Hardy and Littlewood [1929], who also considered higher-order Stieltjes transforms. The simple method of proof above is due to Doetsch [1930] and Karamata [1931]. The latter showed that the condition '$s(\cdot)$ nondecreasing' in Theorem 21.2 can be relaxed. Assuming $s(\cdot)$ to be locally of bounded variation, it is sufficient if $s(v) \geq -Cv^{\alpha}$ when $0 < \alpha < 1$, and slowly decreasing when $\alpha = 0$; cf. Section VII.18.

Valiron [1914] and Titchmarsh [1927] had already used arguments of Tauberian character for transforms of Stieltjes type in their early work on the zeros of entire functions of regular growth; cf. Section III.19.

22 General Dirichlet Series

As we saw in Section 8, such series have the form

$$f(t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t}, \quad \text{with } 0 = \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_n \to \infty. \tag{22.1}$$

If the Dirichlet series converges for $t > 0$ and $f(t) \to A$ as $t \searrow 0$, the numerical series $\sum a_n$ is sometimes called (A, λ) *summable*; cf. Hardy's book [1949] (section 4.7). One may write

$$f(t) = \int_{0-}^{\infty-} e^{-tv} ds(v), \quad \text{where } s(v) = \sum_{\lambda_n \leq v} a_n. \tag{22.2}$$

What would the Tauber-type Theorem 14.1 give here? In the present case, condition (14.4) for convergence $s(u) \to A$ becomes

$$\psi(u) = \int_0^u v ds(v) = \sum_{\lambda_n \leq u} \lambda_n a_n = o(u) \quad \text{as } u \to \infty.$$

As a corollary one obtains an early result of Landau [1907]:

Proposition 22.1. *Suppose that*

$$a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \quad \textit{as } n \to \infty, \tag{22.3}$$

and that the sum function $f(t)$ in (22.1) *tends to A as $t \searrow 0$. Then $\sum_0^{\infty} a_n$ converges to A.*

Indeed, if λ_k is the largest number $\lambda_n \leq u$, then under condition (22.3)

$$\psi(u) = \sum_{n=1}^{k} \lambda_n a_n = \sum_{n=1}^{k} o(\lambda_n - \lambda_{n-1}) = o(\lambda_k) = o(u) \quad \text{as } u \to \infty.$$

The following refinement had its origin in work by Littlewood and Hardy; for the complicated history, see Remarks 22.3 below.

Theorem 22.2. *Let the general Dirichlet series* (22.1) *converge to the sum* $f(t)$ *for* $t > 0$ *and let* $f(t) \to A$ *as* $t \searrow 0$. *Suppose that the coefficients* a_n *satisfy at least one of the following two conditions* (22.4), (22.5)*:*

$$|a_n| \le C \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}, \quad n \ge 1; \tag{22.4}$$

$$a_n \ge -C \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}, \quad n \ge 1 \quad \text{AND} \quad \liminf_{n\to\infty} a_n \ge 0. \tag{22.5}$$

Then $s_N = \sum_{n=0}^{N} a_n \to A$ *as* $N \to \infty$.

Observe that in the classical case $\lambda_n = \log n$ $(n \ge 1)$, one has the sufficient condition

$$a_n \ge -C/(n \log n), \quad n \ge 2.$$

Proof. Here we establish only the sufficiency of condition (22.5). To that end we apply Theorem 21.1 to the present function $f(t)$ which we represent as in (22.2). It is then enough to show that $s(\cdot)$ is slowly decreasing. One may assume $C > 0$ and for given $\varepsilon > 0$, take $B > 0$ so large that $a_n > -\varepsilon$ for $\lambda_n > B$. Next, let $u > B$, $v = (1+\varepsilon)u$ and choose $p = p(u)$, $q = q(v)$ such that $\lambda_{p-1} \le u < \lambda_p$ and $\lambda_q \le v < \lambda_{q+1}$. Then by (22.5)

$$\begin{aligned} s(v) - s(u) &= \sum_{u<\lambda_n\le v} a_n = a_p + \sum_{n=p+1}^{q} a_n \\ &> -\varepsilon - C \sum_{n=p+1}^{q} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \ge -\varepsilon - (C/u)(\lambda_q - \lambda_p) \\ &> -\varepsilon - (C/u)(v-u) = -(1+C)\varepsilon. \end{aligned} \tag{22.6}$$

It follows that $s(\cdot)$ is indeed slowly decreasing. □

For the sufficiency of condition (22.4) we refer to Section II.15 (where we prove a more general result), or to Hardy's book (p. 176).

Remarks 22.3. Littlewood [1911] proved the sufficiency of the two-sided condition in (22.4) under the hypothesis that $\lambda_{n+1}/\lambda_n \to 1$ as $n \to \infty$. For the same case, Hardy and Littlewood [1914b] proved the sufficiency of the first condition in (22.5). That (22.4) is sufficient by itself was proved by Ananda-Rau [1928]; cf. also our general Theorem II.15.1. The first condition in (22.5) is not enough by itself; the combined condition is due to Szász [1929], [1936]. For more details see Hardy's book (theorems 103, 104 and comments).

23 The High-Indices Theorem

We now turn to general Dirichlet series (22.1) in which the indices $\lambda_n > 0$ increase at least geometrically:

$$\lambda_{n+1} \geq \rho\lambda_n \quad (n \geq 1) \quad \text{for some number } \rho > 1. \tag{23.1}$$

One speaks of *high indices* λ_n, *Hadamard sequences* $\{\lambda_n\}$ or *Hadamard gaps* $(\lambda_n, \lambda_{n+1})$. Here the Tauberian condition $a_n = \mathcal{O}(\{(\lambda_n - \lambda_{n-1})/\lambda_n\})$ of (22.4) reduces to $a_n = \mathcal{O}(1)$. However, in the present case *no order condition* on the sequence $\{a_n\}$ is necessary! Hardy and Littlewood[1926]) proved:

Theorem 23.1. *Let the sequence* $\{\lambda_n\}$ *with* $\lambda_0 = 0$ *satisfy the high-indices condition* (23.1), *let the corresponding series of powers*

$$\sum_{n=0}^{\infty} a_n x^{\lambda_n} \tag{23.2}$$

converge to $f(x)$ *for* $0 \leq x < 1$ *and let* $f(x) \to A$ *as* $x \nearrow 1$. *Then*

$$s_n = \sum_{k=0}^{n} a_k \to A \quad \text{as } n \to \infty.$$

The original proof was simplified by Ingham [1937]. He used peak functions of the type $\{4x(1-x)\}^q$ to show that the sequence $\{a_n\}$ is bounded and in fact, tends to zero. To complete the proof he then applied Proposition 22.1. Our proof is similar, but more direct. Like Ingham we start with finite linear combinations of powers, but we deal directly with the partial sums $s_n = \sum_{k=0}^{n} a_k$.

Proposition 23.2. (Boundedness theorem) *Let*

$$f(x) = f_N(x) = \sum_{n=0}^{N} a_n x^{\lambda_n},$$

where the exponents λ_n *satisfy condition* (23.1). *Then there is a number* C_ρ *depending only on* ρ *such that*

$$|s_m| \leq C_\rho M, \quad \forall m \geq 0, \quad \text{where } M \stackrel{\text{def}}{=} \sup_{0 \leq x < 1} |f(x)|. \tag{23.3}$$

The proof will be based on integrated peak functions or 'ramp functions': smooth monotonic approximations to a unit step function. Such functions were used by Zygmund [1959] (chapter 3, (1.38)) in a proof for Littlewood's Theorem 7.1; cf. also Tietz and Zeller [1998a].

Lemma 23.3. *For given* $0 < \alpha < 1/2 < \beta < 1$ *and arbitrary* $\delta > 0$, *there is a polynomial* P *which is increasing on* $[0, 1]$ *and such that*

(i) $0 \leq P(x) \leq 1$ *for* $0 \leq x \leq 1$,
(ii) $P(x) \leq \delta x$ *for* $0 \leq x \leq \alpha$,
(iii) $1 - P(x) \leq \delta(1-x)$ *for* $\beta \leq x \leq 1$.

One may take the polynomial P of the form

$$P(x) = \sum_k b_k x^k = \frac{\int_0^x p^q(y)dy}{\int_0^1 p^q(y)dy}, \quad \text{where} \quad p(y) = 4y(1-y), \tag{23.4}$$

and q is a positive integer depending on α, β, δ. For use below we write $\sum_k |b_k| = B_q$, a number which behaves roughly like 8^q. The proof of the Lemma will be completed later.

Proof of Proposition 23.2. With P as above we define

$$F(x) = \sum_{n=0}^{N} a_n P(x^{\lambda_n}) = \sum_{n,k} a_n b_k x^{\lambda_n k} = \sum_k b_k f(x^k).$$

It follows that

$$\sup_{0 \le x < 1} |F(x)| \le M \sum_k |b_k| = B_q M. \tag{23.5}$$

Suppose now that s_m is the partial sum $s_n = \sum_{k=0}^n a_k$ of maximum absolute value (or one of those partial sums). If $m = 0$ one has $|s_n| \le M$ for all n since $|s_0| = |a_0| = |f(0)|$. From here on we assume $m > 0$. Clearly $|a_n| \le 2|s_m|$ for all n. Thus for $0 < x < 1$

$$\begin{aligned} |s_m - F(x)| &= \left| \sum_{1 \le n \le m} a_n\{1 - P(x^{\lambda_n})\} - \sum_{m < n \le N} a_n P(x^{\lambda_n}) \right| \\ &\le 2|s_m| \sum_{1 \le n \le m} \{1 - P(x^{\lambda_n})\} + 2|s_m| \sum_{n > m} P(x^{\lambda_n}). \end{aligned} \tag{23.6}$$

We will choose α, β and x in $(0, 1)$ such that $x^{\lambda_n} \ge \beta$ when $1 \le n \le m$ and $x^{\lambda_n} \le \alpha$ when $n > m$. To this end we take $x^{\lambda_m} = \beta$ and $\alpha = \beta^\rho$, so that $x^{\lambda_{m+1}} = \beta^{\lambda_{m+1}/\lambda_m} \le \beta^\rho = \alpha$.

Observe that $1 - e^{-t} < t$ and $e^{-t} < 1/t$ when $t > 0$. We thus have

$$\begin{aligned} 1 - x^{\lambda_n} &< \lambda_n \log(1/x) = (\lambda_n/\lambda_m)\lambda_m \log(1/x) \\ &\le \rho^{n-m} \log(1/\beta) \quad \text{for} \quad 1 \le n \le m, \\ x^{\lambda_n} &< 1/\{\lambda_n \log(1/x)\} = (\lambda_m/\lambda_n)/\{\lambda_m \log(1/x)\} \\ &\le \rho^{m-n}/\log(1/\beta) \quad \text{for} \quad n > m. \end{aligned} \tag{23.7}$$

A good choice for $\log(1/\beta)$ will be $(1/\sqrt{\rho})\log 2$, which makes

$$x^{\lambda_m} = \beta = 2^{-1/\sqrt{\rho}}, \quad \alpha = 2^{-\sqrt{\rho}}. \tag{23.8}$$

Using inequalities (iii) and (ii) of the Lemma and the monotonicity of P, we conclude from (23.6)–(23.8) and (23.5) that

$$|s_m - F(x)| \le 2|s_m| \left\{ \sum_{1 \le n \le m} \delta(1 - x^{\lambda_n}) + \sum_{n > m} \delta x^{\lambda_n} \right\},$$

$$
\begin{aligned}
|s_m| &\le |F(x)| + 2\delta|s_m|\Bigl\{\sum_{n\le m}\rho^{n-m-1/2}\log 2 + \sum_{n>m}\rho^{m-n+1/2}/\log 2\Bigr\} \\
&\le B_q M + 5\delta|s_m|\sum_{\nu=0}^{\infty}\rho^{-\nu-1/2} = B_q M + 5\delta|s_m|\sqrt{\rho}/(\rho-1). \qquad (23.9)
\end{aligned}
$$

In Lemma 23.3 we now take δ such that the coefficient of $|s_m|$ in the last member of (23.9) is equal to $1/2$. The final conclusion is that for sufficiently large q depending on ρ,

$$|s_m| \le 2B_q M = 2B_q \sup_{0\le x<1}|f(x)|.$$

Observe that the coefficient $2B_q$ of M depends on ρ but not on N. □

We also need an extension of Proposition 23.2 to infinite sums:

Proposition 23.4. (Boundedness theorem) *The inequality for the partial sums $s_m = \sum_{k\le m} a_k$ in Proposition 23.2 is also true for the case of infinite series $\sum_{n=0}^{\infty} a_n x^{\lambda_n}$, whose sum function $f(x)$ is bounded on the interval $\{0 \le x < 1\}$.*

Proof. Choose any $\varepsilon > 0$ and any $r \in (0,1)$. Then the series for $f(rx)$ converges uniformly for $0 \le x \le 1$. Thus we can find $N_0 = N_0(\varepsilon, r)$ such that for $N \ge N_0$

$$\left|f(rx) - \sum_{n=0}^{N} a_n r^{\lambda_n} x^{\lambda_n}\right| \le \varepsilon, \quad \forall x \in [0,1]. \qquad (23.10)$$

It follows in particular that

$$\left|\sum_{n=0}^{N} a_n r^{\lambda_n} x^{\lambda_n}\right| \le M + \varepsilon, \quad \forall x \in [0,1],$$

where $M = \sup_{0\le x<1}|f(x)|$. Hence by Proposition 23.2

$$\left|\sum_{n=0}^{m} a_n r^{\lambda_n}\right| \le C_\rho(M+\varepsilon), \quad \forall m \le N \text{ and } N \ge N_0(\varepsilon, r).$$

The desired inequality for (fixed) s_m follows by letting ε go to 0 and r to 1.

Proof of Theorem 23.1. We will use the convergence $f(x) \to A$ as $x \nearrow 1$ to conclude that $s_N \to A$ as $N \to \infty$. Define $f(1) = A$, so that the function f becomes uniformly continuous on $[0,1]$. Then for given $\varepsilon > 0$ we can choose $r \in (0,1)$ such that

$$|f(x) - f(rx)| \le \varepsilon, \quad \forall x \in [0,1]. \qquad (23.11)$$

Now for $x \in [0,1)$,

$$|f(x) - f(rx)| = \left|\sum_{n=0}^{\infty} a_n(1 - r^{\lambda_n})x^{\lambda_n}\right|,$$

hence by Proposition 23.4 applied to $f(x) - f(rx)$,

$$\left|\sum_{n=0}^{N} a_n(1 - r^{\lambda_n})\right| \le C_\rho \varepsilon, \quad \forall N. \tag{23.12}$$

For $N \ge N_0(\varepsilon, r)$ it thus follows from (23.10) and (23.11) with $x = 1$ that

$$\begin{aligned}\left|\sum_{n=0}^{N} a_n - f(1)\right| \le \left|\sum_{n=0}^{N} a_n(1 - r^{\lambda_n})\right| + \left|\sum_{n=0}^{N} a_n r^{\lambda_n} - f(r)\right| \\ + |f(r) - f(1)| \le (C_\rho + 2)\varepsilon.\end{aligned}$$

□

Proof of Lemma 23.3. We take P as in (23.4). Then P is increasing on $[0, 1]$ and part (i) of the Lemma is an immediate consequence. As to part (ii), for $0 \le y \le x \le \alpha < 1/2$,

$$p(y) \le 4\alpha(1-\alpha) < 1, \qquad \int_0^x p^q(y)dy \le \{4\alpha(1-\alpha)\}^q x.$$

On the other hand, for $q \ge 2$,

$$I_q = \int_0^1 p^q(y)dy \ge \int_{(1/2)-1/(2q)}^{(1/2)+1/(2q)} \left(1 - \frac{1}{q^2}\right)^q dy \ge \frac{1}{q}\left(1 - \frac{1}{2^2}\right)^2 > \frac{1}{2q}.$$

Hence

$$P(x) \le 2q\{4\alpha(1-\alpha)\}^q x, \quad \text{which is} \quad \le \delta x$$

when q is sufficiently large. Part (iii) follows by a similar argument: for $1/2 < \beta \le x \le y \le 1$ one has $p(y) \le 4\beta(1-\beta)$, so that

$$1 - P(x) = \int_x^1 p^q(y)dy \Big/ I_q \le 2q\{4\beta(1-\beta)\}^q(1-x) \le \delta(1-x)$$

when q is sufficiently large. □

Remarks 23.5. Meyer-König and Zeller [1956] gave a proof for the high-indices theorem with the aid of functional analysis; cf. Section V.21. See also the extension in Meyer-König and Zeller [1960a]. There is an interesting proof of Theorem 23.1 by Halász [1967a] which is based on complex analysis; cf. Section VII.11 and Landau and Gaier [1986] (appendix 2, section 1E). The gap condition (23.1) cannot be relaxed; see Proposition 24.3 and Remarks 24.4 below.

One may ask for which functions g on $[0, 1]$ there is a 'high-indices theorem' involving series

$$h(x) = \sum_{n=0}^{\infty} a_n g(x^{\lambda_n}) \tag{23.13}$$

with Hadamard indices λ_n. The Lambert kernel (Example 2.5) provides an admissible function g, because the Lambert summability of any series $\sum a_n$ implies its Abel summability; cf. Remarks 10.3. For other functions g the following observation may be useful. *Boundedness* of sum functions h on $[0, 1)$ will imply *boundedness* of the corresponding sequence $\{a_n\}$ if (and only if) the L^∞ approximation to $g(x^{\lambda_n})$ by linear combinations of other functions $g(x^{\lambda_k})$ is uniformly bad:

$$d_n = \inf_{\{b_k\}} \sup_{0 \le x < 1} \left| g(x^{\lambda_n}) - \sum_{k \ne n} b_k g(x^{\lambda_k}) \right| \ge c > 0, \quad \forall n. \tag{23.14}$$

Indeed, if $a_n \ne 0$,

$$\sup_{0 \le x < 1} |h(x)| = |a_n| \sup_{0 \le x < 1} \left| g(x^{\lambda_n}) - \sum_{k \ne n} (-a_k/a_n) g(x^{\lambda_k}) \right| \ge |a_n| d_n;$$

cf. Korevaar [1970], [2001c]. In the case of 'bad approximability' one will thus have a high-indices theorem for series (23.13) with Hadamard exponents if boundedness of the sequence $\{a_n\}$ is a Tauberian condition for such series. For the latter see Section II.15.

Levinson [1938], [1940], [1964a] obtained refined high-indices theorems for series related to Wiener kernels. Even for 'non-Wiener' kernels there may be a high-indices theorem for special sequences $\{\lambda_n\}$; cf. Halász [1967b], Johansson [1979].

High-indices theorems involving (weighted) absolute convergence were obtained by Zygmund [1944] and Waterman [1950], [1999]; see also Johansson [1981] for a generalization. The so-called high-indices theorem for Borel summability involves index sequences with much smaller gaps ('square-root gaps'); see Section VI.15.

24 Optimality of Tauberian Conditions

Simple constructions will show that the Tauberian conditions in the most important theorems for power series are optimal conditions of their kind.

Proposition 24.1. *In Littlewood's Theorem 7.1, the condition $|a_n| \le C/n$ is an optimal order condition. Indeed, for every positive increasing function $\phi(n)$ tending to ∞, there is a* DIVERGENT *Abel summable series $\sum_0^\infty a_n$ such that*

$$|a_n| \le \phi(n)/n, \quad \forall n. \tag{24.1}$$

Proof. Let $0 < \phi(n) \nearrow \infty$. We will construct a generating function $f(x) = \sum_{n=0}^\infty a_n x^n$ by adding nonoverlapping blocks of the form

$$\begin{aligned} f_{p,q}(x) &= \frac{\phi(p)}{p+2q}(x^{p+1} + \cdots + x^{p+q} - x^{p+q+1} - \cdots - x^{p+2q}) \\ &= \frac{\phi(p)}{p+2q} x^{p+1} \frac{(1-x^q)^2}{1-x}, \quad \text{with integers } p, q \ge 1. \end{aligned} \tag{24.2}$$

It is clear that the coefficients a_n of the powers x^n in the polynomial $f_{p,q}(x)$ satisfy inequality (24.1). The corresponding partial sums s_n are ≥ 0. They attain their maximum for $n = p + q$ while $s_n = 0$ at the end of the block. We want $\max s_n$ close to 1:

$$s_{p+q} = \frac{q\phi(p)}{p+2q} \approx 1 \quad \text{for large } p. \tag{24.3}$$

Replacing the original function ϕ by a smaller positive function $\phi \nearrow \infty$ if necessary, one may assume that $1 \geq \phi(n)/n \to 0$. We now ensure (24.3) by taking

$$q = [p/\phi(p)] \qquad \text{(the integral part)}. \tag{24.4}$$

For $0 \leq x < 1$, (24.2) and (24.4) give the inequality

$$0 \leq f_{p,q}(x) \leq \frac{\phi(p)}{p} q^2 x^p (1-x) < \phi(p)\frac{q^2}{p^2} \leq \frac{1}{\phi(p)}. \tag{24.5}$$

Now define the power series for f by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=1}^{\infty} f_{p_k,q_k}(x), \tag{24.6}$$

where $q_k = [p_k/\phi(p_k)]$ and the sequence $\{p_k\}$ is chosen such that

$$p_{k+1} \geq p_k + 2q_k \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\phi(p_k)} < \infty. \tag{24.7}$$

Then the sets of powers in the different blocks $f_{p,q}$ do not overlap, so that for the calculation of a_n and s_n one never has to deal with more than one block. Hence

$$\limsup_{n\to\infty} s_n = 1, \quad \liminf_{n\to\infty} s_n = 0; \tag{24.8}$$

cf. (24.3). Finally, by dominated convergence, see (24.5) and (24.7),

$$f(x) = \sum_{k=1}^{\infty} f_{p_k,q_k}(x) \to 0 \quad \text{as } x \nearrow 1 \tag{24.9}$$

because each term here has limit 0. Thus the divergent series $\sum_0^\infty a_n$ is Abel summable to 0.

The series $\sum_0^\infty a_n$ is also *Cesàro* summable to 0:

$$(s_0 + s_1 + \cdots + s_{n-1})/n \to 0.$$

This may be shown by direct estimation, or one may appeal to Theorem 7.3 which implies that an Abel summable series with uniformly bounded partial sums is Cesàro summable.

Proposition 24.2. *In the 'Abel to Cesàro' Theorem 7.3 of Hardy and Littlewood, the condition $s_n \geq -C$ is an optimal order condition. Indeed, for every positive increasing function $\phi(n) \nearrow \infty$, there is an Abel limitable sequence $\{s_n\}$ such that*

$$|s_n| \leq \phi(n), \quad \forall n \tag{24.10}$$

which FAILS *to be* CESÀRO *limitable.*

Proof. Let $0 < \phi \nearrow \infty$. Using the same method as before we now construct a function $g(x) = \sum_{n=0}^{\infty} s_n x^n$ from blocks

$$\begin{aligned} g_{p,q}(x) &= \phi(p)(x^{p+1} + \cdots + x^{p+q} - x^{p+q+1} - \cdots - x^{p+2q}) \\ &= \phi(p)\, x^{p+1}\, \frac{(1-x^q)^2}{1-x}. \end{aligned} \tag{24.11}$$

In $g_{p,q}$ the coefficients s_n of the powers x^n satisfy (24.10). The sums

$$s_n^{(-1)} = s_0 + s_1 + \cdots + s_n \tag{24.12}$$

are ≥ 0 and attain their maximum for $n = p + q$; at the end of the block $s_n^{(-1)} = 0$. We want

$$s_{p+q}^{(-1)} = q\phi(p) \sim p \quad \text{as} \quad p \to \infty. \tag{24.13}$$

To ensure this we again take $1 \geq \phi(n)/n \to 0$ and $q = [p/\phi(p)]$. Then by (24.11)

$$0 \leq (1-x)g_{p,q}(x) \leq \phi(p)x^p(1-x^q)^2 < \phi(p)\frac{4q^2}{p^2} \leq \frac{4}{\phi(p)}. \tag{24.14}$$

Using sequences $\{p_k\}$ and $\{q_k\}$ as in the preceding proof, we now define the power series for g by

$$g(x) = \sum_{n=0}^{\infty} s_n x^n = \sum_{k=1}^{\infty} g_{p_k,q_k}(x). \tag{24.15}$$

Then the numbers s_n satisfy (24.10) and by dominated convergence, the sequence $\{s_n\}$ is Abel limitable to 0:

$$(1-x)g(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n \to 0 \quad \text{as} \quad x \nearrow 1;$$

cf. (24.14). However, the sequence $\{s_n\}$ is not Cesàro limitable:

$$\limsup_{n\to\infty} \frac{s_n^{(-1)}}{n} = 1, \quad \liminf_{n\to\infty} \frac{s_n^{(-1)}}{n} = 0.$$

Proposition 24.3. *In the High-Indices Theorem 23.1, Hadamard sequences* (23.1) *are optimal. Indeed, for any positive increasing sequence $\lambda_n \nearrow \infty$ for which*

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1, \tag{24.16}$$

there is a DIVERGENT *series* $\sum_{n=1}^{\infty} a_n$ *which is* (A, λ) *summable to* 0*:*

$$f(x) = \sum_{n=1}^{\infty} a_n x^{\lambda_n} \to 0 \quad as \;\; x \nearrow 1. \tag{24.17}$$

Proof. Let $0 < \lambda_n \nearrow \infty$ be as in (24.16). Setting $\lambda_{n+1} = \lambda_n + \mu_n$ one has

$$0 \le x^{\lambda_n} - x^{\lambda_{n+1}} = x^{\lambda_n}(1 - x^{\mu_n}) < \frac{\mu_n}{\lambda_n} \quad \text{for } 0 \le x < 1. \tag{24.18}$$

We will focus on values of n for which μ_n/λ_n goes to 0. More precisely, we choose a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that

$$\lambda_{n_{k+1}} > \lambda_{n_k+1} = \lambda_{n_k} + \mu_{n_k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\mu_{n_k}}{\lambda_{n_k}} < \infty. \tag{24.19}$$

Now define f by the following series of powers:

$$f(x) = \sum_{n=1}^{\infty} a_n x^{\lambda_n} = \sum_{k=1}^{\infty} (x^{\lambda_{n_k}} - x^{\lambda_{n_k+1}}). \tag{24.20}$$

By (24.18) and (24.19) $f(x) \to 0$ as $x \nearrow 1$ (again by dominated convergence), but the series $\sum_1^{\infty} a_n$, which consists of terms ± 1, is of course divergent.

Remarks 24.4. Littlewood [1911] proved the optimality of his order condition $|a_n| \le C/n$ as part of a more general result which involves condition (22.4) for a class of Dirichlet series (22.1). The order condition $|a_n| \le C/n$ is still optimal for the implication 'Abel summability $\Rightarrow$ convergence' if one imposes the additional requirement that $f(x) = \sum_0^{\infty} a_n x^n$ be of bounded variation on [0, 1], or even on all radii of the unit disc; see H.S. Shapiro [1965], Kennedy and Szüsz [1966], and Piranian [1966]. However, weaker order conditions may suffice for convergence of $\sum a_n$ if one has more information on the behavior of $f(x)$ for x close to 1; cf. Fatou's theorem in Section III.12 and a certain convergence condition in Example VII.2.4.

It is interesting that the order condition $a_n = \mathcal{O}(1/n)$ is the 'right' Tauberian condition also for some very different summability methods. We mention the *statistical convergence* which was studied from a Tauberian point of view by, among others, Fridy [1985] and Maddox [1989]: $s_n = \sum_{k=0}^{n} a_k \to A$ statistically if the set $\{n \in \mathbb{N} : |s_n - A| \ge \varepsilon\}$ has density zero for every number $\varepsilon > 0$. (The concept goes back to Fast [1951] and Schoenberg [1959].)

The one-sided condition $a_n \ge -C/n$ in Theorem 7.2 is *a fortiori* optimal as to order. Similarly, the Schmidt condition of *slow decrease* of the partial sums s_n (7.1) is an optimal condition. Cf. the corresponding result for general Wiener kernels in Section II.16.

Incidentally, several authors have found (unusual) Tauberian conditions which work for *Cesàro* summability but not for *Abel* summability; see Pitt [1955], Butzer and Neuheuser [1965], Kuttner [1977]. Cf. also Móricz [1994].

Although by Proposition 24.2 the one-sided condition $s_n \geq -C$ is an optimal order condition in the 'Abel to Cesàro' Theorem 7.3, it may be replaced there by a weaker *average-type condition* given by Bingham [1985]:

$$\liminf_{m\to\infty,\, \rho\searrow 1} \inf_{m<n\leq \rho m} \frac{1}{m} \sum_{m<k\leq n} s_k \geq 0. \tag{24.21}$$

There is a related 'Abel to (C, k)' result by Badiozzaman and Thorpe [1996].

Concerning Proposition 24.3: the optimality of Hadamard sequences or gaps in the high-indices theorem has been proved by various constructions; cf. Rudin [1966], P. Borwein and Erdélyi [1995] (theorem 6.2.7).

In a series of papers, Lorentz [1948], [1949], [1951] developed a general theory of permissible Tauberian conditions; cf. Peyerimhoff [1969].

For many Tauberian problems so-called little o-conditions are easy to obtain. They are rarely optimal, but may be used as the starting point for new (weaker) Tauberian conditions. Thus the condition $na_n = o(1)$ in Tauber's first theorem can be relaxed to $(a_1 + 2a_2 + \cdots + na_n)/n = o(1)$ in his second theorem (Section 5), as well as to $na_n = \mathcal{O}(1)$. Results related to the former hold for all regular linear summability methods; see Meyer-König and Tietz [1967], [1968], [1969]. This work was extended by Stieglitz [1969] and Leviatan [1971].

Kangro [1970] proposed a general approach to the problem of the relaxation of little o-Tauberian conditions; cf. Baron [1966/77] (section 27). Several authors have pursued these ideas, among them Sõrmus [1982].

25 Tauberian Theorems of Nonstandard Type

In this section we discuss some Tauberian theorems of Classical Form 2.7:

$$\sum a_n \text{ is } Q\text{-summable } \ \& \ T\{a_n\} \ \Rightarrow \sum a_n \text{ is } P\text{-summable}$$

for the case where *not all* P-summable series are Q-summable, although method Q is more powerful at least in the sense that it sums many series whose terms grow too rapidly for method P. It will turn out that in many cases, the rapid growth is the only 'obstruction'.

In a different sense, Wiener's principal Tauberian theorem (Section II.8) is also of nonstandard type.

The first example below involves Dirichlet series with different index sequences $\lambda = \{\lambda_n\}$ and the corresponding (A, λ) summabilities (Section 22). We mention only a special case; more general results are in Hardy [1910] and Cartwright [1930]; cf. Hardy [1949] (section 4.8 and appendix 5), Korevaar [1954e], and D. Borwein [1990].

Theorem 25.1. *Let $\sum_{n=1}^{\infty} a_n$ be Abel summable, or (A, n) summable, to A:*

$$\sum_{n=1}^{\infty} a_n e^{-nt} \text{ converges to } f(t) \text{ for } t > 0 \text{ and } f(t) \to A \text{ as } t \searrow 0.$$

Suppose that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x} \quad \textit{converges for} \quad x > 0, \quad \textit{to} \quad g(x), \quad \textit{say}.$$

Then $\sum_1^\infty a_n$ *is* $(A, \log n)$ *summable to* A, *that is,* $g(x) \to A$ *as* $x \searrow 0$.

Proof. The result will be derived from the formula

$$g(x) = \sum_1^\infty \frac{a_n}{n^x} = \frac{1}{\Gamma(x)} \int_0^\infty \left(\sum_1^\infty a_n e^{-nt} \right) t^{x-1} dt, \quad x > 0, \tag{25.1}$$

which will be justified below. Since $1/\Gamma(x) = x/\Gamma(x+1)$ we have to show that

$$g_1(x) = x \int_0^\infty f(t) t^{x-1} dt \to A \quad \text{as } x \searrow 0.$$

By changing a_1 one may assume $A = 0$. Then to given $\varepsilon > 0$ one can choose $\delta \in (0, 1)$ such that $|f(t)| < \varepsilon$ for $0 < t \le \delta$. There will be constants M_j such that $|f(t)| \le M_1$ for $0 < t \le 1$ and $|f(t)| \le M_2 e^{-t}$ for $t > 1$. Hence if $0 < x < 1$,

$$\begin{aligned} |g_1(x)| &< \varepsilon \int_0^\delta x t^{x-1} dt + M_1 \int_\delta^1 x t^{x-1} dt + M_2 x \int_1^\infty e^{-t} dt \\ &< \varepsilon + M_1(1 - \delta^x) + M_2 x < 3\varepsilon \quad \text{for } 0 < x < x_0. \end{aligned}$$

Hence $g_1(x) \to A = 0$ as $x \searrow 0$.

The termwise integration in (25.1) is delicate and deserves comment. For $0 < b < B < \infty$ it is clear that $\int_b^B \sum_1^\infty = \sum_1^\infty \int_b^B$. For large B, $\int_B^\infty \sum_1^\infty$ will be small. To show that $\sum_1^\infty \int_B^\infty$ is small one may observe that

$$\sum_1^\infty a_n \int_B^\infty e^{-nt} t^{x-1} dt = \sum_1^\infty \frac{a_n}{n^x} \int_{nB}^\infty e^{-v} v^{x-1} dv,$$

after which one can use standard partial summation. In the case of $\sum_1^\infty \int_0^b$ with small b one can base partial summation on the remainders r_n in the series $\sum a_n/n^x$, setting $a_n/n^x = r_{n-1} - r_n$. □

There *are* series $\sum_1^\infty a_n$ which are $(A, \log n)$ summable but not (A, n) summable. An example is given by $\sum a_n = \sum n^{-1-iy}$ with $y \ne 0$. Here

$$g(x) = \sum_{n=1}^\infty n^{-1-x-iy} = \zeta(1 + x + iy) \to \zeta(1 + iy) \quad \text{as } x \searrow 0,$$

since the zeta function is finite and continuous everywhere except at the point 1 (cf. Section 26). However, by Littlewood's Theorem 7.1, the series $\sum a_n$ with terms that are $\mathcal{O}(1/n)$ cannot be Abel summable, because it is not convergent. Indeed,

$$\sum_1^N n^{-1-iy} = \int_{1-}^N v^{-1-iy} d[v] = \int_1^N v^{-1-iy} dv + \int_{1-}^N v^{-1-iy} d([v] - v)$$
$$= \frac{N^{-iy}}{-iy} + \frac{1}{iy} + 1 + (1+iy) \int_1^N ([v] - v) v^{-2-iy} dv$$
$$= \frac{N^{-iy}}{-iy} + \phi(y) + \mathcal{O}\left(\frac{1}{N}\right); \tag{25.2}$$

the first term on the right fails to have a limit as $N \to \infty$. Here we have written

$$\phi(y) \quad \text{for} \quad \frac{1}{iy} + 1 + (1+iy) \int_1^\infty ([v] - v) v^{-2-iy} dv,$$

which is in fact equal to $\zeta(1+iy)$; cf. formula (26.4) below.

Our second example involves a result of Doetsch [1931]:

Theorem 25.2. *Let $\sum_0^\infty a_n$ be Borel summable to A and let $\sum_0^\infty a_n t^n$ converge for $|t| < 1$. Then $\sum_0^\infty a_n$ is Abel summable to A.*

Proof. By the definition of Borel summability (Example 2.4)

$$F(x) = e^{-x} \sum_{n=0}^\infty s_n (x^n/n!) \to A \quad \text{as } x \to \infty. \tag{25.3}$$

By the hypothesis the numbers a_n and hence also the numbers s_n will be $\mathcal{O}\{(1+\varepsilon)^n\}$ as $n \to \infty$ for every number $\varepsilon > 0$. One may then form the Laplace transform $f(y) = \mathcal{L}F(y)$ by termwise integration when $y > 0$:

$$f(y) = \int_0^\infty F(x) e^{-yx} dx$$
$$= \sum_0^\infty \frac{s_n}{n!} \int_0^\infty x^n e^{-(1+y)x} dx = \sum_0^\infty \frac{s_n}{(1+y)^{n+1}}. \tag{25.4}$$

(Justification by absolute convergence of the sum of integrals.) Now by (25.3)

$$yf(y) = \frac{\int_0^\infty F(x) e^{-yx} dx}{\int_0^\infty e^{-yx} dx} \to A \quad \text{as } y \searrow 0. \tag{25.5}$$

Thus if we set $1/(1+y) = t$ so that $y = (1-t)/t$, relations (25.4), (25.5) show that

$$yf(y) = \frac{1-t}{t} \sum_0^\infty s_n t^{n+1} \to A \quad \text{as } t \nearrow 1.$$

It follows that $\sum_0^\infty a_n$ is Abel summable to A:

$$\sum_0^\infty a_n t^n = (1-t) \sum_0^\infty s_n t^n \to A \quad \text{as } t \nearrow 1.$$

□

One may derive from Section 3 that the series $\sum_0^\infty z^n$ is Borel summable for certain numbers z of absolute value greater than 1. (It is sufficient if $\operatorname{Re} z < 1$.) The terms of such a series grow too rapidly for Abel summability. There are also Abel summable series which are not Borel summable. Indeed, there are divergent Abel summable series $\sum a_n$ with $a_n = \mathcal{O}(1/\sqrt{n})$; cf. Proposition 24.1. Such series cannot be Borel summable or else they would have to converge; see the Hardy–Littlewood Theorem 9.1. More on the relation between Abel and Borel summability may be found in Karamata [1938], Shawyer and Watson [1994] (chapter 9).

There are related nonstandard theorems for integrals of which we mention a special case involving Gauss and Abel summability. For a more general result of this type see Bochner and Chandrasekharan [1949] (chapter 1, section 14).

Let $a(\cdot)$ be locally integrable on $(0, \infty)$. The formal integral $\int_0^\infty a(v)dv$ may be called *Gauss summable* to A if

$$g(s) = \int_0^{\infty-} a(v)e^{-s^2v^2}dv \;\text{ exists for } s > 0 \text{ and } g(s) \to A \text{ as } s \searrow 0. \tag{25.6}$$

Theorem 25.3. *Let $\int_0^\infty a(v)dv$ be Gauss summable to A and suppose that the 'Abel means'*

$$f(t) = \int_0^\infty a(v)e^{-tv}dv, \quad t > 0$$

exist as absolutely convergent integrals. Then $\int_0^\infty a(v)dv$ is Abel summable to A, that is, $f(t) \to A$ as $t \searrow 0$.

Proof. Following Bochner and Chandrasekharan we derive a suitable integral representation:

$$\begin{aligned} e^{-\alpha} &= \frac{2}{\pi}\int_0^\infty \frac{\cos\alpha x}{1+x^2}dx = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,dx \int_0^\infty e^{-(1+x^2)y}dy \\ &= \frac{2}{\pi}\int_0^\infty e^{-y}dy \int_0^\infty e^{-yx^2}\cos\alpha x\,dx, \quad \alpha \ge 0. \end{aligned}$$

Here the first step may be verified by Fourier inversion or complex integration. The final inner integral can be evaluated by moving a path of integration from a line parallel to the real axis back to the axis: for $y > 0$,

$$\begin{aligned} &2\int_0^\infty e^{-yx^2}\cos\alpha x\,dx = \int_{-\infty}^\infty e^{-yx^2+i\alpha x}dx \\ &= e^{-\alpha^2/(4y)}\int_{\mathbb{R}} e^{-y\{x-i\alpha/(2y)\}^2}dx = e^{-\alpha^2/(4y)}\int_{\mathbb{R}-i\alpha/(2y)} e^{-yz^2}dz \\ &= e^{-\alpha^2/(4y)}\int_{\mathbb{R}} e^{-yz^2}dz = \sqrt{\frac{\pi}{y}}e^{-\alpha^2/(4y)}. \end{aligned} \tag{25.7}$$

It follows that

$$e^{-\alpha} = \int_0^\infty \frac{e^{-y-\alpha^2/(4y)}}{\sqrt{\pi y}}dy = \int_0^\infty h(w)e^{-\alpha^2w^2}dw, \quad h(w) = \frac{e^{-1/(4w^2)}}{\sqrt{\pi}w^2}. \tag{25.8}$$

Observe that h is in $L^1(0,\infty)$ and $\int_0^\infty h(w)dw = 1$.

By (25.8) and (25.6) one may thus write

$$\begin{aligned} f(t) &= \int_0^\infty a(v)e^{-tv}dv = \int_0^\infty a(v)dv \int_0^\infty h(w)e^{-t^2v^2w^2}dw \\ &= \int_0^\infty h(w)g(tw)dw \end{aligned}$$

(justified by absolute convergence). By the hypotheses g is bounded and $g(tw) \to A$ for every number $w > 0$ as $t \searrow 0$. Hence by dominated convergence

$$f(t) \to A \int_0^\infty h(w)dw = A \quad \text{as } t \searrow 0.$$

26 Important Properties of the Zeta Function

We first discuss the analytic character of $\zeta(z)$ for $\operatorname{Re} z \le 1$.

Theorem 26.1. *The analytic function*

$$F(z) \stackrel{\text{def}}{=} \zeta(z) - \frac{1}{z-1}, \quad \operatorname{Re} z > 1 \tag{26.1}$$

can be continued analytically to the whole complex z-plane. One special value of interest is

$$F(1) = \lim_{z\to 1} \left\{ \zeta(z) - \frac{1}{z-1} \right\} = \gamma, \tag{26.2}$$

Euler's constant.

Proof. The definition of $\zeta(z)$ as the sum of the Dirichlet series $\sum_1^\infty n^{-z}$ for $\operatorname{Re} z > 1$ (4.1) may be changed to integral form as follows:

$$\zeta(z) = \int_{1-}^\infty v^{-z}d[v] = z\int_1^\infty [v]v^{-z-1}dv. \tag{26.3}$$

Comparison with the formula

$$\int_1^\infty v \cdot v^{-z-1}dv = \frac{1}{z-1}$$

shows that for $\operatorname{Re} z > 1$,

$$F(z) = \zeta(z) - \frac{1}{z-1} = 1 + z\int_1^\infty ([v] - v)v^{-z-1}dv. \tag{26.4}$$

Since $[v] - v$ is bounded this formula gives an analytic continuation of $F(\cdot)$ to the half-plane $\{\operatorname{Re} z > 0\}$. We now introduce the periodic function

$$\omega(v) = [v] - v + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2n\pi v}{n\pi} \qquad (v \neq \text{integer})$$

and its periodic antiderivatives $\omega^{(-k)}(v)$ with average zero,

$$\omega^{(-k)}(v) = 2\sum_{n=1}^{\infty} \frac{\sin(2n\pi v - k\pi/2)}{(2n\pi)^{k+1}}.$$

Starting with (26.4) one may express F in terms of ω and then apply repeated integration by parts. Step by step one thus obtains an analytic continuation of $F(z)$ to every half-plane $\{\operatorname{Re} z > -k\}$:

$$\begin{aligned}
F(z) &= 1 - \frac{1}{2} + z\int_1^{\infty} \omega(v)v^{-z-1}dv \\
&= \frac{1}{2} - \omega^{(-1)}(1)z + z(z+1)\int_1^{\infty} \omega^{(-1)}(v)v^{-z-2}dv = \cdots \qquad (26.5)\\
&= (\text{polynomial in } z) + z(z+1)\cdots(z+k)\int_1^{\infty} \omega^{(-k)}(v)v^{-z-k-1}dv.
\end{aligned}$$

The value $F(1)$ may be derived from (26.4):

$$\begin{aligned}
F(1) &= \lim_{z\to 1} F(z) = 1 + \int_1^{\infty} ([v] - v)v^{-2}dv \\
&= 1 + \lim_{n\to\infty} \int_1^{n} ([v] - v)v^{-2}dv \\
&= 1 + \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} k\int_k^{k+1} v^{-2}dv - \log n \right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) = \gamma. \qquad (26.6)
\end{aligned}$$

What can one say about the zeros of $\zeta(z)$? The Euler product shows that $\zeta(z)$ is free of zeros in the half-plane $\{\operatorname{Re} z > 1\}$; cf. Section 4.

Theorem 26.2. *The function $\zeta(z)$ has no zeros on the line* $\{\operatorname{Re} z = 1\}$.

This theorem was obtained by Hadamard [1896] and de la Vallée Poussin [1896] as an important step in their proofs of the prime number theorem (PNT). It has been an essential ingredient of all subsequent proofs for the PNT except the so-called elementary proof by Selberg [1949] and Erdős [1949a]; cf. Chapters II and III.

In a sense, Theorem 26.2 is equivalent to the PNT – it can be derived from the PNT; see Section III.3. The following is essentially Hadamard's proof of Theorem 26.2 as simplified by Mertens [1898].

Proof of Theorem 26.2. For $z = x + iy$ with $x > 1$ one may represent $\zeta(z)$ by the Euler product $\prod_p 1/(1 - p^{-z})$ taken over all primes p; cf. (4.2). Thus

$$\log|\zeta(x+iy)| = \operatorname{Re}\sum_p -\log(1-p^{-z}) = \operatorname{Re}\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\,p^{-kz}$$
$$= \sum_{p,k}\frac{1}{k}\,p^{-kx}\cos(ky\log p). \tag{26.7}$$

One now uses the wonderful relation

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \ge 0 \quad \text{for all (real) } \theta.$$

For the values $\theta = ky\log p$ it shows in combination with (26.7) that

$$3\log\zeta(x)+4\log|\zeta(x+iy)|+\log|\zeta(x+2iy)| \ge 0,$$

so that

$$\zeta^3(x)|\zeta^4(x+iy)||\zeta(x+2iy)| \ge 1 \qquad (x>1). \tag{26.8}$$

Suppose for a moment that the analytic function $\zeta(\cdot)$ would have a zero at the point $1+iy$ (naturally, with $y\neq 0$). Then $\zeta^4(\cdot)$ would have a zero there of order ≥ 4, so that $\zeta^4(x+iy) = \mathcal{O}\{(x-1)^4\}$ as $x\searrow 1$. Since $\zeta(x)=\mathcal{O}\{1/(x-1)\}$, the left-hand side of (26.8) would then tend to 0 as $x\searrow 1$, an obvious contradiction! □

Remark 26.3. The zeta function has zeros at the points $-2, -4, -6, \cdots$. All its other zeros lie in the so-called critical strip $\{0<\operatorname{Re}z<1\}$. The famous *Riemann hypothesis* asserts that all these zeros lie exactly on the central line $\{\operatorname{Re}z = 1/2\}$. For more information about the zeta function, see Titchmarsh [1951/86], H.M. Edwards [1974], Ivić [1985] or Patterson [1988].

II

Wiener's Theory

1 Introduction

One of the most interesting results of Hardy and Littlewood was their Tauberian theorem for Lambert summability [1921]; see Section I.10, or Theorem 12.1 below. This result implies the prime number theorem (PNT), but the proof made use of estimates from number theory somewhat stronger than the PNT! There was another Tauberian-type theorem which implies the PNT and whose state was unsatisfactory. That was Landau's complex Tauberian theorem [1908], [1909] for Dirichlet series, which required more information about the Riemann zeta function in the half-plane $\{\operatorname{Re} z \geq 1\}$ than seemed reasonable (see Section III.2).

Around 1925, Wiener was developing his general harmonic analysis. There he encountered a problem which required the comparison of two different 'asymptotic averages' of the same function. While held up on the problem, he talked to Ingham during a stay in Göttingen. Noticing the resemblance of Wiener's question to certain Tauberian theorems of Hardy and Littlewood, Ingham suggested that Wiener look at their work. Subsequently Wiener [1927a] solved his special problem by direct representation of one 'limitation kernel' in terms of another. His tools were convolution and Fourier transformation (see Example 6.4 below).

In Germany, Wiener also met R. Schmidt, who had developed a unified Tauberian theory [1925a], [1925b] based on the theory of moments. However, Schmidt could not deal with the 'Lambert Tauberian'. He encouraged Wiener to investigate if he could extend his Fourier integral theory so that it would include that important theorem.

Adding L^1 approximation by 'good' kernels to his arsenal, Wiener achieved the desired breakthrough [1927b], [1928]. The Lambert Tauberian theorem and hence the PNT now followed directly from the nonvanishing of $\zeta(z)$ on the line given by $\{\operatorname{Re} z = 1\}$. Moreover, Wiener's student Ikehara [1931] at M.I.T. could use the new method to eliminate the vexing growth condition on $\zeta'(z)/\zeta(z)$ in Landau's theorem; see Chapter III.

Removing unnecessary restrictions on his limitation kernels, Wiener published a definitive paper on his theory in the Annals of Mathematics [1932]. The long article contains a powerful unified treatment of Tauberian theory based on Fourier transforms.

The central theme remained the comparison of different asymptotic averages. A simple illustration is provided by the step from Abel to Cesàro limitability of a (bounded) sequence $\{s_n\}$; see Example 6.3. An important addition to Wiener's theory by Pitt [1938a] later provided a more direct link to Tauberian theorems such as Littlewood's. Whereas Wiener's principal Tauberian theorem is not of the classical form described in Section I.2, the 'Wiener–Pitt theorem' is.

Wiener's work has had tremendous impact. The Tauberian theorems found wide application. His 'Lemma' on the reciprocal of an absolutely convergent Fourier series led to the theory of Banach algebras. At the hands of a variety of authors, the Tauberian theory was extended to the setting of topological groups.

Wiener's Tauberian work has been discussed in many books (cf. Section 8) and was reproduced in volume 2 of his 'Collected works with commentaries' [1979]. For historical remarks, see also volume 2 of Wiener's autobiography [1956], Levinson's article [1966], the author's commentary (Korevaar [1979]), and the Proceedings of the 1994 Wiener centennial meetings, listed under Wiener [1997a], [1997b].

In the following exposition, Wiener kernels – L^1 kernels for which one has a Wiener-type Tauberian theorem – are introduced with the aid of a testing equation. The equation was mentioned by Karamata [1937a], [1937b] and used by Carleman [1944] and Beurling [1945]. We derive the testing equation in connection with Pitt's form of Wiener's problem. The aim is to go directly from 'limitability' to convergence (Sections 2, 3). The Wiener–Pitt theorem is then made the basis for the Wiener theorems which compare different asymptotic averages (Sections 4, 5). Section 6 shows how the principal Hardy–Littlewood theorems for Abel summability can be obtained from simple Wiener theory.

The reader who is interested primarily in the *final form* and the *more important applications* of Wiener theory may skip most of the material in Sections 2–7. The principal Wiener theorems are collected in Section 8.

Wiener's characterization of 'good' kernels in terms of Fourier transforms is derived both classically (Sections 7–10) and with the aid of distribution theory (Section 11). The author's distributional approach works especially well in the case of families of kernels. A more algebraic treatment of Wiener theory is given in Chapter V. In Section 16 we construct examples which show that the Tauberian conditions in Wiener theory are essentially optimal.

We will discuss several applications, among them a general form of the Lambert Tauberian theorem for Stieltjes integrals (Section 12). Wiener's so-called 'second Tauberian theorem', which involves measures, is discussed in Section 13. It is used in Section 14 to obtain a Wiener Tauberian theorem for series and will also be applied in subsequent chapters. The Wiener theorem for series readily implies the standard form of the Lambert theorem. It is used in Section 15 to obtain an extension of the Tauberian theorem for general Dirichlet series to series involving Wiener kernels. Next we derive Tauberian theorems for 'Landau–Ingham asymptotics' and Ingham summability (Sections 17, 18). The latter also lead quickly to the prime number theorem. The final Section 19 contains an application of Wiener theory to harmonic functions.

Ikehara's theorem – sometimes called the Landau–Ikehara theorem, but more properly called the Wiener–Ikehara theorem – will be treated in Chapter III, in the context of complex Tauberian theory. Extensions of Wiener theory may be found in Chapters IV and V, and applications to Borel summability are made in Chapter VI.

As usual, functions which occur under an integral sign are supposed to be measurable.

2 Wiener Problem: Pitt's Form

Some of the classical Tauberian theorems encountered in Chapter I may serve as introduction to the general theory. Cesàro summability, Abel summability and Lambert summability of a series $\sum_0^\infty a_n$ to the value A can be described by the respective limit relations

$$\sum_{n\le 1/t} a_n(1-nt) \to A, \quad \sum_{n=0}^{\infty} a_n e^{-nt} \to A, \quad \sum_{n=0}^{\infty} a_n \frac{nt}{e^{nt}-1} \to A \quad \text{as } t \searrow 0;$$

cf. Section I.13. An appropriate Tauberian condition for each method was

$$|na_n| \le C, \quad \forall n. \tag{2.1}$$

If one sets $t = 1/u$, the limit relations take the form

$$f(u) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n k_0\left(\frac{n}{u}\right) \to Ak_0(0) \quad \text{as } u \to \infty, \tag{2.2}$$

with, respectively,

$$k_0(v) = \begin{cases} 1-v & \text{for } 0 \le v \le 1 \text{ and } 0 \text{ for } v > 1 \text{ (Cesàro)}, \\ e^{-v} & \text{for } v \ge 0 \quad \text{(Abel)}, \\ v/(e^v-1) & \text{for } v > 0 \text{ and } 1 \text{ for } v = 0 \quad \text{(Lambert)}. \end{cases}$$

At this point one may ask the following general Tauberian question.

Question 2.1. *For which kernels k_0 on $[0, \infty)$ do the conditions (2.2) and (2.1) imply convergence of $\sum_0^\infty a_n$ to A ?*

In terms of the partial sum function

$$s(v) = \sum_{n\le v} a_n, \quad -\infty < v < \infty, \tag{2.3}$$

(for which $s(v) = 0$ when $v < 0$), relation (2.2) may be written with the aid of an improper Stieltjes integral:

$$f(u) = \int_{0-}^{\infty-} k_0\left(\frac{v}{u}\right) ds(v) = \int_{0-}^{\infty-} k_0(v) d_v s(uv) \to Ak_0(0) \quad \text{as } u \to \infty; \tag{2.4}$$

cf. Section I.13. For suitable k_0 and s one can integrate by parts to obtain an (absolutely convergent) ordinary integral for f,

$$f(u) = \int_0^\infty k_1(v)s(uv)dv, \quad \text{where } k_1(v) = -k_0'(v). \tag{2.5}$$

In the classical cases considered, the representation (2.5) is valid with, respectively,

$$k_1(v) = \begin{cases} 1 & \text{for } 0 \le v < 1 \text{ and } 0 \text{ for } v > 1 \text{ (Cesàro)}, \\ e^{-v} & \text{for } v \ge 0 \qquad \text{(Abel)}, \\ \{-v/(e^v - 1)\}' & \text{for } v \ge 0 \qquad \text{(Lambert)}; \end{cases} \tag{2.6}$$

cf. Corollary I.13.2. Since $k_0(0) = \int_0^\infty k_1(v)dv$, relation (2.4) now becomes

$$f(u) = \int_0^\infty k_1(v)s(uv)dv \to A \int_0^\infty k_1(v)dv \quad \text{as } u \to \infty. \tag{2.7}$$

In the special cases above, the boundedness of the function f in (2.2) and the Tauberian condition (2.1) imply that the function s is bounded; cf. Sections I.5 and I.6. For the following theory it is convenient to replace condition (2.1) by the weaker condition that $s(\cdot)$ be *slowly oscillating* on $\mathbb{R}^+ = (0, \infty)$:

$$s(\rho v) - s(v) \to 0 \text{ as } v \to \infty \text{ and } \rho \to 1; \tag{2.8}$$

cf. Definition I.16.1. Under this condition the boundedness of f still implies the boundedness of s for a large class of kernels; see Section I.20.

After this introduction it is natural to pose the following

Problem 2.2. (*Wiener–Pitt question formulated for* $\mathbb{R}^+$) Determine the kernels $k(\cdot)$ in $L^1(0, \infty)$ for which one has a *Tauberian theorem* of the following form: The limit relation

$$\int_0^\infty k\left(\frac{v}{u}\right) s(v)\frac{dv}{u} = \int_0^\infty k(v)s(uv)dv \to A \int_0^\infty k(v)dv \quad \text{as } u \to \infty \tag{2.9}$$

for bounded functions s and corresponding constants A, together with the Tauberian condition

$$s(\cdot) \text{ slowly oscillating (on } \mathbb{R}^+), \tag{2.10}$$

implies

$$s(u) \to A \quad \text{as } u \to \infty. \tag{2.11}$$

Observe that in the 'Abelian direction', the conditions '$|s(\cdot)| \le M$' and '$s(u) \to A$ as $u \to \infty$' imply (2.9) whenever $k(\cdot)$ is in $L^1(0, \infty)$. This follows by dominated convergence: one would have $s(uv) \to A$ as $u \to \infty$ for every $v > 0$, and $|k(v)s(uv)|$ would be majorized by the integrable function $M|k(v)|$.

The Tauberian problem becomes nicer if it is formulated for $\mathbb{R} = (-\infty, \infty)$ instead of $\mathbb{R}^+$ and this is what one usually does. For the translation of Problem 2.2 to $\mathbb{R}$ we set

$$u = e^x, \quad v = e^y, \quad s(v) = s(e^y) = S(y), \tag{2.12}$$
$$k\left(\frac{v}{u}\right)\frac{dv}{u} = k(e^{y-x})e^{y-x}dy = K(x-y)dy, \quad \text{or} \quad k(v)dv = K(-y)dy.$$

The condition 'k in $L^1(\mathbb{R}^+)$' is equivalent to 'K in $L^1(\mathbb{R})$'. [If one starts out with a kernel $K \in L^1(\mathbb{R})$, it is sometimes more natural to define the kernel $k(v)$ as $K(-y)$ or $K(y)$. In that case $k(v)$ is of class $L^1(\mathbb{R}^+, \omega)$ with weight function $\omega(v) = 1/v$; cf. Section 14. The class $L^1(\mathbb{R}^+, \omega)$ occurs naturally in connection with convolution on $\mathbb{R}^+$; cf. Remarks IV.9.5.]

The requirement that s should be slowly oscillating takes on a new form appropriate to $\mathbb{R}$:

Definition 2.3. (*Tauberian conditions for* $\mathbb{R}$) A function $S(\cdot)$ on $\mathbb{R}$ is called *slowly oscillating* on $\mathbb{R}$ (towards $+\infty$) if

$$S(y) - S(x) \to 0 \quad \text{as } x \to \infty \text{ and } y - x \to 0.$$

Similarly, S is said to be *slowly decreasing* on $\mathbb{R}$ if

$$\liminf \{S(y) - S(x)\} \geq 0 \quad \text{for } x \to \infty \text{ and } 0 < y - x \to 0.$$

Finally, we say that S satisfies the *step function condition* on $\mathbb{R}$ if S is piecewise constant, with the intervals of constancy having length $\geq c > 0$.

If S is differentiable and $|S'(x)| \leq C$, then S is uniformly continuous and *a fortiori* slowly oscillating on $\mathbb{R}$. Notice that for slowly decreasing functions, only the decrease is limited, not the increase. If S is differentiable and its derivative is bounded from below, $S'(x) \geq -C$, then S is slowly decreasing on $\mathbb{R}$.

In view of (2.12), Problem 2.2 now has the following equivalent formulation for $\mathbb{R}$ (cf. Pitt [1938a]):

Problem 2.4. (*Wiener–Pitt question for* $\mathbb{R}$) Determine the functions ('limitation kernels') K in $L^1(\mathbb{R})$ for which one has a *Tauberian theorem* of the following form: The limit relation

$$\int_{\mathbb{R}} K(x-y)S(y)dy = \int_{\mathbb{R}} K(y)S(x-y)dy \to A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty \tag{2.13}$$

for bounded functions S and corresponding constants A, together with the Tauberian condition

$$S(\cdot) \text{ slowly oscillating (on } \mathbb{R}), \tag{2.14}$$

implies

$$S(x) \to A \quad \text{as } x \to \infty. \tag{2.15}$$

A kernel $k(v)$ in $L^1(\mathbb{R}^+)$ will be *admissible* in Problem 2.2 if and only if the corresponding kernel $K(y) = k(e^{-y})e^{-y}$ in $L^1(\mathbb{R})$ is admissible in Problem 2.4.

Observe that K *cannot be admissible* if its Fourier transform

$$\hat{K}(t) = \int_{\mathbb{R}} K(y)e^{-ity}dy, \quad -\infty < t < \infty \tag{2.16}$$

vanishes at some point $t = \alpha$. In that case the bounded function $S(x) = e^{i\alpha x}$ satisfies (2.13) with $A = 0$ and (2.14), but not (2.15).

For a first form of the answer to Problems 2.4 and 2.2 see Theorem 3.4. It will ultimately follow from Section 8 that K is admissible in Problem 2.4 if and only if $\hat{K}(t) \neq 0$ for all t (Wiener's condition).

Wiener had actually asked a different question (which has the same answer); see Problem 4.1 below.

3 Testing Equation for Wiener Kernels

We begin with a reduction of Problem 2.4 to a testing equation for admissible kernels. Such an approach to Wiener theory was sketched in slightly more general context by Beurling [1945]; cf. also Karamata [1937a], [1937b], Delange [1950] and Korevaar [1955]. Widder [1971] used the testing equation as the basis for a limited Tauberian theory without Fourier transforms; cf. Section 6.

Reduction 3.1. Suppose that the L^1 kernel K is *not admissible* in Problem 2.4 for $\mathbb{R}$ (the Wiener–Pitt Problem). Then there is a bounded function $S(\cdot)$ which satisfies (2.13) for some constant A and (2.14), but not (2.15). Replacing this function $S(\cdot)$ by $S(\cdot) - A$ we may and will assume in the following that $A = 0$. Then

$$\int_{\mathbb{R}} K(y)S(x-y)dy \to 0 \quad \text{as } x \to \infty, \tag{3.1}$$

but $S(x)$ *does not tend to* 0 as $x \to \infty$. Thus there exist a number $\beta > 0$ and a sequence $x_n \to \infty$ such that

$$|S(x_n)| \geq \beta, \quad n = 1, 2, \ldots. \tag{3.2}$$

The 'slow oscillation' of S means that for every $\varepsilon > 0$, there are positive numbers $B = B(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that

$$|S(y) - S(x)| \leq \varepsilon \ \text{ whenever } \ x \geq B \ \text{ and } \ |y - x| \leq \delta. \tag{3.3}$$

We now consider the sequence of functions $\{S_n\}$ defined by

$$S_n(x) = S(x_n + x), \quad -\infty < x < \infty, \quad n = 1, 2, \ldots.$$

For this sequence one has

$$|S_n(0)| \geq \beta, \quad n = 1, 2, \ldots;$$

the family $\{S_n\}$ is uniformly bounded ($|S_n(x)| \leq M = \sup |S(y)|$ for all x) and it has a property closely related to equicontinuity on every interval $[-C, \infty)$ (although the functions S_n need not be continuous!):

$$|S_n(y) - S_n(x)| = |S(x_n + y) - S(x_n + x)| \le \varepsilon$$

for $x \ge B - x_n$ and $|y - x| \le \delta$. One now chooses a subsequence $\{S_{n_j}\}$ of $\{S_n\}$ which converges at all rational points on $(-\infty, \infty)$. By the standard proof of Arzelà's theorem for bounded equicontinuous families, such a subsequence will converge uniformly on every finite interval; cf. Rudin [1953/76] (section 7.23), Korevaar [1968] (section 2.5.8). We define

$$\Phi(x) = \lim_{j\to\infty} S_{n_j}(x), \quad -\infty < x < \infty. \tag{3.4}$$

The function Φ will be bounded, it is uniformly continuous:

$$|\Phi(y) - \Phi(x)| \le \varepsilon \text{ for } |y - x| \le \delta, \tag{3.5}$$

but it is not the zero function:

$$|\Phi(0)| \ge \beta.$$

Finally, by (3.1),

$$\int_{\mathbb{R}} K(y)\Phi(x-y)dy = \lim_{j\to\infty} \int_{\mathbb{R}} K(y) S_{n_j}(x-y)dy$$
$$= \lim_{j\to\infty} \int_{\mathbb{R}} K(y) S(x_{n_j} + x - y)dy = 0, \quad \forall x \in \mathbb{R}. \tag{3.6}$$

Indeed, for fixed x the sequence $\{S_{n_j}(x-y)\}$ is pointwise convergent to the function $\Phi(x-y)$ and the products $|K(y)S_{n_j}(x-y)|$ are majorized by the integrable function $M|K(y)|$.

We thus have the following

Conclusion 3.2. *Suppose that the L^1 kernel K is not admissible in the Wiener–Pitt Problem 2.4 for $\mathbb{R}$. Then the* TESTING EQUATION

$$K * \Phi(x) = \int_{\mathbb{R}} K(x-y)\Phi(y)dy = \int_{\mathbb{R}} K(y)\Phi(x-y)dy = 0, \quad \forall x \in \mathbb{R} \tag{3.7}$$

has a bounded uniformly continuous solution $\Phi \not\equiv 0$.

If equation (3.7) has a bounded (measurable) solution Φ, one may verify by substitution into (3.7) that

$$\Psi(y) = \int_y^{y+c} \Phi(z)dz = \int_0^c \Phi(y+w)dw \tag{3.8}$$

is also a solution for any fixed $c > 0$. Such a function Ψ is bounded and uniformly continuous, and if Φ is not equal to 0 almost everywhere, one may choose c such that $\Psi \not\equiv 0$.

The reduction of Problem 2.4 to the testing equation works also if in the Tauberian condition, slow oscillation is replaced by the step function condition of Definition 2.3. In that case the limit function Φ will also be piecewise constant.

We now turn things around. By Conclusion 3.2, the *absence* of a nonzero bounded solution Φ of the testing equation (3.7) for $K \in L^1(\mathbb{R})$ implies that K is admissible in the Wiener–Pitt Problem 2.4, so that there is a Tauberian theorem for the kernel K. At this stage it is convenient to give a first definition of Wiener kernels. Equivalent descriptions of such kernels (including Wiener's own definition in terms of the Fourier transform) will be given in subsequent sections.

Definition 3.3. A function K on $\mathbb{R}$ will be called a WIENER KERNEL, denoted $K \in W$, if K is in $L^1(\mathbb{R})$ and the *testing equation* (3.7) has *no* bounded (continuous or measurable) solution Φ other than the zero solution.

Similarly, a function k will be called a *Wiener kernel* for $\mathbb{R}^+$, denoted $k \in W^+$, if k is in $L^1(\mathbb{R}^+)$ and the corresponding *testing equation*

$$\int_0^\infty k\left(\frac{v}{u}\right)\phi(v)\frac{dv}{u} = \int_0^\infty k(v)\phi(uv)dv = 0, \quad \forall u \in \mathbb{R}^+, \tag{3.9}$$

has no bounded solution ϕ other than the zero solution.

The relation $K(x) = k(e^{-x})e^{-x}$ sets up a one-to-one correspondence between kernels $k \in W^+$ and kernels $K \in W$; see (2.12) and the respective testing equations.

By the preceding we have the following Tauberian theorem; cf. Pitt [1938a] (theorem α).

Theorem 3.4. *(i) Let $K \in L^1(\mathbb{R})$ be a Wiener kernel according to Definition 3.3. Then the limit relation*

$$\int_{\mathbb{R}} K(x-y)S(y)dy \to A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty \tag{3.10}$$

for a bounded, slowly oscillating function S on $\mathbb{R}$ implies that $S(x) \to A$.

(ii) Equivalently, let k be in W^+ [so that in particular k is in $L^1(\mathbb{R}^+)$]. Then k is admissible in the Wiener–Pitt Problem 2.2, so that the limit relation

$$\int_0^\infty k(v)s(uv)dv \to A \int_0^\infty k(v)dv \quad \text{as } u \to \infty \tag{3.11}$$

for a bounded, slowly oscillating function s on $\mathbb{R}^+$ implies that $s(u) \to A$.

In each part the condition of slow oscillation may be replaced by the appropriate step function condition from Definition 2.3 or I.16.2.

4 Original Wiener Problem

Suppose that one starts again with a limit relation (3.10) or (3.11) involving an L^1 function and a bounded function. Then it may be that one is interested not in convergence, but in another kind of summability, for example, Cesàro summability; cf. also Wiener's special problem in Example 6.4 below. More generally Wiener wanted to compare asymptotic averages involving arbitrary L^1 kernels.

Problem 4.1. (*Wiener's question*) Determine the functions ('limitation kernels') $K \in L^1(\mathbb{R})$ for which one has a *Tauberian theorem* of the following form: The limit relation

$$\int_{\mathbb{R}} K(x-y)S(y)dy = \int_{\mathbb{R}} K(y)S(x-y)dy \to A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty \tag{4.1}$$

for bounded functions S and corresponding constants A implies

$$\int_{\mathbb{R}} H(x-y)S(y)dy = \int_{\mathbb{R}} H(y)S(x-y)dy \to A \int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{4.2}$$

for EVERY function ('limitation kernel') H in $L^1(\mathbb{R})$.

Wiener's work [1932] implies the following Tauberian theorem for which proofs are given below.

Theorem 4.2. *Suppose that $K \in L^1(\mathbb{R})$ is a Wiener kernel according to Definition 3.3. Then K is admissible in Problem 4.1, so that the limit relation* (4.1) *for bounded S and corresponding A implies* (4.2) *for every function H in $L^1(\mathbb{R})$.*

As in Problem 2.4, K cannot be admissible in Problem 4.1 if $\hat{K}(\alpha) = 0$ for some real number α. In that case $S(x) = e^{i\alpha x}$ satisfies (4.1) with $A = 0$, but it will not satisfy (4.2) unless $\hat{H}(\alpha) = 0$.

It will turn out that the condition $K \in W$ is necessary and sufficient for the admissibility of K in Problems 4.1 and 2.4; see Section 8.

We give two proofs for Theorem 4.2 which both provide additional information. The first proof is functional-analytic, while the second (in Section 5) is based on Theorem 3.4.

First proof. Let S be bounded. Replacing $S(\cdot)$ by $S(\cdot) - A$ we may and will assume that (4.1) holds with $A = 0$. Observe that (4.2) will then hold for every finite linear combination $H_1(y) = \sum_j b_j K(y - \lambda_j)$:

$$\begin{aligned} \int_{\mathbb{R}} H_1(y)S(x-y)dy &= \sum_j b_j \int_{\mathbb{R}} K(y-\lambda_j)S(x-y)dy \\ &= \sum_j b_j \int_{\mathbb{R}} K(z)S(x-\lambda_j - z)dz \to 0 \quad \text{as } x \to \infty. \end{aligned} \tag{4.3}$$

More generally suppose that H belongs to the closed (linear) span $\Sigma(K)$ of the translates of K in L^1, that is, for every number $\varepsilon > 0$ there is a linear combination H_1 such that

$$\int_{\mathbb{R}} |H(y) - H_1(y)|dy \leq \varepsilon.$$

Then

$$\left| \int_{\mathbb{R}} \{H(y) - H_1(y)\}S(x-y)dy \right| \leq \varepsilon \sup |S(z)|,$$

hence by (4.3)

$$\limsup_{x\to\infty} \left| \int_{\mathbb{R}} H(y)S(x-y)dy \right| \le \varepsilon \sup |S(z)|. \tag{4.4}$$

This proves (4.2) for all functions H in $\Sigma(K)$.

By the Hahn–Banach theorem, a function $H(y)$ is the L^1 limit of finite sums $\sum b_j K(y-\lambda_j)$ if and only if every *continuous linear functional* on L^1 which vanishes on all translates of K also vanishes on H; cf. Rudin [1966/87], Korevaar [1968]. Now these functionals have the form

$$l(F) = l_\Phi(F) = \int_{\mathbb{R}} F(y)\Phi(y)dy \quad \text{with bounded } \Phi,$$

and $l_\Phi = 0$ if and only if $\Phi = 0$. Hence the condition for H to be in $\Sigma(K)$ is that the relation

$$\int_{\mathbb{R}} K(y-\lambda)\Phi(y)dy = 0, \quad \forall\lambda \in \mathbb{R} \tag{4.5}$$

should imply

$$l_\Phi(H) = \int_{\mathbb{R}} H(y)\Phi(y)dy = 0 \quad \text{for every bounded } \Phi. \tag{4.6}$$

In particular $\Sigma(K) = L^1$ if and only if (4.5) for bounded Φ implies $l_\Phi = 0$ or $\Phi = 0$.

Renaming $\Phi(y) : \Psi(-y)$ and setting $\lambda = -x$ one finds that (4.5) is equivalent to the testing equation (3.7): the condition

$$0 = \int_{\mathbb{R}} K(y+x)\Psi(-y)dy = \int_{\mathbb{R}} K(x-z)\Psi(z)dz, \quad \forall x \in \mathbb{R}$$

for bounded Ψ should imply $\Psi = 0$. Conclusion: $\Sigma(K) = L^1$ if and only if K is in W, and then relation (4.1) implies (4.2) for all functions H in L^1.

Corollary 4.3. *Let K be in $L^1(\mathbb{R})$. Then the limit relation* (4.1) *for bounded S implies* (4.2) *for all functions H in $\Sigma(K)$, the closed span of the translates of K.*

Furthermore, the function K is a Wiener kernel if and only if the finite linear combinations of its translates are dense in L^1 (so that the translates of K 'span' the space L^1).

Cf. Wiener [1932]. For the functional-analytic approach to the spanning property, cf. Carleman [1944].

5 Wiener's Theorem With Additions by Pitt

Theorem 3.4 does not handle the important one-sided Tauberian condition $na_n \ge -C$ encountered in Sections I.6, I.7 and I.10. That condition implies that the partial sum function $s(v) = \sum_{n\le v} a_n$ is *slowly decreasing* on $\mathbb{R}^+$ (Definition I.16.3):

$$\liminf \{s(\rho v) - s(v)\} \ge 0 \quad \text{for} \quad v \to \infty \quad \text{and} \quad 1 < \rho \to 1. \tag{5.1}$$

For a second proof of Theorem 4.2 we choose the equivalent result for $\mathbb{R}^+$ and we include Pitt's important addition [1938a] for slowly decreasing functions s.

Theorem 5.1. *Suppose that k is in W^+ according to Definition 3.3, so that in particular k is in $L^1(\mathbb{R}^+)$. Then the limit relation*

$$\int_0^\infty k(v)s(uv)dv \to A \int_0^\infty k(v)dv \quad \textit{as } u \to \infty \tag{5.2}$$

for bounded functions s and corresponding constants A implies

$$\int_0^\infty h(v)s(uv)dv \to A \int_0^\infty h(v)dv \quad \textit{as } u \to \infty \tag{5.3}$$

for every function h in $L^1(\mathbb{R}^+)$.

Furthermore, if a bounded function s satisfies (5.2) *and is in addition slowly decreasing on $\mathbb{R}^+$* (5.1) *or satisfies the step function condition for $\mathbb{R}^+$ of Definition I.16.2, then $s(u) \to A$ as $u \to \infty$.*

Pitt also obtained convergence $s(u) \to A$ under more general (but complicated) Tauberian conditions; cf. Pitt [1938a] (theorem 8), [1940] and [1958] (chapter 4, theorem 10) for the case of $\mathbb{R}$.

Proof of the Theorem. Let k be in W^+ and let s be bounded and satisfy limit relation (5.2) with $A = 0$.

(i) In order to establish conclusion (5.3) it will be enough to treat the special case where h is the step function m:

$$m(v) = \begin{cases} 1 \text{ for } 0 < v \le 1, \\ 0 \text{ for } v > 1. \end{cases}$$

Indeed, if (5.3) holds for $h(v) = m(v)$, then it holds also for $h(v) = m_c(v) = m(v/c)$ with $c > 0$. Moreover, every function h in $L^1(\mathbb{R}^+)$ is the limit, in L^1, of piecewise constant functions with bounded support. The latter are (almost everywhere) equal to finite linear combinations of simple step functions m_c.

For $h = m$ the left-hand side of (5.3) becomes the Cesàro mean σ of s introduced in Section I.13:

$$\int_0^\infty m(v)s(uv)dv = \int_0^1 s(uw)dw = \frac{1}{u}\int_0^u s(v)dv = \sigma(u). \tag{5.4}$$

We will show that σ satisfies conditions (2.9), (2.10) of Problem 2.2 with $A = 0$. First of all,

$$\begin{aligned} \int_0^\infty k(v)\sigma(uv)dv &= \int_0^\infty k(v)dv \int_0^1 s(uvw)dw \\ &= \int_0^1 dw \int_0^\infty k(v)s(uvw)dv \to 0 \quad \text{as} \quad u \to \infty. \end{aligned} \tag{5.5}$$

Indeed, $\int_0^\infty k(v)s(uvw)dv \to 0$ pointwise and boundedly on $\{0 < w < 1\}$ as $u \to \infty$. It is clear that $|\sigma(\cdot)|$ is bounded by $\sup|s(\cdot)|$. The function σ is also slowly oscillating: by (5.4), $\rho v\sigma(\rho v) - v\sigma(v) = \int_v^{\rho v} s(w)dw$, so that

$$\begin{aligned}|\sigma(\rho v) - \sigma(v)| &= \left|\frac{1}{v}\int_v^{\rho v} s(w)dw - (\rho - 1)\sigma(\rho v)\right| \\ &\leq 2|\rho - 1| \sup|s(w)| \to 0 \quad \text{as} \quad \rho \to 1. \end{aligned} \tag{5.6}$$

Theorem 3.4, part (ii) applied to σ now shows that $\sigma(u) \to A = 0$ as $u \to \infty$. Thus (5.3) holds for $h = m$ (with $A = 0$) and this completes the proof of part (i).

(ii) The relation $\sigma(u) \to 0$ corresponds to Cesàro limitability of s with Cesàro limit 0; cf. Section I.18. If s is slowly decreasing or satisfies the step function condition, convergence $s(u) \to 0$ now follows from Theorem I.18.1.

□

Problems of Boundedness. For the application of Wiener theory under general Tauberian conditions one must be able to establish *boundedness* of the functions S or s. In the case of positive monotonic kernels one may appeal to the general boundedness theorem of Section I.20.

The following Wiener-type Tauberian for positive kernels, given in Hardy [1949] (section 12.12), is useful if one knows that s is *bounded from below:*

Theorem 5.2. *Let $k(\cdot)$ be in W^+, $k(\cdot) \geq 0$ on $\mathbb{R}^+$ and $k(v) \geq \beta > 0$ for $0 < v < a$. Let $s(\cdot)$ be $\geq -C$ and such that $\int_0^\infty k(v)s(uv)dv$ exists, is bounded on $\{0 < u < \infty\}$ and tends to $A\int_0^\infty k(v)dv$ as $u \to \infty$. Then the Cesàro means $\sigma(u) = \int_0^1 s(uv)dv$ tend to A as $u \to \infty$.*

Proof. Adding C to $s(\cdot)$ if necessary one may suppose that $s(\cdot) \geq 0$. Then

$$\int_0^\infty k(v)s(uv)dv \geq \int_0^a \beta s(uv)dv = \beta\int_0^1 s(uaw)adw = \beta a\sigma(au) \geq 0,$$

and hence $\sigma(\cdot)$ is bounded, say by b. Thus by (5.4)

$$\sigma'(u) = \{s(u) - \sigma(u)\}/u \geq -b/u \quad \text{almost everywhere},$$

so that σ is slowly decreasing; cf. (5.1). Finally, by (5.2)

$$\int_0^\infty k(v)\sigma(uv)dv \to A\int_0^\infty k(v)dv \quad \text{as} \quad u \to \infty;$$

cf. the derivation of (5.5). Applying the final part of Theorem 5.1 to σ one concludes that $\sigma(u) \to A$. □

For other results under a one-sided Tauberian condition, see Pitt [1938b], Diamond and Essén [1978]. In Chapter V we will treat a boundedness theorem of Pitt for rapidly decreasing kernels.

6 Direct Applications of the Testing Equations

The testing equations in Section 3 may be used to treat a number of kernels without Wiener's 1932 Fourier method.

Example 6.1. For Cesàro summability, the kernel $k(v)$ is equal to 1 on the interval $\{0 \le v < 1\}$ and equal to 0 for $v > 1$; see (2.6). Here the testing equation (3.9) for $\mathbb{R}^+$ becomes

$$\int_0^1 \phi(uv)dv = 0 \quad \text{or} \quad \int_0^u \phi(w)dw = 0, \quad \forall u > 0.$$

By differentiation this implies that $\phi(u) = 0$ almost everywhere, hence the Cesàro kernel is in W^+.

Example 6.2. In the case of Abel summability, the kernel $k(v)$ is equal to e^{-v}; see (2.6). Here the testing equation for $\mathbb{R}^+$ becomes

$$\int_0^\infty e^{-v}\phi(uv)dv = 0 \quad \text{or} \quad \int_0^\infty \phi(w)e^{-w/u}dw = 0, \quad \forall u > 0. \tag{6.1}$$

In other words, one requires that the Laplace transform of ϕ should be equal to zero. It is well-known that this condition implies $\phi = 0$. To verify this, note that by (6.1) for $u = 1/n$,

$$\int_0^\infty \phi(w)e^{-nw}dw = 0 \quad \text{or} \quad \int_0^1 \phi\{\log(1/x)\}x^{n-1}dx = 0, \quad n = 1, 2, \ldots .$$

An appropriate moment theorem then shows that $f(x) = \phi\{\log(1/x)\}$ is (equivalent to) the zero function on (0, 1). Indeed, by integration by parts, the continuous function $F(x) = \int_0^x f(y)dy$ will also be orthogonal to the powers $1, x, x^2, \cdots$, hence $\int_0^1 F(x)P(x)dx = 0$ for all polynomials P. One may then derive from Weierstrass's approximation theorem that F is orthogonal to itself, $\int_0^1 F\overline{F} = 0$, so that $F(x) \equiv 0$. Conclusion: $f(x) = F'(x) = 0$ almost everywhere on (0, 1).

It follows that $k(v) = e^{-v}$ is in W^+. One can now use Theorems 5.1 and 5.2 to derive the principal Hardy–Littlewood theorems for Abel summability. In Littlewood's Theorem I.7.1 one knows that $|na_n| \le C$, hence the partial sum function $s(v) = \sum_{n \le v} a_n$ is slowly oscillating. To verify its boundedness one may use Corollary I.5.2. In the Hardy–Littlewood Theorem I.7.2 one has $na_n \ge -C$, so that $s(\cdot)$ is slowly decreasing. The boundedness of $s(\cdot)$ can again be derived from Section I.5; cf. Wiener [1933] (section 16). Under the weaker Schmidt condition of slowly decreasing $\{s_n\}$ or $s(\cdot)$ one may appeal to Boundedness Theorem I.19.1; cf. Remarks I.19.4.

The 'Abel to Cesàro' Theorem I.7.3 provides the simplest illustration of Wiener's comparison of asymptotic averages:

Example 6.3. Let the sequence $\{s_n\}$ be Abel limitable to A, or equivalently, let $\sum a_n = \sum(s_n - s_{n-1})$ have Abel sum A. By Example 6.2, the relevant kernel $k(v) = e^{-v}$ is in W^+. Hence if the sequence $\{s_n\}$ is bounded, the first (Wiener) part of Theorem 5.1 implies its Cesàro limitability to A: just take $h(v) = 1$ for $0 \le v < 1$ and 0 for $v > 1$. If one knows only that the sequence $\{s_n\}$ is bounded from below, $s_n \ge -C$, its Cesàro limitability follows from Theorem 5.2.

Theorem I.21.1 for the Laplace transform can be derived from the same theorem.

Example 6.4. Before he developed his general theory, Wiener [1927a] dealt with the following problem which arose in his work on general harmonic analysis; cf. Levinson [1966] (pp. 17, 18). Let $s(\cdot)$ be integrable over every finite interval $(0, B)$. Does the existence of one of the asymptotic averages

$$\lim \sigma(u) = \lim \frac{1}{u} \int_0^u s(v)dv = \lim \int_0^1 s(uv)dv, \tag{6.2}$$

$$\lim \frac{2}{\pi u} \int_0^\infty \left(\frac{\sin v/u}{v/u} \right)^2 s(v)dv = \lim \int_0^\infty \frac{2\sin^2 v}{\pi v^2} s(uv)dv \tag{6.3}$$

as $u \to \infty$ imply the existence of the other, and their equality? In the desired application the function s is nonnegative.

In (6.2) the relevant kernel is the Cesàro kernel which is in W^+, but the kernel

$$k(v) = \frac{2\sin^2 v}{\pi v^2} \tag{6.4}$$

in (6.3) is a little more difficult. We will use Example 6.2 to verify that k is also in W^+. Thus one may apply Theorems 5.1 and 5.2 to treat Wiener's 1927 questions.

Let ϕ be any bounded solution of the testing equation (3.9) for k. Then

$$\int_0^\infty k(tv)\phi(uv)dv = \int_0^\infty k(w)\phi(uw/t)dw/t = 0, \quad \forall t, u > 0. \tag{6.5}$$

Below we will determine $f \in L^1(0, \infty)$ such that

$$\int_0^\infty tf(t)k(tv)dt = e^{-v}. \tag{6.6}$$

Multiplying equation (6.5) by $tf(t)$ and integrating over $0 < t < \infty$, one may deduce from (6.6) that ϕ satisfies equation (6.1). Hence $\phi = 0$ and k is indeed in W^+.

The condition on f may be written as

$$\int_0^\infty f(t)\frac{1 - \cos 2vt}{\pi t} dt = v^2 e^{-v},$$

so that by differentiation with respect to v,

$$\frac{2}{\pi} \int_0^\infty f(t) \sin 2vt \, dt = (2v - v^2)e^{-v}.$$

The final equation can be solved by Fourier inversion; the solution

$$f(t) = \frac{8t(12t^2 - 1)}{(4t^2 + 1)^3} \tag{6.7}$$

also satisfies equation (6.6).

Remarks 6.5. For our treatment of the Hardy–Littlewood theorems in Example 6.2, cf. Delange [1949] and Widder [1971].

Bochner and Hardy [1926] answered half of Wiener's questions in Example 6.4: they showed that $\lim \sigma(u) = A$ in (6.2) implies that the limit in (6.3) exists and is equal to A. In his complete treatment of the problem, Wiener [1927a] used the solution (6.7) to equation (6.6). In the terminology of Hardy's book [1949] (sections 12.10, 12.12), the kernel k in (6.4) defines Riemann summability of order two.

7 Fourier Analysis of Wiener Kernels

In order to decide whether a given L^1 function K is a Wiener kernel according to Definition 3.3, we have to investigate if the zero function is the only bounded solution Φ of the testing equation

$$K * \Phi(x) = \int_{\mathbb{R}} K(x - y)\Phi(y)dy = 0, \quad \forall x \in \mathbb{R}. \tag{7.1}$$

FOURIER TRANSFORMATION. Before we discuss the testing equation we list some basic facts about Fourier transformation on $L^1 = L^1(\mathbb{R})$ which will be used later on. Proofs may be found in many books, for example, Titchmarsh [1937/86].

For $F \in L^1$ the Fourier transform $\hat{F}$ is the function

$$\hat{F}(t) = \int_{\mathbb{R}} F(x)e^{-itx}dx, \quad -\infty < t < \infty. \tag{7.2}$$

[Some authors use e^{itx} instead of e^{-itx}, but this makes no essential difference in the theory.] By (7.2) the transform of $F(x)e^{icx}$ is $\hat{F}(t - c)$, the transform of $F(x - c)$ is $\hat{F}(t)e^{-ict}$ and the transform of $F(x/\lambda)$ with $\lambda > 0$ is $\lambda\hat{F}(\lambda t)$.

The Fourier transform $\hat{F}(t)$ is bounded and continuous; for $|t| \to \infty$ it tends to 0 by the Riemann–Lebesgue lemma; cf. Korevaar [1968] (section 3.2.3). If $\hat{F}$ is also in L^1 one has the inversion formula

$$F(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{F}(t)e^{ixt}dt, \tag{7.3}$$

valid for almost all x. In any case F can be recovered as a Cesàro-type limit: for almost all x,

$$F(x) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) \hat{F}(t)e^{ixt}dt. \tag{7.4}$$

Thus Fourier transformation on L^1 is one to one: if $\hat{F} = 0$, then $F = 0$.

CONVOLUTION. Assuming that both F and G are L^1 functions, the product $H(x, y) = F(x - y)G(y)$ will be measurable; cf. Rudin [1966/87] (chapter 7). Knowing this, one may use Fubini's theorem to show that the convolution

$$F * G(x) = (F * G)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} F(x - y)G(y)dy \tag{7.5}$$

is also in L^1. Indeed, on the basis of positivity one has

$$\begin{aligned} \int_{\mathbb{R}^2} |H(x, y)|dxdy &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} |F(x - y)G(y)|dy \\ &= \int_{\mathbb{R}} |G(y)|dy \int_{\mathbb{R}} |F(x - y)|dx, \end{aligned} \tag{7.6}$$

and the right-hand side is finite. This proves the existence of the inner integral in the second member and of the integral (7.5) for $F * G(x)$ for almost all x, as well as the integrability of $F * G$ over $\mathbb{R}$. Fubini's 'inversion of order of integration theorem' next shows that the Fourier transform of $F * G$ is equal to the product of the transforms:

$$\begin{aligned} (F * G)\hat{}(t) &= \int_{\mathbb{R}} e^{-itx}dx \int_{\mathbb{R}} F(x - y)G(y)dy \\ &= \int_{\mathbb{R}} G(y)e^{-ity}dy \int_{\mathbb{R}} F(x - y)e^{-it(x-y)}dx = \hat{G}(t)\hat{F}(t). \end{aligned} \tag{7.7}$$

The convolution of an L^1 function F and a bounded function G exists for all x (and is continuous).

Formula (7.7) implies that there is no unit element in L^1 relative to convolution. For if there would be $E \in L^1$ such that $E * F = F$ for all $F \in L^1$, then the formula $\hat{E}\hat{F} = \hat{F}$ would show that $\hat{E} \equiv 1$, which contradicts the Riemann–Lebesgue lemma.

Example 7.1. An important Fourier pair has as transform part the so-called *triangle function*

$$\Delta_{\lambda,c}(t) = \Delta_\lambda(t - c) = \begin{cases} 1 - |t - c|/\lambda & \text{for } |t - c| \le \lambda, \\ 0 & \text{for } |t - c| > \lambda, \end{cases}$$

where c is real and $\lambda > 0$. The 'Fourier original' is

$$D_{\lambda,c}(x) = D_\lambda(x)e^{icx}, \quad D_\lambda(x) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \Delta_\lambda(t)e^{ixt}dt = \frac{1 - \cos\lambda x}{\pi\lambda x^2}.$$

The directed family $\{D_\lambda\}$, $\lambda \to \infty$, forms a so-called *approximate identity* relative to convolution: for any $F \in L^1$ one has $D_\lambda * F \to F$, both in L^1 and in the sense of almost everywhere convergence. The latter fact plays a role in the proof of formula (7.4). By analogy with the case of Fourier series (Section I.3), the approximate identity $\{D_\lambda\}$ may be called the Fejér kernel for $\mathbb{R}$.

HEURISTICS. In view of the preceding it is natural to try to deal with the testing equation (7.1) by Fourier analysis. Formal application of the transformation rule (7.7) for a convolution gives

$$(K * \Phi)\hat{}(t) = \hat{K}(t)\hat{\Phi}(t) = 0, \quad -\infty < t < \infty. \tag{7.8}$$

Thus if

$$\hat{K}(t) \neq 0 \quad \text{for all (real) } t, \tag{7.9}$$

(Wiener's condition), one would hope to conclude that

$$\hat{\Phi} = 0 \quad \text{and hence} \quad \Phi = 0. \tag{7.10}$$

The final conclusion $\Phi = 0$ is correct, but it requires a fair amount of work to validate the heuristic argument. In classical Fourier theory (as opposed to distributional theory), $\hat{\Phi}$ is not defined for arbitrary bounded Φ, and (7.8) does not make sense unless either K or Φ is better than postulated.

Theorem 7.2. *A function K in $L^1(\mathbb{R})$ is a Wiener kernel* [in the sense of Definition 3.3 that $K * \Phi = 0$ for bounded Φ implies $\Phi = 0$] *if and only if the Fourier transform $\hat{K}$ is zero-free (on $\mathbb{R}$). Equivalently (by Section 3), a function k in $L^1(\mathbb{R}^+)$ is a Wiener kernel for $\mathbb{R}^+$ if and only if its 'Fourier–Mellin transform'*

$$\check{k}(t) \stackrel{\text{def}}{=} \int_0^\infty k(v)v^{it}dv \tag{7.11}$$

does not vanish anywhere.

[If one works with the class $L^1(\mathbb{R}^+, \omega)$ with weight function $\omega(v) = 1/v$, mentioned in Section 2, one would obtain a somewhat different formula for the Fourier–Mellin transform; cf. Section 14.]

Below we derive the difficult part of Theorem 7.2 from the following Division Theorem of Wiener [1932] (p. 18) which will be proved in Section 9. (There is a more general result on division in Theorem 9.3.) The author's (shorter) distributional proof for Theorem 7.2 will be given in Section 11.

Theorem 7.3. (Division Theorem) *Let $\hat{H}$ and $\hat{K}$ be Fourier transforms of L^1 functions H and K on $\mathbb{R}$, let $\hat{H}(t) = 0$ for all t outside the finite interval (a, b) and let $\hat{K}(t)$ be $\neq 0$ for $a \leq t \leq b$. Then*

$$\frac{\hat{H}}{\hat{K}} = \hat{Q} \quad \textit{with} \quad Q \in L^1 \quad (\textit{so that} \;\; H = Q * K). \tag{7.12}$$

Derivation of Theorem 7.2. If $\hat{K}(\alpha) = 0$ (for a real α), then $\Phi(y) = e^{i\alpha y}$ is a nonzero bounded solution of the testing equation (7.1), so that K is not in W:

$$\int_{\mathbb{R}} K(x-y)e^{i\alpha y}dy = \int_{\mathbb{R}} K(z)e^{i\alpha(x-z)}dz = e^{i\alpha x}\hat{K}(\alpha) = 0, \quad \forall x.$$

Suppose now that $\hat{K}(t) \neq 0$ for every t and that Φ is a bounded solution of the testing equation $K * \Phi = 0$. Taking

$$H(x) = H_c(x) = D_{1,c}(x) = \frac{1 - \cos x}{\pi x^2}e^{icx}, \quad \hat{H}(t) = \Delta_1(t - c)$$

where c is arbitrary (Example 7.1), we let $Q = Q_c$ be as in Division Theorem 7.3. Then by Fubini's theorem and (7.12)

$$0 = Q_c * (K * \Phi) = (Q_c * K) * \Phi = H_c * \Phi. \tag{7.13}$$

In particular (since $H_c * \Phi$ is continuous),

$$0 = (H_c * \Phi)(0) = \int_{\mathbb{R}} H_c(-y)\Phi(y)dy = \int_{\mathbb{R}} \frac{1-\cos y}{\pi y^2} e^{-icy}\Phi(y)dy. \tag{7.14}$$

This holds for every real c, so that the L^1 function $(1-\cos y)\Phi(y)/(\pi y^2)$ has Fourier transform identically equal to 0. Hence

$$(1-\cos y)\Phi(y) = 0 \quad \text{for almost all} \quad y,$$

and thus also $\Phi(y) = 0$ almost everywhere. Conclusion: K is a Wiener kernel in the sense of Definition 3.3.

Theorem 7.2 has the following corollary.

Corollary 7.4. *If the testing equation $K * \Phi = 0$ in (7.1) has a bounded solution $\Phi \not\equiv 0$, then the Fourier transform $\hat{K}$ has a zero α, so that the testing equation has an exponential solution $\Phi(y) = e^{i\alpha y}$ with real α.*

8 The Principal Wiener Theorems

We summarize what we have learned about the class W of Wiener kernels K and explicitly state the principal Tauberian theorems.

Theorem 8.1. (Wiener Kernels) *Each of the following conditions may be used to characterize $K \in L^1(\mathbb{R})$ as a Wiener kernel:*

(1) The Fourier transform

$$\hat{K}(t) = \int_{\mathbb{R}} K(x)e^{-itx}dx \tag{8.1}$$

of K is different from 0 for all (real) t [this was Wiener's definition]*;*

(2) The testing equation $K * \Phi = 0$ for bounded Φ implies $\Phi = 0$ [this was our original definition for a Wiener kernel, Definition 3.3]*;*

(3) The translates $K(\cdot - \lambda)$ of K, $-\infty < \lambda < \infty$, span the whole space L^1;

(4) K is admissible in the Wiener Problem 4.1;

(5) K is admissible in the Wiener–Pitt Problem 2.4.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is Theorem 7.2. For (2) $\Leftrightarrow$ (3), see Corollary 4.3. For (2) $\Rightarrow$ (4) $\Rightarrow$ (1), see Theorem 4.2 and the observation following it. Similarly, for (2) $\Rightarrow$ (5) $\Rightarrow$ (1), see Theorem 3.4 and the observation following Problem 2.4.

We can now formulate Wiener's Principal Tauberian Theorem [1932] (theorem 8) as follows; cf. Theorem 4.2 and the lines below it. The theorem compares *asymptotic averages*:

Theorem 8.2. *Let K be in L^1. Then the condition $K \in W$ is necessary and sufficient in order that the relation*

$$\int_{\mathbb{R}} K(x-y)S(y)dy \to A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty, \tag{8.2}$$

for (otherwise arbitrary) bounded functions S and corresponding constants $A = A_S$, imply

$$\int_{\mathbb{R}} H(x-y)S(y)dy \to A \int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{8.3}$$

for every function H in $L^1(\mathbb{R})$.

We next state WIENER'S APPROXIMATION THEOREM [1932] (theorem 2), which emphasizes the equivalence (3) $\Leftrightarrow$ (1) in Theorem 8.1.

Theorem 8.3. *For $K \in L^1$, the finite linear combinations of the translates $K(\cdot - \lambda)$, $\lambda \in \mathbb{R}$ are dense in L^1 if and only if $\hat{K}(t) \neq 0$ for all $t \in \mathbb{R}$.*

Being equivalent to Theorem 8.2, this result is also sometimes called Wiener's Tauberian theorem, especially in the context of extensions to groups. (This might appear mysterious to those looking for a direct connection with summability!) For a quick proof of the Theorem one may use the 'continuous linear functionals test' for a spanning set in L^1 to obtain the testing equation (cf. Section 4), and then treat the latter with the aid of distributions; see Section 11. An algebraic proof of Theorem 8.3 will be given in Chapter V.

There is a corresponding (but simpler) approximation theorem of Wiener [1932] for L^2. The translates of an L^2 function K span L^2 if and only if $\hat{K}(t) \neq 0$ for almost all $t \in \mathbb{R}$. Uniform approximation has been considered by Boas and Bochner [1938]. In particular, if K is of bounded variation on $\mathbb{R}$, the translates span the space $C_0(\mathbb{R})$ of the continuous functions which vanish at $\pm\infty$ if and only if the Fourier transform $\widehat{dK}$ does not vanish on an interval. Approximation involving special families of translates occurs in work of Korevaar [1947], Ganelius [1972], Atzmon and Olevskiĭ [1996], and N. Nikolski [1999]; cf. also related work by H.S. Shapiro [1968].

Going back to L^1, one may ask which functions H are in the closed span $\Sigma(K)$ of translates of K if $\hat{K}$ *does* have zeros. (Such functions H can then be used to go from (8.2) to (8.3); cf. Section 4.) If H is L^1 limit of linear combinations of translates of K, then $\hat{H}(t) = 0$ everywhere on the zero set of $\hat{K}$, but the latter property is not enough for H to be in $\Sigma(K)$. Wiener [1932] proved that $\Sigma(K)$ contains all functions H for which supp $\hat{H}$ belongs to the open set where $\hat{K}(t) \neq 0$; cf. Corollary 9.4. For more refined results see, for example, Rudin [1962/90] (chapter 7).

It is useful to include a statement of Pitt's additions [1938a], [1958] to Wiener's theorem for $\mathbb{R}$. Here we use the Tauberian conditions of Definition 2.3.

Theorem 8.4. *For K in $L^1(\mathbb{R})$, the condition $K \in W$ is necessary and sufficient in order that the limit relation* (8.2), *for (otherwise arbitrary) bounded, slowly oscillating or slowly decreasing functions $S(\cdot)$ on $\mathbb{R}$ and corresponding constants $A = A_S$, imply $S(x) \to A$ as $x \to \infty$.*

For $K \in W$ and bounded S satisfying relation (8.2), *the step function condition for S on $\mathbb{R}$ also implies convergence $S(x) \to A$.*

As in the case of Theorem 8.2, the necessity of the condition '$\hat{K}$ zero-free' in the first part follows by considering the exponentials $S(x) = e^{i\alpha x}$; they are slowly oscillating. For the 'sufficiency', cf. Theorem 5.1. The optimality of the Tauberian conditions in Wiener theory will be discussed in Section 16.

There are also Wiener theorems for *families of kernels;* see Section 10.

RESULTS FOR $\mathbb{R}^+$. We have seen in the preceding sections that the Theorems for $\mathbb{R}$ have analogs for $\mathbb{R}^+$. By Theorem 7.2 a function k in $L^1(\mathbb{R}^+)$ is in the Wiener class W^+ if and only if its Fourier–Mellin transform

$$\check{k}(t) = \int_0^\infty k(v) v^{it} dv \tag{8.4}$$

is different from 0 for all (real) t. Because it is so often used in applications, we explicitly state the Wiener–Pitt theorem for $\mathbb{R}^+$ (cf. Theorem 5.1):

Theorem 8.5. *Let k be in W^+, let s be bounded and let*

$$\int_0^\infty k(v)s(uv)dv \to A \int_0^\infty k(v)dv \quad \text{as } u \to \infty. \tag{8.5}$$

Then

$$\int_0^\infty h(v)s(uv)dv \to A \int_0^\infty h(v)dv \quad \text{as } u \to \infty \tag{8.6}$$

for every function h in $L^1(\mathbb{R}^+)$. If moreover s is slowly decreasing on $\mathbb{R}^+$ as in (5.1) *or satisfies the step function condition for $\mathbb{R}^+$ of Definition I.16.2, then $s(u) \to A$ as $u \to \infty$.*

Remarks 8.6. (Cf. the comments in Korevaar [1979]) There are numerous expositions of Wiener's Tauberian theory in lesser or greater generality. Although the details of the derivations vary, the underlying ideas are usually the same.

For given bounded S, the limit relation (8.2) may be extended from K to other kernels H by convolution and approximation. In his first paper on general Tauberian theorems, Wiener [1928] required not only that K be in L^1, but also that $xK(x)$ be in L^1. Such a hypothesis simplifies the analysis; cf. Bochner [1933b], Levinson [1966], [1973]. One can achieve further simplification by requiring that $(1 + x^2)K(x)$ be in L^1; see Kac [1965]. This condition is satisfied in many applications, for example in the case of the Lambert kernel for $\mathbb{R}$.

In his definitive article, Wiener [1932] made the L^1 Approximation Theorem 8.3 the cornerstone of his general Tauberian theory. It may be remarked that he did not base the proof on the dual approach via the testing equation: Wiener described explicitly how to obtain approximating linear combinations. An important step in his proof was the observation that Fourier transforms of L^1 functions are characterized locally by absolutely convergent Fourier series. Division then appeared explicitly in

his famous 'Lemma' for such series: If f is periodic, continuous and nowhere equal to zero, and has absolutely convergent Fourier series, the same is true for the reciprocal $1/f$. See Theorem V.5.3 in this book.

The Lemma, and contributions by several authors, foremost among them Beurling [1938] and Gel'fand [1939], led to the theory of Banach algebras; see Gel'fand [1941a], Gel'fand, Raikov and Shilov [1964/01], Naimark [1956/72]; cf. also Palmer [1994]. In that context Wiener's Fourier series lemma and L^1 approximation theorem attain a particularly elegant form; see Chapter V. There are further extensions in the setting of harmonic analysis on groups. See for example the books by Loomis [1953], Rickart [1960], R.E. Edwards [1967], Rudin [1962/90], [1973/91], Katznelson [1968/76], Reiter and Stegeman [2000]. Chapter V also contains an extension of Theorem 8.3 to weighted L^1 approximation, which goes back to Beurling [1938].

Of the alternative approaches to Wiener's general theory we also mention those by Pitt [1938a], Carleman [1944], Beurling [1945], [1947] and Pollard [1953]. Van Neerven [1997] gave a proof in the setting of operator theory. Korevaar's short distributional proof [1965] is given in Section 11 below.

Besides Wiener's own book [1933] and books already referred to, we mention expositions of Wiener's theory in books by Widder [1941], Hardy [1949], Hille and Phillips [1957/74], Pitt [1958], Goldberg [1961], Hewitt and Ross [1970], and van de Lune [1986]. As indicated before, most of the treatments involve related ideas, but one proof by Beurling is quite different: it starts with a direct proof of Corollary 7.4. See Beurling [1945] and cf. Garsia [1963].

Ganelius formulated a Wiener-type Tauberian theorem for distributions [1971] (section 1.4); see Johansson [1995] for a more precise statement. Wiener theory for distributions and various classes of generalized functions has been the subject of numerous papers by Stanković and coauthors; see in particular Pilipović and Stanković [1993].

9 Proof of the Division Theorem

The main purpose of this section is to prove Division Theorem 7.3. The present proof is essentially due to Pitt [1938a], but the basic ideas go back to Wiener, who employed absolutely convergent Fourier series.

Under the norm

$$\|F\| = \int_{\mathbb{R}} |F(x)|dx, \tag{9.1}$$

the linear space L^1 becomes a complete normed space or Banach space. The norm satisfies the inequalities

$$\|F + G\| \le \|F\| + \|G\|, \quad \|F * G\| \le \|F\|\,\|G\|; \tag{9.2}$$

cf. formula (7.6). [One may conclude that with convolution as multiplication, L^1 becomes a *Banach algebra*. More on this in Chapter V.] Observe also that

$$\sup_t |\hat{F}(t)| = \sup_t \left| \int_{\mathbb{R}} F(x)e^{-itx}dx \right| \le \|F\|. \tag{9.3}$$

Proposition 9.1. (Division Theorem, special case) *Let F and G be in L^1 with $\|G\| < 1$. Then*

$$\frac{\hat{F}}{1+\hat{G}} = \hat{P} \quad \textit{with} \quad P \in L^1. \tag{9.4}$$

Proof. Consider the series

$$F - F * G + F * G * G - \cdots + (-1)^n F * G^{*n} + \cdots. \tag{9.5}$$

Since by (9.2)

$$\|F * G^{*n}\| \leq \|F\| \|G\|^n,$$

the sum of the norms of the terms in (9.5) is finite. Because L^1 is complete, it follows that the series (9.5) converges in L^1 to a function P. Since the Fourier transform of the sum function P may be obtained by termwise Fourier transformation, one finds that $\hat{P}$ is equal to $\hat{F} - \hat{F}\hat{G} + \hat{F}(\hat{G})^2 - \cdots = \hat{F}/(1+\hat{G})$.

We will occasionally use the symbol $\widehat{L^1}$ for the class of Fourier transforms of L^1 functions. Thus the assertion in Theorem 7.3 is that certain quotients $\hat{H}/\hat{K}$ are in $\widehat{L^1}$. The proof will use some special functions in $\widehat{L^1}$ and their translates. The basic building block is the triangle function Δ_ε with small $\varepsilon > 0$; cf. Example 7.1. It has support $[-\varepsilon, \varepsilon]$ and involves an isosceles triangle with base $[-\varepsilon, \varepsilon]$ and height 1; the Fourier original D_ε has norm 1. We also need the elementary *trapezoidal function*

$$\tau_\varepsilon(t) = \Delta_\varepsilon(t+\varepsilon) + \Delta_\varepsilon(t) + \Delta_\varepsilon(t-\varepsilon). \tag{9.6}$$

It has support $[-2\varepsilon, 2\varepsilon]$ and has the value 1 on $[-\varepsilon, \varepsilon]$; the slanted edges run up from the point $(-2\varepsilon, 0)$ to $(-\varepsilon, 1)$ and down from $(\varepsilon, 1)$ to $(2\varepsilon, 0)$; see Figure II.9. The Fourier inverse T_ε of τ_ε is given by

$$T_\varepsilon(x) = D_\varepsilon(x)(e^{-i\varepsilon x} + 1 + e^{i\varepsilon x}) = \frac{\cos \varepsilon x - \cos 2\varepsilon x}{\pi \varepsilon x^2}; \tag{9.7}$$

the norm $\|T_\varepsilon\|$ is bounded by $3\|D_\varepsilon\| = 3$.

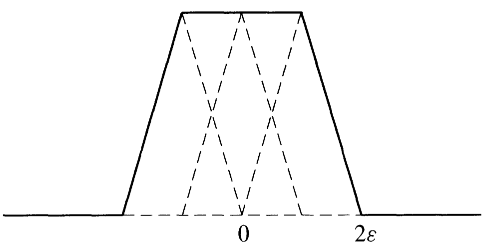

Fig. II.9. The function τ_ε

Proposition 9.2. (Division Theorem, special case) *For* $K \in L^1$ *with* $\hat{K}(c) \neq 0$ *and sufficiently small* $\varepsilon > 0$ (so that inequality (9.14) below is satisfied), *one has*

$$\frac{\Delta_\varepsilon(t-c)}{\hat{K}(t)} = \hat{R}_{\varepsilon,c}(t) \quad \textit{with} \;\; R_{\varepsilon,c} \in L^1. \tag{9.8}$$

If $\hat{K}(t) \neq 0$ *for* $a \le t \le b$ *one can find* $\varepsilon_0 > 0$ *such that there is a formula* (9.8) *for every* $\varepsilon \le \varepsilon_0$ *and every* $c \in [a,b]$.

Proof. For small ε one has $\hat{K}(t) \neq 0$ on the support $[c-\varepsilon, c+\varepsilon]$ of $\Delta_\varepsilon(t-c)$. Since $\tau_\varepsilon(t-c) = 1$ on that support, we may write

$$\frac{\Delta_\varepsilon(t-c)}{\hat{K}(t)} = \frac{\Delta_\varepsilon(t-c)}{\hat{K}(c)} \Bigg/ \left\{1 + \frac{\hat{K}(t) - \hat{K}(c)}{\hat{K}(c)} \tau_\varepsilon(t-c)\right\}. \tag{9.9}$$

Proposition 9.1 guarantees that this quotient is in $\widehat{L^1}$ if

$$\hat{G}(t) \stackrel{\text{def}}{=} \frac{\hat{V}_{\varepsilon,c}(t)}{\hat{K}(c)} \stackrel{\text{def}}{=} \frac{\{\hat{K}(t) - \hat{K}(c)\}\tau_\varepsilon(t-c)}{\hat{K}(c)} \tag{9.10}$$

is the Fourier transform of a function G of norm less than 1. Direct computation shows that $\hat{V}_{\varepsilon,c}(t)$ is the Fourier transform of

$$V_{\varepsilon,c}(x) = \int_{\mathbb{R}} \{T_\varepsilon(x-y) - T_\varepsilon(x)\} K(y) e^{ic(x-y)} dy.$$

Thus

$$\|V_{\varepsilon,c}\| \le \int_{\mathbb{R}} dx \int_{\mathbb{R}} |T_\varepsilon(x-y) - T_\varepsilon(x)||K(y)| dy = \int_{\mathbb{R}} |K(y)| \rho_\varepsilon(y) dy, \tag{9.11}$$

where

$$\begin{aligned} \rho_\varepsilon(y) &= \int_{\mathbb{R}} |T_\varepsilon(x-y) - T_\varepsilon(x)| dx = \int_{\mathbb{R}} |\varepsilon T_1(\varepsilon x - \varepsilon y) - \varepsilon T_1(\varepsilon x)| dx \\ &= \int_{\mathbb{R}} |T_1(z - \varepsilon y) - T_1(z)| dz. \end{aligned}$$

By taking B large one readily verifies that

$$\int_{\mathbb{R}} |T_1(z-\gamma) - T_1(z)| dz = \int_{-B}^{B} + \left(\int_{-\infty}^{-B} + \int_{B}^{\infty}\right) \to 0 \quad \text{as } \gamma \to 0.$$

It follows that $\rho_\varepsilon(y) \to 0$ as $\varepsilon \searrow 0$ for every y; since $\|T_1\| \le 3$, the supremum of $\rho_\varepsilon(y)$ is bounded by 6. Hence by dominated convergence, the final integral

$$I(\varepsilon) \stackrel{\text{def}}{=} \int_{\mathbb{R}} |K(y)| \rho_\varepsilon(y) dy \tag{9.12}$$

in (9.11) tends to 0 as $\varepsilon \searrow 0$. We record that

$$\|V_{\varepsilon,c}\| \le I(\varepsilon) \quad \textit{for every} \;\; c. \tag{9.13}$$

Conclusion. Let $\varepsilon > 0$ be so small that

$$I(\varepsilon) < |\hat{K}(c)|. \tag{9.14}$$

Then the inverse transform G of $\hat{G}$ in (9.10) has norm less than 1, hence by (9.9) and Proposition 9.1, $\Delta_\varepsilon(t-c)/\hat{K}(t)$ is in $\widehat{L^1}$. (A worried reader may verify directly that $\hat{K}(t)$ cannot vanish on $[c-\varepsilon, c+\varepsilon]$, since on that interval

$$|\hat{K}(t) - \hat{K}(c)| = |\hat{V}_{\varepsilon,c}(t)| \le \|V_{\varepsilon,c}\| < |\hat{K}(c)|.)$$

For the second part of the Proposition one takes ε_0 so small that

$$I(\varepsilon) < \min_{a \le t \le b} |\hat{K}(t)| \quad \text{for every} \;\; \varepsilon \le \varepsilon_0.$$

Proof of Wiener's Division Theorem 7.3. Multiplying the original H and K by a suitable exponential $e^{i\alpha x}$ in order to translate $\hat{H}$ and $\hat{K}$, one may assume that the interval (a, b) in the Theorem is symmetric about the origin. Having $\hat{H}(t) = 0$ for $|t| \ge b$ and $\hat{K}(t) \ne 0$ for $|t| \le b$ we have to show that $\hat{H}/\hat{K}$ is in $\widehat{L^1}$. Continuing with the notation introduced above, we use a sum of translates of the triangle function Δ_ε to construct a trapezoidal function $\hat{U}$ which is equal to 1 on $[-b, b]$:

$$\hat{U}(t) = \sum_{n=-N}^{N} \Delta_\varepsilon(t - n\varepsilon), \qquad \varepsilon = b/N. \tag{9.15}$$

Here we take ε so small (N so large) that $\Delta_\varepsilon(t-c)/\hat{K}(t)$ is in $\widehat{L^1}$ for every $c \in [-b, b]$; cf. Proposition 9.2. Then

$$\frac{\hat{U}(t)}{\hat{K}(t)} = \sum_{n=-N}^{N} \frac{\Delta_\varepsilon(t-n\varepsilon)}{\hat{K}(t)} = \hat{R}(t) \quad \text{with} \;\; R \;\; \text{in} \;\; L^1. \tag{9.16}$$

Finally, since $\hat{U} = 1$ on the interval $[-b, b]$ which contains supp $\hat{H}$,

$$\frac{\hat{H}}{\hat{K}} = \hat{H}\frac{\hat{U}}{\hat{K}} = \hat{H}\hat{R} = \hat{Q}, \quad \text{with} \;\; Q = H * R \;\; \text{in} \;\; L^1.$$

□

A slightly more general form of the Division Theorem is

Theorem 9.3. *Let $\hat{H}$ and $\hat{K}$ be Fourier transforms of L^1 functions, let $\hat{H}$ have compact support and let $\hat{K}(t)$ be different from 0 for all t in the support of $\hat{H}$. Then $\hat{H}/\hat{K} = \hat{Q}$ with Q in L^1.*

Proof. The (open) set on which $\hat{K}(t) \neq 0$ can be represented as a countable union of (disjoint) maximal open intervals. By the hypothesis, finitely many of these will cover supp $\hat{H}$, say (α_1, β_1), (α_2, β_2), $\cdots$, with $\beta_1 \leq \alpha_2$, etc. The part of supp $\hat{H}$ in (α_1, β_1) must have positive distance to the part in (α_2, β_2), etc. Tentatively writing $\hat{H}_j$ for the restriction of $\hat{H}$ to (α_j, β_j), one obtains a decomposition $\hat{H} = \hat{H}_1 + \hat{H}_2 + \cdots$ such that supp $\hat{H}_j$ belongs to a compact subinterval $[a_j, b_j]$ of (α_j, β_j). Representing $\hat{H}_j$ as $\hat{H} \cdot \hat{U}_j$ with a trapezoidal function $\hat{U}_j$ which is equal to 1 on $[a_j, b_j]$ and has support in (α_j, β_j), one finds that $\hat{H}_j$ is in $\hat{L}^1$. With the aid of Theorem 7.3, the division of $\hat{H} = \sum \hat{H}_j$ by $\hat{K}$ may now be carried out term by term.

The following related result was also mentioned by Wiener [1932].

Corollary 9.4. *Let H and K be L^1 functions such that $\hat{K}(t) \neq 0$ for all t in the support of $\hat{H}$. Then H belongs to the closed span $\Sigma(K)$ of the translates of K in L^1. Thus for these H and K, relation* (8.2) *for bounded S implies* (8.3).

Indeed, H can be approximated in L^1 by functions H_ω for which $\hat{H}_\omega$ has compact support contained in supp $\hat{H}$. (One may take $H_\omega = H * D_\omega$ with large ω.) By the Theorem, $H_\omega = Q_\omega * K$ with $Q_\omega \in L^1$. Thus for any bounded function Φ such that $K * \Phi = 0$, also $H_\omega * \Phi = 0$. It follows that H_ω is in $\Sigma(K)$, etc.; cf. Section 4.

10 Wiener Families of Kernels

For some applications one needs an extension of Theorem 8.2 to the case of families of kernels.

Definition 10.1. A family of functions $\mathcal{F} = \{K_\nu\}$ on $\mathbb{R}$ will be called a *Wiener family of kernels*, notation $\mathcal{F} = \{K_\nu\} \subset WF$, if each K_ν is in $L^1(\mathbb{R})$ and the system of testing equations

$$\int_{\mathbb{R}} K_\nu(x-y)\Phi(y)dy = 0, \quad \forall x \in \mathbb{R}, \ \forall K_\nu \in \mathcal{F} \tag{10.1}$$

has no bounded solution $\Phi \neq 0$.

One can of course also consider Wiener families for $\mathbb{R}^+$.

Theorem 10.2. *The L^1 family $\mathcal{F} = \{K_\nu\}$ is in WF if and only if the Fourier transforms $\hat{K}_\nu$ have no common (real) zero.*

Classical proof. If $\hat{K}_\nu(\alpha) = 0$ for all $K_\nu \in \mathcal{F}$, then $\Phi(y) = e^{i\alpha y}$ is a solution of the system (10.1). For the other direction, suppose that Φ is any bounded solution of the system (10.1) and that for each t, $\hat{K}_\nu(t) \neq 0$ for some $K_\nu \in \mathcal{F}$. We will show that for every interval $[-b, b]$ there is then an L^1 function U with $\hat{U} = 1$ on $[-b, b]$ such that $U * \Phi = 0$.

Indeed, let the continuous function $\hat{K}_\nu(t)$ be $\neq 0$ for $a_\nu \leq t \leq b_\nu$. Then by Proposition 9.2 one can find $\varepsilon_\nu > 0$ such that for every $\varepsilon \leq \varepsilon_\nu$, the quotient $\Delta_\varepsilon(t-c)/\hat{K}_\nu(t)$

is equal to a function $\hat{R}_{\varepsilon,c} \in \widehat{L^1}$ for all $c \in [a_\nu, b_\nu]$. It follows that $D_{\varepsilon,c} = R_{\varepsilon,c} * K_\nu$, hence $D_{\varepsilon,c} * \Phi = 0$. One can cover $[-b, b]$ by finitely many open intervals, and next by closed intervals, throughout each of which some $\hat{K}_\nu$ is $\neq 0$. Each closed interval has associated functions $\Delta_\varepsilon(t-c)$ with $D_{\varepsilon,c} * \Phi = 0$ for all sufficiently small ε and all c in the interval. One can then obtain a suitable function $\hat{U}$ by adding congruent triangular functions $\Delta_\varepsilon(\cdot - c)$; cf. (9.15). The corresponding function U is a sum of L^1 functions $D_{\varepsilon,c}$ and hence $U * \Phi = 0$.

Having such a function U for arbitrary $b > 0$, one may convolve with $H_c = D_{1,c}$ for $-b+1 \le c \le b-1$ (so that $\hat{H}_c\hat{U} = \hat{H}_c$) to conclude that $H_c * \Phi = 0$ for every c. For the proof that $\Phi = 0$ one may then use the final part of the derivation of Theorem 7.2.

We now verify Wiener's Tauberian theorem [1932] for families of kernels.

Theorem 10.3. *Let $\mathcal{F}$ be a family of L^1 kernels K_ν. Then the condition $\mathcal{F} \subset WF$ is necessary and sufficient in order that the relations*

$$\int_{\mathbb{R}} K_\nu(x-y)S(y)dy \to A \int_{\mathbb{R}} K_\nu(y)dy \quad \text{as } x \to \infty, \quad \forall K_\nu \in \mathcal{F} \tag{10.2}$$

for (otherwise arbitrary) bounded functions $S(\cdot)$ and corresponding constants $A = A_S$ imply

$$\int_{\mathbb{R}} H(x-y)S(y)dy \to A \int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{10.3}$$

for every function H in L^1.

Proof. The necessity is clear. Suppose now that the L^1 family $\mathcal{F} = \{K_\nu\}$ is such that for some bounded S and some $H \in L^1$, one has (10.2) with $A = 0$ but not (10.3). Then consider $T = H * S$. This function is bounded and uniformly continuous, $K_\nu * T(x) = H * K_\nu * S(x) \to 0$ as $x \to \infty$, $\forall \nu$, but for some $\beta > 0$ one has $|T(x_n)| \ge \beta$ for a sequence $x_n \to \infty$. As in Reduction 3.1 one now constructs a bounded uniformly continuous function $\Phi(\cdot) = \lim T(x_{n_j} + \cdot) \not\equiv 0$ for which $K_\nu * \Phi = 0$ for all ν. Conclusion: $\mathcal{F}$ is not a Wiener family.

Remarks 10.4. There is of course also a Wiener approximation theorem and a Wiener–Pitt theorem for Wiener families; cf. Theorem 8.3 or Section V.3 and Theorem 8.4. The results for families also have analogs for $\mathbb{R}^+$.

11 Distributional Approach to Wiener Theory

The proof of the Fourier transform characterization of Wiener kernels, Theorem 7.2, can be simplified if one uses distributional Fourier transforms (Korevaar [1965]). The argument then becomes local and requires no global division theorem. This approach is particularly efficient in the case of Wiener families; see below.

Distributional Fourier Transformation; cf. the books Schwartz [1966], Hörmander [1983–85]. The standard *testing functions* in distribution theory for $\mathbb{R}$

are the C^∞ functions ϕ of compact support, notation $\phi \in C_0^\infty$. For Fourier transformation one enlarges this class to the Schwartz space $\mathcal{S}$: $\phi \in \mathcal{S}$ on $\mathbb{R}$ if $\phi \in C^\infty$ and all derivatives $D^q\phi$ are *rapidly decreasing*. That is, they tend to zero at $\pm\infty$ more rapidly than any negative power of the variable. Convergence in $\mathcal{S}$ is defined with the aid of the family of seminorms

$$M_{pq}(\phi) = \sup_{x\in\mathbb{R}} |x^p D^q \phi(x)|, \quad p, q = 0, 1, 2, \ldots.$$

On $\mathcal{S}$, Fourier transformation is one to one and onto. It is also continuous in the sense of $\mathcal{S}$, as may be derived from the rules

$$(\widehat{D\phi})(t) = it\hat{\phi}(t), \quad (\widehat{x\phi})(t) = iD\hat{\phi}(t).$$

One now introduces the dual space $\mathcal{S}'$ of $\mathcal{S}$ of so-called *tempered distributions*. These are the continuous linear functionals T on $\mathcal{S}$ to $\mathbb{C}$:

$$T(c_1\phi_1 + c_2\phi_2) = c_1T(\phi_1) + c_2T(\phi_2),$$
$$\text{if } \phi_\lambda \to \phi \text{ in } \mathcal{S}, \text{ then } T(\phi_\lambda) \to T(\phi).$$

Every locally integrable function F on $\mathbb{R}$ of at most polynomial growth is represented in $\mathcal{S}'$ through the formula $F(\phi) = \int_{\mathbb{R}} F\phi$. It is convenient to write $T(\phi) = < T, \phi >$ and for the rules, to think of $T(\phi)$ as some kind of integral $\int_{\mathbb{R}} T\phi$. Convergence in $\mathcal{S}'$ is defined by the rule

$$T_\lambda \to T \quad \text{if} \quad < T_\lambda, \phi > \to < T, \phi >, \quad \forall \phi \in \mathcal{S}.$$

One says that $T = 0$ on (a, b) if $< T, \phi > = 0$ for all $\phi \in \mathcal{S}$ with support in (a, b). If $T = 0$ in a neighborhood of every point $c \in \mathbb{R}$, then $T = 0$ (on $\mathbb{R}$). For locally integrable functions F in $\mathcal{S}'$, being equal to 0 on (a, b) in the sense of $\mathcal{S}'$ means that $F(x) = 0$ almost everywhere on (a, b), as in the case of $L^1(a, b)$.

Tempered distributions can be multiplied by C^∞ functions g whose derivatives grow at most polynomially. The product $gT \in \mathcal{S}'$ is defined by

$$< gT, \phi > = < Tg, \phi > = < T, g\phi >, \quad \forall \phi \in \mathcal{S}.$$

For rapidly decreasing integrable functions g there is a convolution $g * T$ in $\mathcal{S}'$, given by

$$< g * T, \phi > = < T * g, \phi > = < T, g_R * \phi >, \quad \text{where } g_R(x) = g(-x).$$

For a tempered distribution T, $T \in \mathcal{S}'$, the *Fourier transform* $\hat{T}$ is the tempered distribution (continuous linear functional on $\mathcal{S}$) defined by

$$< \hat{T}, \phi > = < T, \hat{\phi} >, \quad \forall \phi \in \mathcal{S}.$$

In particular every locally integrable function F of at most polynomial growth has a Fourier transform in $\mathcal{S}'$. If $\hat{T} = U$ then $T = \{1/(2\pi)\}\hat{U}_R$. Fourier transformation on $\mathcal{S}'$ is one to one, onto and continuous. For rapidly decreasing functions g,

$$(g * T)\hat{} = \hat{g}\hat{T}. \tag{11.1}$$

Distributional Treatment of the Testing Equation. We now give another proof of Theorems 7.2 and 10.2. It will be sufficient to prove the more general Theorem 10.2. Accordingly, suppose that for the L^1 family of kernels $\mathcal{F} = \{K_\nu\}$, the Fourier transforms $\hat{K}_\nu$ have no common zero. Let Φ be any bounded solution of the system of testing equations

$$K_\nu * \Phi = 0, \quad \forall K_\nu \in \mathcal{F}.$$

We will prove that $\hat{\Phi} = 0$ (hence $\Phi = 0$) by showing that for any given point $c \in \mathbb{R}$, one has $\hat{\Phi} = 0$ in a neighborhood of c.

To this end we choose a 'trapezoidal' C^∞ function $\tau(t) = \tau_1(t)$ which is equal to 1 for $|t| \le 1$ and equal to 0 for $|t| \ge 2$. [One may, but need not, require that $0 \le \tau(t) \le 1$ for all t.] For any $\varepsilon > 0$ we set $\tau_\varepsilon(t) = \tau(t/\varepsilon)$. Observe that

$$\tau_{2\varepsilon}(t-c) = 1 \quad \text{on the support of} \quad \tau_\varepsilon(t-c).$$

We next choose a function $K = K_\nu$ in $\mathcal{F}$ such that $\hat{K}(c) \neq 0$ and take $\varepsilon > 0$ so small that $\hat{K}(t) \neq 0$ for $c - 2\varepsilon \le t \le c + 2\varepsilon$. Then

$$\frac{\tau_\varepsilon(t-c)}{\hat{K}(t)} = \frac{\tau_\varepsilon(t-c)}{\hat{K}(c)} \frac{1}{1+\hat{G}(t)}, \qquad \hat{G}(t) = \frac{\hat{K}(t) - \hat{K}(c)}{\hat{K}(c)} \tau_{2\varepsilon}(t-c).$$

The same argument that was used for Proposition 9.2 will show that $\|G\| < 1$ when ε is sufficiently small. For such ε, by Proposition 9.1,

$$\frac{\tau_\varepsilon(t-c)}{\hat{K}(t)} = \hat{R}(t) \quad \text{with} \quad R \text{ in } L^1.$$

Let us denote the inverse Fourier transform of $\tau_\varepsilon(t-c)$ by g so that g is in $\mathcal{S}$. Then since $(\hat{R}\hat{K})(t) = \tau_\varepsilon(t-c)$ one has $R * K = g$. Now $K * \Phi = 0$, so that

$$0 = R * (K * \Phi) = (R * K) * \Phi = g * \Phi.$$

Hence by (11.1), $0 = (\hat{g}\hat{\Phi})(t) = \tau_\varepsilon(t-c)\hat{\Phi}(t)$. Finally, since $\tau_\varepsilon(t-c) = 1$ for $c - \varepsilon \le t \le c + \varepsilon$,

$$\hat{\Phi}(t) = 0 \quad \text{throughout the open interval} \quad (c-\varepsilon, c+\varepsilon). \tag{11.2}$$

□

Before we continue with more theory we discuss an important application.

12 General Tauberian for Lambert Summability

We recall the famous Tauberian theorem of Hardy and Littlewood [1921] for Lambert series (Section I.10), which implies the prime number theorem as we saw.

Theorem 12.1. *Let $\sum_0^\infty a_n$ be Lambert summable to A, that is,*

$$\sum_{n=0}^{\infty} a_n \frac{nt}{e^{nt}-1} \quad \textit{converges for } t>0 \textit{ and the sum function tends to } A \tag{12.1}$$

as $t \searrow 0$. Suppose also that

$$|na_n| \le C \quad \textit{or} \quad na_n \ge -C. \tag{12.2}$$

Then $\sum_0^\infty a_n$ converges to A.

Wiener [1928], [1932] gave the first proof of the Theorem independent of prime number theory. His proof used only the nonvanishing of the zeta function on the line $\{\mathrm{Re}\, z = 1\}$. Thus Wiener became the first to show that the prime number theorem is a consequence of that property of $\zeta(z)$.

Wiener also showed that the Tauberian conditions (12.2) may be relaxed to '$\{s_n\}$ slowly decreasing':

$$\liminf\,(s_m - s_n) \ge 0 \quad \text{for } n \to \infty \text{ and } 1 < m/n \to 1.$$

Motivated by the special case $s(v) = \sum_{n \le v} a_n$, we consider an extension of Theorem 12.1 to the case of Stieltjes integrals with slowly decreasing functions $s(\cdot)$:

$$\liminf\{s(\rho v) - s(v)\} \ge 0 \quad \text{as } v \to \infty \text{ and } 1 < \rho \to 1. \tag{12.3}$$

Suppose also that $s(v)$ vanishes for $v < 0$, is of bounded variation on every finite interval, continuous from the right and such that the integral $\int_{0-}^{\infty} ds(\cdot)$ is Lambert summable to A. That is, the improper integral

$$\int_{0-}^{\infty-} \frac{tv}{e^{tv}-1} ds(v) \text{ exists for } t>0 \text{ and has limit } A \text{ as } t \searrow 0.$$

Setting $t = 1/u$ and integrating by parts one arrives at the limit relation

$$f(u) \stackrel{\text{def}}{=} \int_{0-}^{\infty-} k(v) d_v s(uv) = \int_0^\infty k_1(v) s(uv) dv \to A \quad \text{as } u \to \infty, \tag{12.4}$$

where

$$k(v) = \frac{v}{e^v - 1}, \quad k_1(v) = \frac{d}{dv} \frac{-v}{e^v - 1}; \tag{12.5}$$

cf. Section 2 or I.13. Is it true that $s(u) \to A$ as $u \to \infty$?

For the application of standard Wiener theory one needs boundedness of the function $s(\cdot)$. In the case of series and the Tauberian conditions in (12.2), the boundedness may be proved by relatively simple arguments; cf. Section I.5. In the general case of (12.3) one may appeal to Boundedness Theorem I.20.1; we check the conditions. The kernel $k(\cdot)$ in (12.5) qualifies: it is positive, continuous and decreasing with $k(\infty-) = 0$. Furthermore, the function $f(u)$ in (12.4) is well-defined and bounded for $u > u_0$. Hence by Theorem I.20.1 our slowly decreasing s will indeed be bounded; cf. Remark I.20.2.

Theorem 12.2. (General Lambert Tauberian) *Let $s(v)$ vanish for v negative, be of bounded variation on every finite interval, continuous from the right and such that the 'Lambert transform'*

$$f(u) = \int_{0-}^{\infty-} \frac{v}{e^v - 1} d_v s(uv) \tag{12.6}$$

exists for $u > 0$. Suppose that $f(u) \to A$ as $u \to \infty$ and that s is slowly decreasing on $\mathbb{R}^+$ (12.3). Then $s(u) \to A$ as $u \to \infty$.

Proof. By the preceding, the relation $f(u) \to A$ can be written in the form (12.4) with $k_1(\cdot)$ as in (12.5), and $s(\cdot)$ is bounded. For the application of Theorem 8.5 we still have to show that k_1 is a Wiener kernel on $\mathbb{R}^+$. This will be the case if (and only if) the Fourier–Mellin transform of k_1 is different from 0 for all real t. The transform is

$$\check{k}_1(t) = \int_0^\infty k_1(v) v^{it} dv = \int_0^\infty \left(\frac{d}{dv} \frac{-v}{e^v - 1} \right) v^{it} dv. \tag{12.7}$$

It is clear that $\check{k}_1(0) = 1$. For the computation of $\check{k}_1(t)$ we introduce an ordinary Mellin transform which involves a complex variable z:

$$g(z) = \int_0^\infty k_1(v) v^{z-1} dv = \lim_{\beta \searrow 0,\, B \to \infty} \int_\beta^B v^{z-1} \frac{d}{dv} \frac{-v}{e^v - 1} dv. \tag{12.8}$$

Observe that $\check{k}_1(t) = g(1 + it)$.

The function $g(z)$ is continuous (in fact, analytic) throughout the half-plane $\{\operatorname{Re} z > 0\}$. Indeed, the limit in (12.8) exists uniformly in every strip $\{\varepsilon \le \operatorname{Re} z \le E\}$ with $\varepsilon > 0$. In order to evaluate $g(z)$ we take $\operatorname{Re} z > 1$, so that one can integrate by parts:

$$\begin{aligned} g(z) &= \int_0^\infty v^{z-1} d\frac{-v}{e^v - 1} = (z-1) \int_0^\infty \frac{v^{z-1}}{e^v - 1} dv \\ &= (z-1) \sum_{n=1}^\infty \int_0^\infty v^{z-1} e^{-nv} dv = (z-1) \sum_{n=1}^\infty \frac{\Gamma(z)}{n^z} \\ &= (z-1)\Gamma(z)\zeta(z); \end{aligned} \tag{12.9}$$

cf. the definition of the zeta function in Section I.4. Now $\Gamma(z)$ is never equal to 0: the function $1/\Gamma$ is entire (analytic everywhere); cf. Whittaker and Watson [1927/96] (section 12.1). Thus it follows from (12.9) and (12.8) that $G(z) = (z-1)\zeta(z)$ has a continuous (in fact, analytic) extension $g(z)/\Gamma(z)$ to the half-plane $\{\operatorname{Re} z > 0\}$. [Since $g(1) = \Gamma(1) = 1$, so that $G(1) = 1$, one obtains confirmation that the quotient $\zeta(z) = G(z)/(z-1)$ has a first order pole at $z = 1$ with residue 1; cf. Section I.26.]

We now know that

$$\check{k}_1(t) = g(1 + it) = it\Gamma(1 + it)\zeta(1 + it), \quad t \ne 0; \quad \check{k}_1(0) = 1. \tag{12.10}$$

To show that $\check{k}_1(t) \neq 0$ for all (real) t we use the following fact about the zeta function which was verified in Theorem I.26.2:

$$\zeta(z) \neq 0 \quad \text{for } \operatorname{Re} z = 1.$$

Theorem 7.2 thus shows that k_1 is a Wiener kernel, $k \in W^+$. We can then apply the Wiener–Pitt Theorem 8.5 to relation (12.4). Since the function s is bounded and slowly decreasing, one concludes that $s(u) \to A$ as $u \to \infty$.

Remark 12.3. One can use a similar (but easier) argument to obtain another proof for Schmidt's form of the Hardy–Littlewood Theorem I.7.2 involving power series. The relevant kernels are $k(v) = e^{-v}$ and $k_1(v) = e^{-v}$; cf. Section 2. The Fourier–Mellin transform $\hat{k}_1(t)$ is equal to $\Gamma(1+it)$; further details may be left to the reader.

13 Wiener's 'Second Tauberian Theorem'

In situations where one aims for convergence $S(x) \to A$ or $s(u) \to A$, the Wiener–Pitt Theorems 8.4 and 8.5 are usually the most convenient. Prior to Pitt's work, Wiener [1932] had formulated several theorems to facilitate convergence proofs for series. An advantage of his approach is that (at least) under the simpler Tauberian conditions, it does not require a separate proof that S or s is bounded; see Section 14 below.

In the following S stands for a function on $\mathbb{R}$ which is locally of bounded variation. For such S we consider the 'Wiener Tauberian conditions'

$$\int_x^{x+1} |dS(y)| \leq B, \quad -\infty < x < \infty, \tag{13.1}$$

$$\int_x^{x+1} \{|dS(y)| - dS(y)\} \leq B, \quad -\infty < x < \infty, \tag{13.2}$$

where $B < \infty$. In the special case of series $\sum_0^\infty a_n$ and $S(x) = \sum_{n \leq e^x} a_n$, the Tauberian condition $|na_n| \leq C$ implies a bound of the form (13.1):

$$\int_x^{x+1} |dS(y)| = \sum_{e^x < n \leq e^{x+1}} |a_n| \leq C \sum_{e^x < n \leq e^{x+1}} 1/n \to C \quad \text{as } x \to \infty.$$

Similarly, the Tauberian condition $na_n \geq -C$ implies an inequality (13.2):

$$\int_x^{x+1} \{|dS(y)| - dS(y)\} = \sum_{e^x < n \leq e^{x+1}} (|a_n| - a_n) \leq 2C \sum_{e^x < n \leq e^{x+1}} 1/n.$$

Definition 13.1. (Wiener) Let $M = M^1$ denote the subspace of those continuous functions $H \in L^1$ for which

$$\|H\|_M = \sum_{n=-\infty}^{\infty} \sup_{n \leq z \leq n+1} |H(z)| < \infty. \tag{13.3}$$

For $H \in M$ and S as in (13.1), the integral

$$H * dS(x) = \int_{\mathbb{R}} H(x-y)dS(y) = \sum_{n=-\infty}^{\infty} \int_{x-n-1}^{x-n} H(x-y)dS(y)$$

will exist and be bounded by $\|H\|_M B$.

Theorem 13.2. (Wiener's second Tauberian) *Let K be in M. Then the condition that K is a Wiener kernel, $K \in W$, is necessary and sufficient in order that the relation*

$$S_K(x) = \int_{\mathbb{R}} K(x-y)dS(y) \to A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty, \tag{13.4}$$

for functions $S(\cdot)$ satisfying the Tauberian condition (13.1) *and corresponding constants $A = A_S$, imply*

$$S_H(x) = \int_{\mathbb{R}} H(x-y)dS(y) \to A \int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{13.5}$$

for every function H in M.

Incidentally, for $K \in M$ the nonvanishing of $\hat{K}$ also implies that the translates of K span the normed space M; cf. Wiener [1932] (theorem 7), Hewitt and Ross [1970], Reiter and Stegeman [2000].

Proof of the Theorem. The necessity of the condition '$\hat{K}$ zero-free' follows by considering the functions $S(x) = x$ and $S(x) = e^{i\alpha x}$ with real α.

To prove the sufficiency, we start with any function S that satisfies condition (13.1) and form the function $S_H = H * dS$ with $H \in M$. Then S_H is bounded and will be slowly oscillating (Definition 2.3). Indeed, S_H will be uniformly continuous on $\mathbb{R}$: for every x

$$|S_H(x+\lambda) - S_H(x)| = \left| \int_{\mathbb{R}} \{H(z+\lambda) - H(z)\} dS(x-z) \right|$$
$$\leq B \sum_{n=-\infty}^{\infty} \sup_{n \leq z \leq n+1} |H(z+\lambda) - H(z)|.$$

Here the value of the sum is independent of x and will tend to 0 as $\lambda \to 0$. To verify this one may estimate as follows, taking N large:

$$\sum_{n=-\infty}^{\infty} \sup_{n \leq z \leq n+1} |H(z+\lambda) - H(z)| = \sum_{|n| \leq N} \cdots + \sum_{|n|>N} \cdots$$
$$\leq \sum_{|n| \leq N} \sup_{n \leq z \leq n+1} |H(z+\lambda) - H(z)|$$
$$+ \sum_{|n|>N} \sup_{n \leq z \leq n+1} \{|H(z+\lambda)| + |H(z)|\}.$$

One finally uses the uniform continuity of H on every finite interval and the smallness of the tail in the series for $\|H\|_M$.

Suppose now that $K \in M$ is in W and that S satisfies (13.1) and (13.4). Subtracting Ady from $dS(y)$ if necessary, we may assume that $A = 0$, so that the convolution $S_K(x) = K * dS(x)$ tends to zero. Then by dominated convergence

$$K * S_H(x) = K * (H * dS)(x) = H * (K * dS)(x) = H * S_K(x) \to 0$$

as $x \to \infty$. It now follows from the Wiener–Pitt Theorem 8.4 that $S_H(x) \to 0$.

□

It can be shown that Theorems 13.2 and 8.2 are actually equivalent – each can be derived from the other; cf. R.E. Edwards [1958].

The next result will be formulated for Wiener families of kernels (Definition 10.1).

Theorem 13.3. (Second Tauberian, family form) *Let the functions K_ν be in M and form a Wiener family $\mathcal{F}$. Let S be of bounded variation on every finite interval, satisfy condition* (13.2) *and be such that for some function $H_1 \geq 0$ (but $\not\equiv 0$) in M, the integral $\int_{\mathbb{R}} H_1(x-y)dS(y)$ exists and is bounded from above for x running over $\mathbb{R}$. Then the limit relations*

$$\int_{\mathbb{R}} K_\nu(x-y)dS(y) \to A \int_{\mathbb{R}} K_\nu(y)dy \quad \text{as } x \to \infty, \quad \forall K_\nu \in \mathcal{F} \tag{13.6}$$

imply

$$\int_{\mathbb{R}} H(x-y)dS(y) \to A \int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{13.7}$$

for every function H in M.

Proof. By the hypotheses there is a constant C such that

$$\begin{aligned}\int_{\mathbb{R}} H_1(x-y)|dS(y)| &= \int_{\mathbb{R}} H_1(x-y)\{|dS(y)| - dS(y)\} \\ &\quad + \int_{\mathbb{R}} H_1(x-y)dS(y) \leq \|H_1\|_M B + C, \quad \forall x.\end{aligned}$$

On the other hand there will be an interval $[a, b]$ on which $H_1(z) \geq \beta > 0$. Hence

$$\int_{\mathbb{R}} H_1(x-y)|dS(y)| \geq \beta \int_{x-b}^{x-a} |dS(y)|, \quad \forall x.$$

Combination of these inequalities gives a uniform bound for $\int_{x-b}^{x-a} |dS(y)|$ and this implies a uniform bound for $\int_z^{z+1} |dS(y)|$.

One may now apply a family analog of Theorem 13.2; cf. Theorem 10.3.

Remarks 13.4. In the case of a *nondecreasing* function S, the continuity requirement on H can be relaxed. Assuming that such an S satisfies condition (13.4) or (13.6) and (13.1), conclusion (13.5) will hold for all functions H which satisfy condition (13.3)

and are locally Riemann integrable (or equivalently, continuous almost everywhere); cf. Beneš [1961]. Thinking of S as normalized so that it is continuous from the right, one would interpret dS as a positive measure and $H * dS(x)$ as a Lebesgue–Stieltjes integral. A function H as described can be approximated from above and below by piecewise constant functions of compact support, whose integrals are close to that of H. Thus it is sufficient to prove (13.5) for characteristic functions of finite intervals. The latter can be approximated from above and below by trapezoidal functions, which are in M.

Applications of Wiener's (extended) second theorem will be given in Sections IV.18 and VI.13. Bingham [1989] has used the extension to give a simple proof for Blackwell's [1953] renewal theorem.

14 A Wiener Theorem for Series

The following results are contained in Wiener [1932] (theorems 12, 13); cf. also the exposition in Bochner [1933b].

Starting with a continuous nondecreasing function K on $\mathbb{R}$ such that $K(-\infty+) = 0$ and $K(\infty-) = 1$, we consider the family of nonnegative functions

$$K_\nu(x) = K(x+\nu) - K(x), \quad 0 < \nu < \infty. \tag{14.1}$$

Notice that the functions K_ν are in M (Definition 13.1). We will assume that for some constant $a > 0$,

$$K(x) = \mathcal{O}(e^{-a|x|}) \text{ as } x \to -\infty, \quad K(x) = 1 - \mathcal{O}(x^{-1-a}) \text{ as } x \to \infty. \tag{14.2}$$

Then the two-sided Laplace transform

$$\begin{aligned} \mathcal{L}K(z) &= \int_{\mathbb{R}} K(x)e^{-zx}dx \qquad (14.3) \\ &= \int_{-\infty}^{0} K(x)e^{-zx}dx + \int_0^\infty \{K(x)-1\}e^{-zx}dx + \frac{1}{z} \end{aligned}$$

exists and is analytic in the strip $\{0 < \operatorname{Re} z < a\}$. In terms of this transform,

$$\mathcal{L}K_\nu(z) = \int_{\mathbb{R}} \{K(x+\nu) - K(x)\}e^{-zx}dx = (e^{\nu z} - 1)\mathcal{L}K(z).$$

By (14.2), $\mathcal{L}K(z) - 1/z$ has a continuous extension $\mathcal{L}K(it) - 1/(it)$ onto the imaginary axis $\{z = it\}$. In terms of it, $\mathcal{L}K_\nu(z)$ has the continuous extension given by

$$\begin{aligned} \mathcal{L}K_\nu(it) &= \hat{K}_\nu(t) = (e^{i\nu t} - 1)\mathcal{L}K(it), \quad t \neq 0; \\ \mathcal{L}K_\nu(0) &= \hat{K}_\nu(0) = \nu. \end{aligned} \tag{14.4}$$

We suppose that $\mathcal{L}K(it) \neq 0$ for all $t \neq 0$. Then the functions K_ν of (14.1) form a *Wiener family:* the Fourier transforms $\hat{K}_\nu$ have no common zero.

We now wish to apply Theorem 13.3. It is convenient to require that S be of bounded variation on every interval $(-\infty, C]$ and that $S(-\infty+) = 0$. We impose the Tauberian condition (13.2) and require that the integral $\int_{\mathbb{R}} K(x-y)dS(y)$ exist and define a bounded function on $\mathbb{R}$; as a result, the functions $K_\nu * dS$ will also be bounded. We finally suppose that

$$\int_{\mathbb{R}} K(x-y)dS(y) \to A \quad \text{as } x \to \infty,$$

so that

$$\int_{\mathbb{R}} K_\nu(x-y)dS(y) \to 0 \quad \text{as } x \to \infty,\ \forall\, \nu > 0.$$

Then by Theorem 13.3

$$\int_{\mathbb{R}} H(x-y)dS(y) \to 0 \quad \text{as } x \to \infty,\ \forall\, H \in M.$$

For given $\varepsilon > 0$ one may in particular take

$$H(x) = \begin{cases} K(x) & \text{for } -\infty < x < -\varepsilon, \\ K(x) - 1 - x/\varepsilon & \text{for } -\varepsilon \le x < 0, \\ K(x) - 1 & \text{for } 0 \le x < \infty; \end{cases}$$

it follows from (14.2) that the continuous function H is in M. Conclusion:

$$\begin{aligned} \int_{\mathbb{R}} \{K(x-y) - H(x-y)\}dS(y) &= \int_{-\infty}^{x} dS(y) + \int_{x}^{x+\varepsilon} \left(1 + \frac{x-y}{\varepsilon}\right)dS(y) \\ &= \frac{1}{\varepsilon}\int_{x}^{x+\varepsilon} S(y)dy \to A \quad \text{as } x \to \infty. \end{aligned} \tag{14.5}$$

We will formulate the result as a theorem, but first change variables and notation:

$$\begin{aligned} &x = \log u, \quad y = \log v, \quad S(y) = s(v), \\ &K(x-y) = K\left(\log\frac{u}{v}\right) = k\left(\frac{v}{u}\right), \quad K(-y) = k(v), \\ &\mathcal{L}K(z) = \int_{\mathbb{R}} K(-y)e^{zy}dy = \int_0^\infty k(v)v^{z-1}dv = k^*(z). \end{aligned} \tag{14.6}$$

[Observe that the change of variables differs slightly from the one in Section 2. The function k^* corresponds to the Mellin transform of k encountered in formula (12.8). Later we will prefer to call $k^*(-z)$ the Mellin transform of k; cf. Sections III.3, IV.9.]

Theorem 14.1. (Wiener's second Tauberian, special form) *Let $k(\cdot)$ on $[0, \infty)$ be continuous, nonincreasing and such that*

$$k(v) = 1 - \mathcal{O}(|\log v|^{-1-a}) \ \text{as}\ v \searrow 0, \quad k(v) = \mathcal{O}(v^{-a}) \ \text{as}\ v \to \infty \tag{14.7}$$

for some constant $a > 0$. Then the analytic function

$$k^*(z) = k^*(x+it) = \int_0^\infty k(v)v^{z-1}dv, \quad 0 < x < a \tag{14.8}$$

has finite boundary values $k^*(it)$ for all $t \neq 0$: we suppose that they are $\neq 0$. Let $s(v)$ vanish for $v < 0$, be of bounded variation on every finite interval, continuous from the right and such that

$$\int_u^{eu} \{|ds(v)| - ds(v)\} \leq B < \infty, \quad 1 \leq u < \infty. \tag{14.9}$$

Finally, making $k(v)$ continuous on some interval $(-\delta, 0]$, suppose that for this function $s(\cdot)$, the general-kernel transform $\int_{0-}^\infty k(v/u)ds(v)$ exists as a bounded function for $0 < u < \infty$ and that

$$\int_{0-}^\infty k(v/u)ds(v) \to A \quad as \ u \to \infty. \tag{14.10}$$

Then for every number $\rho > 1$,

$$\frac{1}{\log\rho}\int_u^{\rho u} s(v)\frac{dv}{v} \to A \quad as \ u \to \infty. \tag{14.11}$$

Proof. For the case $s(0) = 0$ the proof follows from the preceding discussion; the case $s(0) \neq 0$ may be reduced to this special case. □

One may apply Theorem 14.1 to the case of series $\sum_{n=0}^\infty a_n k(n/u)$, setting $s(v) = \sum_{n \leq v} a_n$.

Theorem 14.2. *Let k satisfy the conditions in Theorem 14.1, including the nonvanishing of the boundary values $k^*(it)$ for $t \neq 0$. Let $\sum_{n=0}^\infty a_n k(n/u)$ converge for $0 < u < \infty$ to a bounded sum-function and let*

$$\sum_{n=0}^\infty a_n k(n/u) \to A \quad as \ u \to \infty. \tag{14.12}$$

Suppose that

$$na_n \geq -C, \quad \forall n. \tag{14.13}$$

Then

$$\sum_0^\infty a_n = A. \tag{14.14}$$

Proof. We verify condition (14.9). By (14.13), negative numbers a_n are in absolute value bounded by C/n, hence

$$\int_u^{eu} \{|ds(v)| - ds(v)\} = \sum_{u<n\leq eu} (|a_n| - a_n) \leq \sum_{u<n\leq eu} 2C/n \leq 4C \quad \text{for } u \geq 1.$$

It is clear that the other conditions of Theorem 14.1 are satisfied. By another application of (14.13)

$$s(v) \geq s(u) - C\frac{v-u+1}{u} \quad \text{when} \quad v > u. \tag{14.15}$$

Hence by (14.11) for $u \to \infty$,

$$A - \limsup s(u) = \liminf \left\{ \frac{1}{\log \rho} \int_u^{\rho u} s(v)\frac{dv}{v} - s(u) \right\} \geq -C(\rho - 1).$$

Since this holds for every $\rho > 1$ it follows that $\limsup s(u) \leq A$. One similarly shows that $\liminf s(u) \geq A$. □

Applications 14.3. Wiener [1928], and more explicitly, Bochner [1933b], obtained the Lambert Tauberian Theorem I.10.1 from Theorem 14.2 by taking

$$k(v) = \frac{v}{e^v - 1}.$$

In this case

$$k^*(z) = \int_0^\infty \frac{v^z}{e^v - 1} dv = \Gamma(z+1)\zeta(z+1), \quad \operatorname{Re} z > 0,$$

so that $k^*(it) = \Gamma(1+it)\zeta(1+it) \neq 0$ for all real t; cf. Section 12. Assuming the convergence of the Lambert series below for $u > 0$, the conditions

$$a_0 + \sum_{n=1}^\infty a_n \frac{n/u}{e^{n/u} - 1} \to A \quad \text{as} \quad u \to \infty$$

and $na_n \geq -C$ imply $\sum_0^\infty a_n = A$.

This derivation of Theorem I.10.1 does not require a separate proof for the boundedness of the partial sums $s_n = \sum_{k \leq n} a_k$, as was necessary in Section 12.

If one takes $k(v) = e^{-v}$ one obtains another proof for Theorem I.7.2 of Hardy and Littlewood. The choice $k(v) = 1/(1+v)$ gives a Tauberian theorem for 'Stieltjes series' $\sum_{n=0}^\infty a_n/(1 + n/u)$; cf. Section I.21.

15 Extensions

We next use Wiener's Theorem 14.1 to extend Theorem 14.2 to arbitrary series of the form

$$\sum_{n=0}^\infty a_n k(\lambda_n/u), \quad \text{where} \quad 0 = \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_n \to \infty. \tag{15.1}$$

The series reduces to a general Dirichlet series when $k(v) = e^{-v}$. As in Section I.22 we set $s(v) = \sum_{\lambda_n \leq v} a_n$.

Theorem 15.1. *Let the kernel k satisfy the same conditions as in Theorems 14.2 and 14.1. Let the series* (15.1) *converge to a bounded function $f(u)$ on $(0,\infty)$ and let $f(u)$ tend to A as $u\to\infty$. Suppose that the coefficients a_n satisfy at least one of the following two conditions* (15.2), (15.3)*:*

$$|a_n| \le C\,\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}, \quad n\ge 1; \tag{15.2}$$

$$a_n \ge -C\,\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}, \quad n\ge 1 \quad \text{AND} \quad \liminf_{n\to\infty} a_n \ge 0. \tag{15.3}$$

Then $s(u)\to A$ as $u\to\infty$.

Proof. In the proof of Theorem I.22.2 for Dirichlet series we restricted ourselves to condition (15.3); we now suppose that condition (15.2) is satisfied. Then

$$\int_u^{eu} |ds(v)| = \sum_{u<\lambda_n\le eu} |a_n| \le \frac{C}{u} \sum_{u<\lambda_n\le eu} (\lambda_n-\lambda_{n-1}) \le Ce.$$

This implies condition (14.9); the other conditions of Theorem 14.1 are clearly fulfilled. We may thus use conclusion (14.11).

In order to derive the convergence of $s(u)$ to A we distinguish between values of u in larger and smaller gaps of the sequence $\{\lambda_n\}$. Taking $u\ge\lambda_1$ we determine m such that $\lambda_m\le u<\lambda_{m+1}$ and choose $\delta>0$. If λ_{m+1} is larger than $(1+\delta)\lambda_m$, it follows from (14.11) with $\rho=1+\delta$ and interval of integration $\{\lambda_m\le v\le\rho\lambda_m\}$ [so that $s(v)=s(u)$] that $s(u)$ is close to A when u is large.

In the case where $\lambda_{m+1}\le(1+\delta)\lambda_m$ we will have a suitable analog to (14.15). Indeed, by (15.2)

$$\begin{aligned} |s(v)-s(u)| &\le \sum_{u<\lambda_n\le v} |a_n| \le C \sum_{u<\lambda_n\le v} \frac{\lambda_n-\lambda_{n-1}}{\lambda_n} \\ &\le C\frac{v-\lambda_m}{u} \le C\frac{v-u+\lambda_{m+1}-\lambda_m}{u} \le C\frac{v-u}{u} + C\delta. \end{aligned}$$

A short calculation now gives

$$\left|\frac{1}{\log\rho}\int_u^{\rho u} s(v)\frac{dv}{v} - s(u)\right| \le C(\rho-1)+C\delta. \tag{15.4}$$

Since δ could be any positive number and ρ may be taken close to 1, we thus conclude from (14.11) that $s(u)\to A$ also when u goes to ∞ through the 'smaller' gaps $[\lambda_m,\lambda_{m+1})$. □

For the case of Hadamard gaps, that is, $\lambda_{n+1}/\lambda_n\ge\rho$ for a constant $\rho>1$ and all n, (15.2) implies that boundedness of the sequence $\{a_n\}$ is a Tauberian condition. The high-indices theorem (Section I.23) shows that in this case, there is a class of kernels for which no order condition is required on the coefficients a_n. For general high-indices theorems involving Wiener kernels, which require considerable technique, we refer

to Levinson [1938], [1940], [1964a]; cf. Redheffer's commentary in Levinson [1997]. See also the references in Remarks I.23.5.

One can formulate another companion to Theorem 14.2, which extends the Hardy–Littlewood Theorem I.7.4; cf. Wiener [1932], Bochner [1933b]:

Theorem 15.2. *Let $k(v)$ be nonnegative on $(0, \infty)$, continuous and such that for some constant $a > 0$,*

$$k(v) = \mathcal{O}(|\log v|^{-1-a}) \quad \text{both for} \quad v \searrow 0 \quad \text{and for} \quad v \to \infty.$$

Also suppose that

$$k^*(z) = \int_0^\infty k(v)v^{z-1}dv \neq 0 \quad \text{for} \quad z = it.$$

Let α be positive. Suppose that the series $\sum_{n=1}^\infty n^{-\alpha}a_n k(n/u)$ converges to a bounded sum-function on the half-line $\{0 < u < \infty\}$. Then the limit relation

$$\sum_{n=1}^\infty n^{-\alpha}a_n k(n/u) \to A \int_0^\infty k(v)(dv)/v \quad \text{as} \quad u \to \infty$$

and the condition

$$na_n \geq -Cn^\alpha$$

imply that

$$s_n = \sum_{j=1}^n a_j \sim (A/\alpha)n^\alpha.$$

The Hardy–Littlewood theorem follows by taking $k(v) = v^\alpha e^{-v}$, for which $k^*(it) = \Gamma(\alpha + it)$. The choice $k(v) = v^{\alpha+1}/(e^v - 1)$ gives a corresponding result for Lambert series. Like the former, the latter result can be obtained without Wiener theory or Fourier analysis; cf. Korevaar [1946].

16 Discussion of the Tauberian Conditions

In the principal Wiener Theorem 8.2, the boundedness of S (outside a set of measure zero) is a natural condition: without boundedness, the convolution $H * S$ in (8.3) would not exist for all functions H in L^1.

However, for special K and H there may also be unbounded functions S which satisfy (8.2) and for which $H * S$ exists. Even so, some kind of boundedness of S may be necessary for conclusion (8.3); cf. Proposition I.24.2 for the case of Abel and Cesàro limitability. Of course, for special K the boundedness of S may be a consequence of other Tauberian conditions; cf. Section V.10.

We will see that in the Wiener–Pitt Theorem 8.4, it is essential to have some condition on S such as slow decrease or increase. If S is slowly decreasing, then for every number $\varepsilon > 0$ there are positive numbers B and δ such that

$$S(y) - S(x) \geq -\varepsilon \quad \text{whenever} \quad x \geq B \quad \text{and} \quad 0 < y - x \leq \delta. \tag{16.1}$$

For differentiable S the condition is satisfied if $\liminf_{x\to\infty} S'(x) \geq -C$, but this cannot be relaxed to $S'(x) \geq -\phi(x)$ with $\phi(x) \to \infty$ as $x \to \infty$.

Proposition 16.1. *Always requiring that S be bounded, the condition in Theorem 8.4 that S be slowly decreasing cannot be relaxed to a condition of the form*

$$S(y) - S(x) \geq -(y-x)\phi(x) \quad \textit{for} \quad y > x, \quad \textit{with} \quad 1 \leq \phi(x) \to \infty, \tag{16.2}$$

for ANY *Wiener kernel K on $\mathbb{R}$.*

Proof. For given $K \in W$ and $1 \leq \phi \to \infty$ we will construct a bounded function S which satisfies the limit relation $K * S(x) \to 0$ and condition (16.2), but which fails to tend to 0 at ∞. Observe that ϕ may be assumed nondecreasing: before starting the construction, one may replace $\phi(x)$ by its minorant $\inf_{x\geq t} \phi(t)$. Our function S will be obtained as an infinite sum of simple 'zigzags' $Z = Z_{\lambda,\delta}$ with *nonoverlapping* supports $[\lambda, \lambda+4\delta]$. On its support, each zigzag function Z consists of two contiguous congruent isosceles triangles, one above the axis and one below (see Figure II.16):

$$Z_{\lambda,\delta}(x) = \begin{cases} (x-\lambda)/\delta & \text{for } \lambda \leq x \leq \lambda+\delta, \\ -(x-\lambda-2\delta)/\delta & \text{for } \lambda+\delta \leq x \leq \lambda+3\delta, \\ (x-\lambda-4\delta)/\delta & \text{for } \lambda+3\delta \leq x \leq \lambda+4\delta, \\ 0 & \text{for } x \leq \lambda \text{ and } x \geq \lambda+4\delta. \end{cases} \tag{16.3}$$

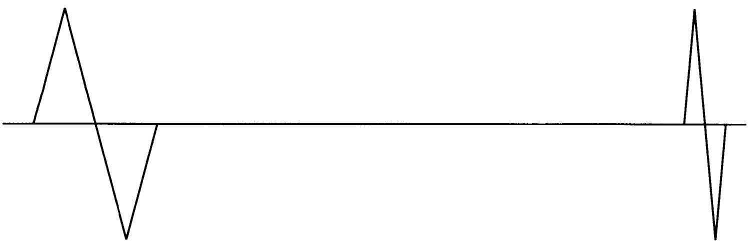

Fig. II.16. Two zigzags $Z_{\lambda,\delta}$

Any infinite sum $S = \sum Z_{\lambda,\delta}$ of nonoverlapping zigzags will satisfy the conditions

$$-1 \leq S \leq 1, \quad \limsup S = 1 \text{ at } \infty \quad \text{and} \quad \liminf S = -1, \tag{16.4}$$

so that S has no limit at ∞. Also

$$|S'(x)| = |Z'_{\lambda,\delta}| \leq 1/\delta \quad \text{almost everywhere for } \lambda \leq x \leq \lambda+4\delta. \tag{16.5}$$

Since we want to satisfy condition (16.2) we take $1/\delta = \phi(\lambda)$. Finally, the terms in the series for S will be chosen such that

$$K * S(x) = \int_{\mathbb{R}} K(x-y)S(y)dy \to 0 \quad \text{as } x \to \infty. \tag{16.6}$$

For the construction of S we use the basic relation

$$\rho(\delta) = \int_{\mathbb{R}} |K(z+\delta) - K(z)|\, dz \to 0 \quad \text{as } \delta \searrow 0, \tag{16.7}$$

which may be derived from the fact that K can be approximated in L^1 by (uniformly) continuous functions with compact support. Observe that for $Z = Z_{\lambda,\delta}$ one has $Z(y+2\delta) = -Z(y)$ when $\lambda \le y \le \lambda + 2\delta$, hence

$$\begin{aligned} K * Z(x) &= \int_{\lambda}^{\lambda+4\delta} K(x-y)Z(y)dy = \left(\int_{\lambda}^{\lambda+2\delta} + \int_{\lambda+2\delta}^{\lambda+4\delta}\right)\cdots \\ &= \int_{\lambda}^{\lambda+2\delta} K(x-y)Z(y)dy - \int_{\lambda}^{\lambda+2\delta} K(x-y-2\delta)Z(y)dy. \end{aligned}$$

It follows that

$$|K * Z(x)| \le \int_{\mathbb{R}} |K(x-y) - K(x-y-2\delta)|dy = \rho(2\delta). \tag{16.8}$$

We also need the simple inequality

$$|K * Z(x)| \le \int_{x-\lambda-4\delta}^{x-\lambda} |K(z)|dz. \tag{16.9}$$

Now let S be any function of the form

$$S(x) = \sum_{n=1}^{\infty} Z_{\lambda_n,\delta_n}(x), \quad \text{where}$$

$$\delta_n = 1/\phi(\lambda_n) \searrow 0, \quad \lambda_{n+1} \ge \lambda_n + 4\delta_n, \quad \sum_{n=1}^{\infty} \rho(2\delta_n) < \infty. \tag{16.10}$$

By (16.5) S will satisfy condition (16.2). We finally verify the limit relation (16.6). For given $\varepsilon > 0$, take B and N so large that

$$\int_B^{\infty} |K(z)|dz < \varepsilon \quad \text{and} \quad \sum_{n>N} \rho(2\delta_n) < \varepsilon.$$

Then by (16.9) and (16.8)

$$\begin{aligned} |K * S(x)| &\le \sum_{n=1}^{N} |(K * Z_{\lambda_n,\delta_n}(x)| + \sum_{n>N} \cdots \\ &\le \int_{x-\lambda_N-4\delta_N}^{\infty} |K(z)|dz + \sum_{n>N} \rho(2\delta_n) < 2\varepsilon \end{aligned} \tag{16.11}$$

whenever $x \ge B + \lambda_N + 4\delta_N$. □

We comment briefly on the results in Sections 13, 14. In Theorem 13.2 the Tauberian condition (13.1) is necessary to ensure the existence of $H * dS$ for all functions H in M.

In Theorem 14.2, the Tauberian condition $a_n \geq -C/n$ is optimal as to order for all 'reasonable' kernels $k(\cdot)$:

Proposition 16.2. *Let $k(\cdot)$ be any nonincreasing kernel on $\mathbb{R}^+$ with $k(0) = 1$ and $k(\infty-) = 0$ as in Theorem 14.2, but with the additional property that for some finite constant M,*

$$k(v) - k(w) \leq M \frac{w - v}{v} \quad \text{for } 0 < v < w < \infty. \tag{16.12}$$

Then for every function $0 < \phi \nearrow \infty$, there is a DIVERGENT *series $\sum_0^\infty a_n$ such that $\sum_{n=0}^\infty a_n k(nt)$ converges to a bounded sum function $F(t)$ on $(0, \infty)$ which tends to 0 as $t \searrow 0$, while*

$$|a_n| \leq \phi(n)/n, \quad \forall n. \tag{16.13}$$

Proof. One may use essentially the same construction as in Section I.24 for the case of power series, where $k(v) = e^{-v}$. Again adjusting ϕ to ensure that $1 \geq \phi(n)/n \to 0$, one starts with the basic building block

$$\begin{aligned} F_{p,q}(t) = \frac{\phi(p)}{p + 2q} &[k\{(p + 1)t\} + \cdots + k\{(p + q)t\} \\ &-k\{(p + q + 1)t\} - \cdots - k\{(p + 2q)t\}]. \end{aligned} \tag{16.14}$$

For $q = [p/\phi(p)]$ it follows from (16.12) that

$$0 \leq F_{p,q}(t) \leq \frac{\phi(p)}{p} \sum_{m=p+1}^{p+q} [k(mt) - k\{(m + q)t\}] \leq M\phi(p)\frac{q^2}{p^2} \leq \frac{M}{\phi(p)}. \tag{16.15}$$

One proceeds to define F with the aid of nonoverlapping blocks $F_{p,q}$ for which $\sum_p 1/\phi(p) < \infty$. The coefficients a_n of the functions $k(nt)$ will satisfy the inequality in (16.13), the partial sums s_n will have lim sup $= 1$ and lim inf $= 0$, and $F(t) \to 0$ as $t \searrow 0$ by dominated convergence. □

Remarks 16.3. Proposition 16.1 for $\mathbb{R}$ is similar to a result for $\mathbb{R}^+$ which Lorentz [1951] (p. 253) obtained as a consequence of his general theory of Tauberian conditions.

The method used for Proposition 16.2 can also be used to show that the Tauberian conditions in Theorem 15.1 are optimal as to order whenever $\lambda_{n+1}/\lambda_n \to 1$ as $n \to \infty$. For the special case of Dirichlet series this optimality may be derived from a result in Littlewood [1911].

17 Landau–Ingham Asymptotics

With the aid of Wiener theory, Ingham [1945] obtained two Tauberian theorems which are important in number theory. In fact, Theorem 17.3 (and a more special early result

of Landau) quickly lead to the prime number theorem. For biographical information on Ingham, see Burkill [1969].

Definition 17.1. We will say that a function S on $[1, \infty)$ has *Landau–Ingham asymptotics* of type (A, B) if

$$F(u) \stackrel{\text{def}}{=} \sum_{1 \le n \le u} S\left(\frac{u}{n}\right) = Au \log u + Bu + o(u) \quad \text{as } u \to \infty. \tag{17.1}$$

Examples 17.2. By the definition of Euler's constant γ, the function $S(v) = v$ has asymptotics of type $(1, \gamma)$; cf. formula (I.26.6).

The Chebyshev function $S(v) = \psi(v) = \sum_{k \le v} \Lambda(k)$ has asymptotics of type $(1, -1)$. Indeed, by Example I.4.2 and Stirling's formula,

$$\begin{aligned} F(u) &= \sum_{n \le u} \psi(\frac{u}{n}) = \sum_{n \le u} \sum_{k \le u/n} \Lambda(k) = \sum_{m \le u} \sum_{nk = m} \Lambda(k) \\ &= \sum_{m \le u} \log m = \log([u]!) = u \log u - u + o(u). \end{aligned} \tag{17.2}$$

Theorem 17.3. *Let S have Landau–Ingham asymptotics of type (A, B) and suppose that S is nonnegative and nondecreasing, or at least, that $S(v) + C[v]$ has that property for some constant C. Then*

$$S(v) \sim Av \quad \textit{as } v \to \infty \tag{17.3}$$

and

$$\int_1^{\infty -} \frac{S(v) - Av}{v^2} dv = B - A\gamma. \tag{17.4}$$

Using a strong number-theoretic estimate, Landau [1909] (p. 599) obtained (17.3) from an estimate (17.1) with a smaller remainder term. Cf. also Gordon [1958], who obtained (17.3) from (17.1) with an intermediate remainder by using a Selberg-type formula.

The function

$$S(v) = \frac{v}{\log(v+2)}, \quad v \ge 1$$

is nonnegative, nondecreasing and $o(v)$. Once the Theorem has been established, (17.4) shows that this function S cannot have Landau–Ingham asymptotics.

Proof of the Theorem. Let S have asymptotics of type (A, B). It is convenient to set $S(v) = 0$ for $v < 1$; for the time being we suppose also that S is nonnegative and nondecreasing.

(i) An argument used in number theory (well-known in the case $S = \psi$) will show that $S(u) = \mathcal{O}(u)$. Indeed, by (17.1) and since $S(u/n)$ is nonincreasing as a function of n,

$$\begin{aligned} &S(u) - S(u/2) + S(u/3) - \cdots = F(u) - 2F(u/2) = Au \log 2 + o(u), \\ &S(u) - S(u/2) \le Au \log 2 + o(u) \le A'u, \quad 1 \le u < \infty, \\ &S(u) \le S(u/2) + A'u \le S(u/4) + A'(u + u/2) \le \cdots \le 2A'u. \end{aligned}$$

It follows that

$$s(u) \stackrel{\text{def}}{=} \frac{S(u)}{u} \le 2A'. \tag{17.5}$$

(ii) Let $[\cdot]$ denote the usual 'integral part' function. We wish to estimate

$$f(u) \stackrel{\text{def}}{=} \int_0^\infty \left[\frac{1}{v}\right] s(uv)dv = \int_{1/u}^1 \left[\frac{1}{v}\right] s(uv)dv = \frac{1}{u}\int_1^u \left[\frac{u}{w}\right] s(w)dw. \tag{17.6}$$

Using (17.1) in the final step below one finds that for $u \to \infty$,

$$\begin{aligned}
\int_1^u \left[\frac{u}{w}\right] s(w)dw &= \int_1^u \left(\sum_{n\le u/w} 1\right) s(w)dw = \sum_{n\le u}\int_1^{u/n} s(w)dw \\
&= \sum_{n\le u}\int_n^u \frac{1}{n} s\left(\frac{v}{n}\right)dv = \int_1^u \sum_{n\le v} S\left(\frac{v}{n}\right)\frac{dv}{v} = \int_1^u F(v)\frac{dv}{v} \\
&= A(u\log u - u) + Bu + o(u).
\end{aligned}$$

Thus it follows from (17.6) that

$$f(u) = \int_0^\infty \left[\frac{1}{v}\right] s(uv)dv \left(= \int_{1/u}^\infty \cdots \right) = A\log u - A + B + o(1). \tag{17.7}$$

(iii) The function $k(v) = [1/v]$ is not integrable over $\mathbb{R}^+$, but the difference

$$k_\lambda(v) = k(v) - \lambda k(\lambda v) = \left[\frac{1}{v}\right] - \lambda\left[\frac{1}{\lambda v}\right] \quad \textit{is in } L^1(\mathbb{R}^+), \;\; \forall \lambda > 0. \tag{17.8}$$

Relation (17.7) now shows that

$$\int_0^\infty k_\lambda(v)s(uv)dv = f(u) - f\left(\frac{u}{\lambda}\right) \to A\log\lambda \quad \text{as } u \to \infty. \tag{17.9}$$

It will follow from (iv) below that $\log\lambda = \int_0^\infty k_\lambda(v)dv$.

(iv) The next step is to compute the Fourier–Mellin transform

$$\check{k}_\lambda(t) = \int_0^\infty k_\lambda(v)v^{it}dv.$$

For $\operatorname{Re} z > 0$,

$$\int_0^\infty \left[\frac{1}{v}\right] v^z dv = \int_0^\infty [x]x^{-z-2}dx = \frac{\zeta(z+1)}{z+1};$$

cf. formula (I.26.3). Thus by (17.8)

$$\int_0^\infty k_\lambda(v)v^z dv = \int_0^\infty \left(\left[\frac{1}{v}\right] - \lambda\left[\frac{1}{\lambda v}\right]\right) v^z dv = (1-\lambda^{-z})\frac{\zeta(z+1)}{z+1}.$$

For $z = x + it$ and $x \searrow 0$, the conclusion is that

$$\check{k}_\lambda(t) = (1 - \lambda^{-it})\frac{\zeta(1+it)}{1+it} \quad \text{for } t \neq 0, \quad \check{k}_\lambda(0) = \log \lambda. \tag{17.10}$$

Let us take $\lambda \neq 1$. Since $\zeta(1+it) \neq 0$ (cf. Theorem I.26.2), the only zeros of $\check{k}_\lambda(t)$ then are the points $t = n2\pi/\log\lambda$, $n = \pm 1, \pm 2, \cdots$. Thus the family $\{k_\lambda\}$, $\lambda \in \mathbb{R}^+$ is a Wiener family for $\mathbb{R}^+$; cf. Section 10.

(v) We finally show that s is slowly decreasing. For $w > v > 0$, by (17.5),

$$\begin{aligned} s(w) - s(v) &= \frac{S(w)}{w} - \frac{S(v)}{v} = \frac{S(w) - S(v)}{v} - \frac{S(w)}{w}\frac{w-v}{v} \\ &\geq -2A'\frac{w-v}{v} \to 0 \quad \text{as } v \to \infty \text{ and } w/v \searrow 1. \end{aligned} \tag{17.11}$$

From (17.9)–(17.11) and the Wiener–Pitt theorem for Wiener families on $\mathbb{R}^+$, we conclude that $s(u) \to A$ as $u \to \infty$; cf. Theorem 8.5. This proves the first part of the Theorem, formula (17.3).

(vi) If $S(v) = v$ or $s(v) = 1$ (for $v \geq 1$), then $F(u) = u\log u + \gamma u + o(u)$, hence $f(u) = \log u - 1 + \gamma + o(1)$; see (17.7). We now use (17.6) and (17.7) for $s(\cdot) - A$ instead of $s(\cdot)$:

$$\begin{aligned} \frac{1}{u}\int_1^u \left[\frac{u}{w}\right]\{s(w) - A\}dw &= A\log u - A + B - A(\log u - 1 + \gamma) + o(1) \\ &= B - A\gamma + o(1). \end{aligned}$$

If one replaces $[u/w]$ in this formula by u/w one finds that

$$\frac{1}{u}\int_1^u \frac{u}{w}\{s(w) - A\}dw = B - A\gamma + o(1) \quad \text{as } u \to \infty, \tag{17.12}$$

because the error is majorized by

$$\frac{1}{u}\int_1^u |s(w) - A|dw = o(1).$$

Formula (17.12) implies (17.4).

(vii) We will verify that for $S_0(v) = [v]$ one has (17.1) with $A = 1$ and $B = 2\gamma - 1$:

$$\begin{aligned} F(u) &= \sum_{n\leq u}\left[\frac{u}{n}\right] = \sum_{n\leq u}\sum_{k\leq u/n} 1 = \sum_{m\leq u}\sum_{nk=m} 1 \\ &= \sum_{m\leq u} d(m) = u\log u + (2\gamma - 1)u + o(u); \end{aligned} \tag{17.13}$$

cf. formula (I.4.9). For S_0 relation (17.3) is obviously correct and (17.4) is valid by part (vi) or formulas (I.26.4) and (I.26.2). It follows that the condition 'S nonnegative and nondecreasing' at the beginning of the proof may be relaxed to the corresponding condition for $S(v) + C[v]$ with some constant C. □

Application 17.4. By (17.2) the function $S = \psi$ has asymptotics of type $(1, -1)$. Hence by Theorem 17.3

$$\psi(v) \sim v \quad \text{as } v \to \infty, \qquad \int_1^{\infty -} \frac{\psi(v) - v}{v^2} dv = -1 - \gamma.$$

The first relation is equivalent to the prime number theorem, Theorem I.10.2. A related derivation of the PNT has been given by Levinson [1964b]. Once again the prime number theorem has been obtained as a consequence of the nonvanishing of $\zeta(z)$ on the line $\{\operatorname{Re} z = 1\}$.

Remark 17.5. One may avoid the use of Wiener *families* in the proof of Theorem 17.3 by employing the single kernel

$$m(v) = 2\left[\frac{1}{v}\right] - 2\left[\frac{1}{2v}\right] - 3\left[\frac{1}{3v}\right].$$

By (17.10) its Fourier–Mellin transform is given by

$$\check{m}(t) = (2 - 2^{-it} - 3^{-it})\frac{\zeta(1+it)}{1+it}, \quad t \neq 0; \quad \check{m}(0) = \log 6.$$

The transform is free of (real) zeros since 2^{-it} and 3^{-it} (both of absolute value 1) cannot be equal to 1 at the same time when $t \neq 0$. This is so because $\log 3/\log 2$ is irrational.

18 Ingham Summability

References: Ingham [1945], Hardy [1949] (appendix 4).

Definition 18.1. One says that $\sum_{n=1}^{\infty} a_n$ is *Ingham summable* to A if

$$\sum_{n \le u} a_n \frac{n}{u}\left[\frac{u}{n}\right] \to A \quad \text{as } u \to \infty. \tag{18.1}$$

Theorem 18.2. *Let $\sum_{n=1}^{\infty} a_n$ be Ingham summable to A and suppose that $na_n \ge -C$. Then $\sum_{n=1}^{\infty} a_n$ converges to A.*

Proof. Set

$$\sum_{n \le v} na_n = S(v) = vs(v), \qquad \sum_{n \le v} a_n = s^*(v).$$

Since $na_n \ge -C$ the function $S(v) + C[v]$ is nonnegative and nondecreasing. The Ingham summability (18.1) implies that S has Landau–Ingham asymptotics of type $(0, A)$:

$$F(u) = \sum_{n \le u} S\left(\frac{u}{n}\right) = \sum_{n \le u} \sum_{k \le u/n} ka_k = \sum_{k \le u} ka_k \left[\frac{u}{k}\right] = Au + o(u).$$

Hence by Theorem 17.3 with $A = 0$ and B replaced by the new A,

$$s(u) = \frac{S(u)}{u} \to 0 \quad \text{and} \quad \int_1^u \frac{S(v)}{v^2} dv \to A \quad \text{as } u \to \infty.$$

Now

$$\int_1^u \frac{S(v)}{v^2} dv = \int_1^u \sum_{n \le v} n a_n \frac{dv}{v^2} = \sum_{n \le u} n a_n \int_n^u \frac{dv}{v^2}$$
$$= \sum_{n \le u} a_n (1 - \frac{n}{u}) = s^*(u) - s(u).$$

It follows that

$$s^*(u) = s(u) + \int_1^u \frac{S(v)}{v^2} dv \to A \quad \text{as } u \to \infty.$$

□

Application 18.3. Let us take $a_n = \{\Lambda(n) - 1\}/n$. It follows from (17.2) and (17.13) that $\sum_{n=1}^{\infty} a_n$ is Ingham summable to -2γ:

$$\sum_{n \le u} \frac{\Lambda(n) - 1}{n} \frac{n}{u} \left[\frac{u}{n}\right] = \frac{1}{u} \left\{ \sum_{n \le u} \Lambda(n) \left[\frac{u}{n}\right] - \sum_{n \le u} \left[\frac{u}{n}\right] \right\}$$
$$= \log u - 1 - (\log u + 2\gamma - 1) + o(1) \to -2\gamma.$$

Hence by Theorem 18.2, the series $\sum_{n=1}^{\infty} a_n$ is convergent (to -2γ). As we know from Section I.10, the convergence of the series implies the prime number theorem.

Remarks 18.4. In his article, Ingham showed that his summability implies Cesàro summability, but the proof of this 'Abelian' result made use of number-theoretic estimates somewhat stronger than the prime number theorem; cf. Hardy [1949] (appendix 4). It follows that the Tauberian condition in Theorem 18.2 can be relaxed to '$s_n = \sum_{k=1}^{n} a_k$ slowly decreasing' (but that could also be proved more directly).

Ingham mentioned also that there are convergent series which are not summable in the sense of Definition 18.1. Explicit examples have been published by Pennington [1955] and Rajagopal [1955]. Ingham or related summability was also considered by Wintner [1943], [1957] and Karamata [1958].

S.L. Segal has written a number of articles on asymptotics of Landau–Ingham type and Ingham summability. We mention Segal [1969], [1979]; cf. also Jukes [1974], Geluk [1979], Rangachari [1985].

19 Application of Wiener Theory to Harmonic Functions

Following Rudin [1978], we use Wiener theory to discuss a converse of one of Fatou's theorems for harmonic functions. If U is a positive harmonic function in the unit disc, it has a Poisson–Stieltjes (or Herglotz) representation

$$U(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{1-r^2}{2\pi(1-2r\cos(\theta-\phi)+r^2)} d\mu(\phi).$$

Here $d\mu(\phi) = \mu(d\phi)$ is a finite positive measure; equivalently, one may think of $\mu(\phi) = (d\mu)[-\pi, \phi]$ as a bounded nondecreasing function on $[-\pi, \pi]$. More generally, let U be the Poisson integral of a complex measure $d\mu$ of finite total variation on $[-\pi, \pi]$. Suppose now that the corresponding function μ has a symmetric derivative at the point $\phi = \theta_0$:

$$D_s\mu(\theta_0) \stackrel{\text{def}}{=} \lim_{\delta\to 0} \frac{\mu(\theta_0+\delta)-\mu(\theta_0-\delta)}{2\delta} = A.$$

Then by a theorem of Fatou [1906], U tends to A for radial approach to the point $e^{i\theta_0}$:

$$U(re^{i\theta_0}) \to A \quad \text{as } r \nearrow 1.$$

If μ has ordinary derivative $\mu'(\theta_0) = A$, the limit of $U(re^{i\theta})$ is equal to A under arbitrary 'nontangential' approach to the point $e^{i\theta_0}$; cf. Zygmund [1959].

For the discussion of a converse it is convenient to consider an analog of Fatou's theorem for a half-plane, and more generally, for the *half-space*

$$\mathbb{R}_+^{n+1} = \{(x, y) : x \in \mathbb{R}^n,\ y > 0\}.$$

The corresponding Poisson kernel has the form

$$P(x, y) = \frac{cy}{(|x|^2 + y^2)^{(n+1)/2}}, \tag{19.1}$$

where $c = c(n)$ is such that

$$\int_{\mathbb{R}^n} P(x, y)dx = \int_{\mathbb{R}^n} P(x, 1)dx = 1. \tag{19.2}$$

[Although we do not need the actual value of $c(n)$, we mention that it is $2/\sigma_{n+1}$, where σ_p stands for the area of the unit sphere in $\mathbb{R}^p$, that is, $\sigma_p = 2\pi^{p/2}/\Gamma(p/2)$.]

Positive harmonic functions U in the half-space $\mathbb{R}_+^{n+1}$ may be characterized by a representation

$$U(x, y) = by + \int_{\mathbb{R}^n} P(x-\xi, y)\mu(d\xi), \tag{19.3}$$

where $b \geq 0$ and μ is a positive measure on $\mathbb{R}^n$ such that the Poisson integral is convergent. References: Loomis and Widder [1942] for the case $n = 1$, Rudin (loc. cit.) and E.M. Stein [1970] (sections 7.1–7.4) for general n.

Let B_r denote the ball $B(0, r)$ in $\mathbb{R}^n$, let $|B_r| = |B_1|r^n$ [$= (\sigma_n/n)r^n$] denote its volume, and set

$$m(r) = \frac{\mu(B_r)}{|B_r|}. \tag{19.4}$$

The *symmetric derivative* of μ at the origin of $\mathbb{R}^n$ is defined by

$$D_s\mu(0) \stackrel{\text{def}}{=} \lim_{r\searrow 0} m(r), \tag{19.5}$$

provided the limit exists (and is finite). By an analog of Fatou's theorem, the existence of $D_s\mu(0)$ implies that U has a 'normal' limit at $x = 0$,

$$\lim_{y \searrow 0} U(0, y) = D_s\mu(0).$$

This result does not require that μ be positive, but for positive μ the converse is true:

Theorem 19.1. *Let $U(x, y)$ be the Poisson integral of a positive measure μ on $\mathbb{R}^n$ which is finite for one, and hence every, point $(x, y) \in \mathbb{R}^{n+1}_+$. Suppose that $U(0, y)$ has finite limit A as $y \searrow 0$. Then $D_s\mu(0) = A$.*

For $n = 1$ this was proved by Loomis [1943]; Theorem 19.1 is due to Rudin (loc. cit.).

Proof of the Theorem. We show that it is enough to suppose that a suitable average of $V(y) = U(0, y)$ has limit A. Let

$$M_\alpha V(r) \stackrel{\text{def}}{=} r^{-\alpha} \int_{y=0}^{y=r} V(y) dy^\alpha \to A \quad \text{as } r \searrow 0 \tag{19.6}$$

for some number $\alpha > 0$. Then for $\beta > 0$,

$$(\beta - \alpha) M_\beta \circ M_\alpha = \beta M_\alpha - \alpha M_\beta, \tag{19.7}$$

as is shown by inversion of the order of integration. By (19.6), the left-hand side of (19.7) applied to V has limit $(\beta - \alpha)A$ as $r \searrow 0$. It follows that $M_\beta V(r) \to A$ as $r \searrow 0$ for every $\beta > 0$. Thus we may assume that (19.6) holds for the value $\alpha = n$, which will turn out to be convenient. The average $(M_n V)(r)$ can be rewritten as follows:

$$\begin{aligned} M_n V(r) &= r^{-n} \int_0^r U(0, y) dy^n = r^{-n} \int_0^r dy^n \int_{\mathbb{R}^n} P(\xi, y) \mu(d\xi) \\ &= r^{-n} \int_{\mathbb{R}^n} \mu(d\xi) \int_0^r P(\xi, y) n y^{n-1} dy. \end{aligned} \tag{19.8}$$

Observe that μ can be replaced by its restriction to a neighborhood of the origin without affecting the hypothesis or the conclusion of the Theorem. Thus one may assume that $\mu(\mathbb{R}^n)$ is finite.

We have to prove that the averages $m(r)$ in (19.4) tend to A and will begin by showing that they are *bounded.* If $|\xi| < r$ then $P(\xi, y) > cy/(r^2 + y^2)^{(n+1)/2}$, so that the final inner integral in (19.8) is larger than

$$\int_0^r \frac{ncy^n dy}{(r^2 + y^2)^{(n+1)/2}} = \int_0^1 \frac{nct^n dt}{(1 + t^2)^{(n+1)/2}} = c_1(n) = c_1,$$

say. Hence

$$M_n V(r) \geq c_1 \frac{\mu(\mathbb{R}^n)}{r^n} \geq c_1 |B_1| \frac{\mu(B_r)}{|B_r|} = c_1 |B_1| m(r). \tag{19.9}$$

Relation (19.6) for $\alpha = n$ now shows that the function $m(r)$ remains bounded as $r \searrow 0$. The inequality $m(r) \le \|\mu\|/|B_r|$ gives boundedness for $r \ge \delta > 0$.

The next step is to convert the relation $M_n V(r) \to A$ to a Wiener-type limit relation involving $m(r)$. To this end we substitute $y = |\xi|v$ in the final inner integral of (19.8) and introduce the kernel

$$k(v) = \frac{c_2(n)}{(1+v^2)^{(n+1)/2}}, \quad c_2(n) = nc|B_1| \quad [= 2\sigma_n/\sigma_{n+1}]. \tag{19.10}$$

This gives

$$P(\xi, |\xi|v) = \frac{cv}{|\xi|^n(1+v^2)^{(n+1)/2}} = \frac{k(v)v}{n|\xi|^n|B_1|}.$$

Hence

$$\begin{aligned} M_n V(r) &= \int_{\mathbb{R}^n} \mu(d\xi) \int_0^{r/|\xi|} P(\xi, |\xi|v) n|\xi|^n v^{n-1}(dv)/r^n \\ &= \int_{\mathbb{R}^n} \mu(d\xi) \int_0^{r/|\xi|} k(v) \frac{dv}{|B_1|(r/v)^n} = \int_0^\infty k(v) \frac{dv}{|B_{r/v}|} \int_{|\xi|<r/v} \mu(d\xi) \\ &= \int_0^\infty k(v) \frac{\mu(B_{r/v})}{|B_{r/v}|} dv = \int_0^\infty k(v) m(r/v) dv. \end{aligned} \tag{19.11}$$

Here the inversion of the order of integration may be justified by the boundedness of $m(\cdot)$, which in particular shows that μ contains no point mass at the origin. The limit relation for $M_n V(r)$ implies that the *final member* of (19.11) *tends to* A as $r \searrow 0$.

Can we now apply Wiener's theorem for $\mathbb{R}^+$, Theorem 8.5 ? Setting $r = 1/u$ and $m(r/v) = m\{1/(uv)\} = s(uv)$, we obtain a limit relation of the right kind:

$$\int_0^\infty k(v) s(uv) dv \to A = A' \int_0^\infty k(v) dv \quad \text{as } u \to \infty. \tag{19.12}$$

Here k is given by (19.10), $s(\cdot)$ is bounded and $A' = A/\check{k}(0)$. But is k a *Wiener kernel* for $\mathbb{R}^+$? The relevant Fourier–Mellin transform $\check{k}(t) = \int_0^\infty k(v) v^{it} dv$ can be computed by the change of variables $1 + v^2 = 1/w$, which gives a beta-function:

$$\begin{aligned} \check{k}(t) &= c_2(n) \int_0^1 w^{(n-2-it)/2}(1-w)^{(-1+it)/2} dw/2 \\ &= c_2(n) \frac{\Gamma\{(n-it)/2\}\Gamma\{(1+it)/2\}}{2\Gamma\{(n+1)/2\}}. \end{aligned} \tag{19.13}$$

Thus the kernel k is in W^+ since the Gamma function is free of zeros; cf. Whittaker-Watson [1927/96] (section 12.1). *Conclusion:* as $r = 1/u \searrow 0$,

$$\int_0^\infty h(v) m(r/v) dv = \int_0^\infty h(v) s(uv) dv \to A' \int_0^\infty h(v) dv \tag{19.14}$$

for every function h in $L^1(\mathbb{R}^+)$.

We will use a special property of $m(\cdot)$ to deduce that $m(r) \to A'$ as $r \searrow 0$. Take $\gamma > 1$. Since $r^n m(r) = \mu(B_r)/|B_1|$ is nondecreasing, one finds that

$$m(r/v) \leq v^n m(r) \leq \gamma^n m(r) \quad \text{for } 1 \leq v \leq \gamma.$$

Now let h be any nonnegative function with support in $[1, \gamma]$ for which $\int_0^\infty h(v)dv = 1$. Then

$$\int_1^\gamma h(v)m(r/v)dv \leq \gamma^n m(r),$$

hence by (19.14)

$$\liminf_{r \searrow 0} \gamma^n m(r) \geq A'.$$

Since this holds for every $\gamma > 1$, $\liminf m(r) \geq A'$. Working with the interval $[1/\gamma, 1]$ and appropriate h, one similarly finds that $\limsup m(r) \leq A'$. Thus

$$\lim_{r \searrow 0} m(r) = A', \quad D_s\mu(0) = A' = A/\check{k}(0). \tag{19.15}$$

It remains to show that $A' = A$. To that end we could use Fatou's theorem or we could compute $\check{k}(0)$ directly, but it is simpler to consider the special case where μ is Lebesgue measure. Then the Poisson integral $U(x, y)$ of $\mu(d\xi) = d\xi$ is identically equal to 1, so that $A = \lim U(0, y) = 1$. Likewise $m(r) \equiv 1$, hence $(D_s\mu)(0) = A' = 1$. Since $A' = A/\check{k}(0)$, it follows that $\check{k}(0) = 1$. Thus $(D_s\mu)(0) = A' = A$ for all positive measures μ whose Poisson integral satisfies the conditions of Theorem 19.1. This completes the proof.

Several authors have obtained converses of Fatou's theorem for non-normal approach in $\mathbb{R}^{n+1}$. We mention Loomis (loc. cit.) and Gehring [1957], [1960] for the case $n = 1$, and Brossard and Chevalier [1990], [1995] who treated general n. Ramey and Ullrich [1982] considered pluriharmonic functions (real parts of holomorphic functions) in $\mathbb{C}^n$.

III

Complex Tauberian Theorems

1 Introduction

In the course of Chapters I and II, the notion of 'Tauberian theorem' has evolved. We would now say that such a theorem involves a class of objects S (functions, series, sequences) and a transformation $\mathcal{T}$. The transformation is an 'averaging operation' with attendant continuity property: certain limit behavior of the original S implies related limit behavior of the image $\mathcal{T}S$. The aim of a Tauberian theorem is to reverse the averaging, or pass to a different average. One wants to go from a limit property of $\mathcal{T}S$ to a limit property of S, or another transform of S. Such theorems typically require an additional condition, a 'Tauberian condition', on S and perhaps a condition on the transform $\mathcal{T}S$.

We recall some examples from Chapter I; cf. Hardy's book [1949]. By Abel's theorem [1826], convergence $\sum_{n=0}^{\infty} a_n = A$, or $s_n = \sum_{k=0}^{n} a_k \to A$, implies that the 'Abel transform'

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{\sum\limits_{n=0}^{\infty} s_n x^n}{\sum\limits_{n=0}^{\infty} x^n} \quad \text{has limit } A \text{ as } x \nearrow 1.$$

Tauber [1897] proved that the condition $na_n \to 0$ as $n \to \infty$ is sufficient for a converse; cf. Section I.5. A corresponding optimal order condition requires that the sequence $\{na_n\}$ be bounded (Littlewood [1911]); see Section I.7. It is appropriate to mention another Tauberian theorem involving power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ that are known to converge for $|x| < 1$. By a continuity theorem of Frobenius [1880], the asymptotic relation $s_n \sim An$ implies that

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n \sim \frac{A}{1-x} \quad \text{as } x \nearrow 1.$$

(Cf. the computation in Example I.2.3.) In this case there is a Tauberian converse under the condition $a_n \geq 0$ (Hardy and Littlewood [1914a]); see Sections I.7 and I.11.

The situation is more delicate in the case of (classical) Dirichlet series $f^*(x) = \sum_{n=1}^{\infty} a_n/n^x$, which are important in number theory. If such a series converges for $x > 1$ and $s_n = \sum_{k=1}^{n} a_k \sim An$, then

$$f^*(x) \sim \frac{A}{x-1} \quad \text{as } x \searrow 1;$$

see Theorem 3.1. However, in this case one needs much more for a converse than the condition $a_n \geq 0$. Here it is desirable to consider the behavior of the sum function $f^*(x)$ for complex values of x; cf. Section 2.

Tauberian theorems in which complex-analytic or related boundary properties of the transform play an important role are called *complex Tauberians*. The first results of that kind were Tauberian theorems *avant la lettre*. For power series there was Fatou's theorem [1905], [1906]: If the transform $f(z) = \sum_0^\infty a_n z^n$ of the series $\sum_0^\infty a_n$ exists for $|z| < 1$ and is *analytic* at the point $z = 1$, then the 'Tauberian' condition $a_n \to 0$ already implies the convergence of $\sum_0^\infty a_n$. Cf. Littlewood's condition $a_n = \mathcal{O}(1/n)$ if one knows only that $f(x) \to A$! Marcel Riesz [1909], [1911], [1916] extended Fatou's theorem to general Dirichlet series $\sum_{n=0}^{\infty} a_n e^{-\mu_n z}$. Both power series and Dirichlet series can be considered as special cases of (possibly improper) Laplace–Stieltjes transforms

$$F(z) = \mathcal{L}dS(z) = \int_{0-}^{\infty-} e^{-zt} dS(t) = \lim_{B\to\infty} \int_{0-}^{B} e^{-zt} dS(t), \tag{1.1}$$

which play an important role in Tauberian theory.

Number theoretic questions have inspired a great deal of complex Tauberian theory for Dirichlet series and Laplace transforms. The early proofs of the prime number theorem (PNT), by Hadamard [1896] and de la Vallée Poussin [1896], required analytic information about the Riemann zeta function $\zeta(z)$ (Sections I.4, I.26) on and to the left of the line $\{\text{Re}\, z = 1\}$. The best estimates known today for the prime counting function $\pi(u)$ still make use of such information. And, for one of the quickest proofs of just the prime number theorem:

$$\pi(u) \sim \frac{u}{\log u} \quad \text{as } u \to \infty, \tag{1.2}$$

Littlewood [1971] went right across the 'critical strip' $\{0 < x = \text{Re}\, z < 1\}$ for the zeta function!

Since 1900, mathematicians have aimed for simple proofs of the PNT which use as little about the zeta function as possible. The ingenious 'elementary' proofs by Selberg [1949] and Erdős [1949a] avoid the zeta function altogether, but they do not qualify as simple. Of numerous expositions of these proofs, we mention Levinson [1969], Diamond [1982], and Nathanson [2000] (section 2.9). See also the general

surveys by Schwarz [1969b], Lavrik [1984], Bateman and Diamond [1996], Apostol [2000], Narkiewicz [2000], and Tenenbaum and Mendès France [2000].

Besides simple analytic properties, the minimal information about the zeta function required for a complex proof of the PNT is the nonvanishing of $\zeta(z)$ on the line $\{\operatorname{Re} z = 1\}$. (The latter property is actually a consequence of the PNT; see Corollary 3.2 or Ingham [1932].) For his relatively simple proof, Landau [1908], [1909] had to know in addition that $\zeta'(z)/\zeta(z)$ does not grow faster than a power of z as $z \to \infty$ in the half-plane $\{\operatorname{Re} z \geq 1\}$; cf. Section 2. Hardy and Littlewood [1918] could relax this to an exponential order condition, but the real *breakthrough* came with Wiener's Tauberian theory [1928], [1932], [1933] (see Chapter II) and the work of his student Ikehara [1931]. Applying Wiener's early Tauberian theory, Ikehara could dispense with the growth condition in the complex approach. The resulting 'Wiener–Ikehara theorem' has long provided the preferred way to the PNT. We discuss two proofs of the Theorem. The traditional proof in Section 4 is based on an approximate identity, and this method will be important for us later on. The elegant newer proof by Graham and Vaaler [1981] in Section 5, which adds precision, is based on one-sided L^1 approximation by Fourier transforms of functions with bounded support.

A very attractive approach to the PNT is due to Newman [1980]. He could prove a useful Tauberian theorem for Dirichlet series with bounded coefficients by simple contour integration. Substituting Laplace transforms of bounded functions, we describe Newman's way to the PNT in Sections 6–8; cf. Korevaar [1982] and Zagier [1997]. This material is largely independent of Sections 2–5.

In the 1930's the Wiener–Ikehara method was refined and extended by a number of authors, most notably Ingham [1935], [1936], [1941]. The first paper contains the Tauberians just referred to, as well as stronger one-sided versions, and includes refinements of Fatou's theorem. Sections 9–12 cover such results, with partly new proofs. The 1941 paper will be discussed in Sections IV.21, IV.22.

Sections 13–15 are devoted to more recent developments in the Fatou–Riesz and Laplace transform area, which were motivated by operator theory and semigroups. In this introduction we only signal the seminal papers by Katznelson and Tzafriri [1986], Allan, O'Farrell and Ransford [1987], and Arendt and Batty [1988]. The (partly new) refinements in this book involve local H^1 and pseudofunction boundary behavior of transforms. Their development was aided by Newman's contour integration method which, in turn, can benefit from the use of pseudofunctions; see Theorem 14.6. For numerous other results and applications, see the book by Arendt, Batty, Hieber and Neubrander [2001].

There is a large body of Tauberian *remainder theory* involving conditions in the complex domain. We present notable examples where 'complex' information gives much stronger results than exclusively 'real' information. Sharp remainder estimates for the theorems of Fatou and Riesz were first discussed by the author in [1954b]; cf. Section 17 where we give a more transparent proof. In the 1950's, Freud, the author and others obtained remainder estimates in Hardy–Littlewood theorems for power series under real conditions; cf. Section 18 and see Chapter VII for a detailed discussion. In Section 18 we discuss stronger estimates under complex conditions, due to Postnikov

[1953] and Subhankulov [1960], [1964]. Finally we consider complex Tauberian results for the Stieltjes transform, due to Malliavin [1962] and Pleijel [1963], which can be applied to study the zero distribution of entire functions and the distribution of eigenvalues (Section 19). For other aspects of remainder theory, see the books by Ganelius [1971], Postnikov [1980], and Subhankulov [1976] (Russian), as well as Chapter VII.

Analyticity of transforms in a strip plays a role in certain Tauberian theorems involving 'regular variation', see Section IV.9 and the book by Bingham, Goldie and Teugels [1987]. For multidimensional complex Tauberians we refer to the book by Vladimirov, Drozhzhinov and Zav'yalov [1986]. Several authors have added something to 'real' Hardy–Littlewood theorems by interesting complex variable proofs. In this context we mention Delange [1952], Jurkat [1956a], [1957], and Halász [1967a]; cf. Landau and Gaier [1986] (appendix 2) and Section VII.11.

2 A Landau-Type Tauberian for Dirichlet Series

Landau [1908], [1909] (section 66) derived the prime number theorem from a Tauberian result of the following type.

Theorem 2.1. *Let $f(z)$ be given for* $\operatorname{Re} z > 1$ *by a convergent Dirichlet series,*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}, \tag{2.1}$$

in which the coefficients satisfy the 'Tauberian condition' $a_n \geq 0$. Suppose that for some constant A, the analytic function

$$g(z) = f(z) - \frac{A}{z-1}, \quad \operatorname{Re} z > 1, \tag{2.2}$$

has an analytic or just continuous extension (also called g) to the closed half-plane $\{\operatorname{Re} z \geq 1\}$. *Finally, suppose that there is a constant M such that*

$$g(z) = \mathcal{O}(|z|^M) \quad \textit{for} \ \operatorname{Re} z \geq 1. \tag{2.3}$$

Then

$$\frac{1}{n} s_n = \frac{1}{n} \sum_{k=1}^{n} a_k \to A \quad \textit{as} \ n \to \infty. \tag{2.4}$$

Proof. One may start with the formula

$$\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{tz}}{z(z+1)} dz = \begin{cases} 1 - e^{-t} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0 \end{cases}$$

($\alpha > 0$), which is an easy application of the residue theorem. We now set $\sum_{n \leq v} a_n = s(v)$ (so that $s(v) = 0$ for $v < 1$) and take $\alpha > 1$, $u > 0$. Then by absolute convergence,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f(z)\frac{u^z}{z(z+1)}dz = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^z}\frac{u^z}{z(z+1)}dz$$
$$= \sum_{n\le u} a_n(1-n/u) = \int_0^u (1-v/u)ds(v) = \frac{1}{u}\int_0^u s(v)dv.$$

Again by the residue theorem,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{z-1}\frac{u^z}{z(z+1)}dz = \begin{cases}(u-1)^2/(2u) & \text{for } u \ge 1,\\ 0 & \text{for } 0 < u < 1.\end{cases}$$

Subtracting A times the last identity from the preceding and multiplying by u, one obtains from (2.2) that

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} g(z)\frac{u^{z+1}}{z(z+1)}dz = \begin{cases}s^{(-1)}(u) - A(u-1)^2/2 & \text{for } u \ge 1,\\ 0 & \text{for } 0 < u < 1,\end{cases} \tag{2.5}$$

where we have written $s^{(-1)}(u)$ for $\int_0^u s(v)dv$; cf. Ingham [1932]. Integrating $k-1$ times with respect to u from 0 on and then dividing by u^{k+1}, one concludes that for $u \ge 1$ and $\alpha > 1$,

$$\frac{s^{(-k)}(u)}{u^{k+1}} - \frac{A}{(k+1)!}\left(1-\frac{1}{u}\right)^{k+1} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{u^{z-1}g(z)dz}{z(z+1)\dots(z+k)}. \tag{2.6}$$

Here $s^{(-k)}(u)$ denotes the k-times iterated integral,

$$s^{(-k)}(u) = \int_0^u dv_1\int_0^{v_1} dv_2 \cdots \int_0^{v_{k-1}} s(v_k)dv_k.$$

We finally use the hypothesis (2.3). Taking $k > M$ and invoking dominated convergence, one may let α go to 1 in (2.6). The resulting right-hand side with $z = 1+iy$ is an absolutely convergent Fourier integral, involving the exponential $\exp(iy\log u)$. By the Riemann–Lebesgue lemma (cf. Rudin [1966/87] or Korevaar [1968]), this right-hand side will tend to 0 as $u \to \infty$. One thus obtains the asymptotic relation

$$s^{(-k)}(u) \sim \frac{A}{(k+1)!}u^{k+1}.$$

By the monotonicity of $s(\cdot)$, one may differentiate k times to conclude that

$$s(u) \sim Au, \tag{2.7}$$

which implies (2.4). (The differentiation may be justified by repeated use of the mean-value theorem; cf. Lemma I.17.1.)

Derivation of the PNT. For $\mathrm{Re}\, z > 1$, logarithmic differentiation of the Euler product for the zeta function, $\zeta(z) = \prod_p 1/(1-p^{-z})$ over the primes p, gives the formula

$$f(z) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}. \tag{2.8}$$

Here $\Lambda(n) = \log p$ if $n = p^\beta$ ($\beta \geq 1$) and $\Lambda(n) = 0$ if n is not a prime power; cf. Section I.4. One knows that $\zeta(z)$ can be continued analytically across every point of the line $\{\operatorname{Re} z = 1\}$ except the point $z = 1$, where $\zeta(z)$ has a simple pole with residue 1; cf. Section I.26. We also need the crucial fact that $\zeta(\cdot)$ has no zeros on the line $\{\operatorname{Re} z = 1\}$, which was verified in Theorem I.26.2. It follows that $-\zeta'(z)/\zeta(z)$ likewise is analytic at every point of the line $\{\operatorname{Re} z = 1\}$ with the exception of a simple pole at $z = 1$ with residue 1. Thus

$$g(z) = f(z) - \frac{1}{z-1} = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1} \tag{2.9}$$

has an analytic extension to the closed half-plane $\{\operatorname{Re} z \geq 1\}$. If one knows that this function g indeed satisfies a growth condition (2.3), the conclusion from Theorem 2.1 is that

$$\psi(u) \stackrel{\text{def}}{=} \sum_{n \leq u} \Lambda(n) = \sum_{p^\alpha \leq n} \log p \sim u; \tag{2.10}$$

cf. (2.7). This relation for Chebyshev's function ψ is equivalent to the PNT (1.2); cf. Section I.10.

For a proof of a suitable growth condition we refer to Landau or Ingham (loc. cit.). However, Wiener's Tauberian theory and Ikehara's article [1931] showed that *no growth condition* is necessary:

Theorem 2.2. *Let $f(z)$ be given for* $\operatorname{Re} z > 1$ *by a Dirichlet series* (2.1) *with $a_n \geq 0$, and let $g(z) = f(z) - A/(z-1)$ have a continuous extension to the closed half-plane* $\{\operatorname{Re} z \geq 1\}$. *Then*

$$s_n = \sum_{k=1}^{n} a_k \sim An \quad \textit{as } n \to \infty. \tag{2.11}$$

The result is sometimes called 'Landau–Ikehara theorem', but the name Wiener–Ikehara theorem seems more appropriate. By the preceding, the Theorem readily provides a proof of the prime number theorem based solely on the nonvanishing of $\zeta(z)$ on the line $\{\operatorname{Re} z = 1\}$. For a proof of the Wiener–Ikehara theorem, see Section 4. The Tauberian condition $a_n \geq 0$ may be relaxed to $a_n \geq -C$. [One may apply the above result to $f^*(z) = f(z) + C\zeta(z)$.]

3 Mellin Transforms

Instead of sums of Dirichlet series, one may more generally consider functions f which for $\operatorname{Re} z > 1$ are given by a (possibly improper: only conditionally convergent) Mellin–Stieltjes transform,

$$f(z) = \int_{1-}^{\infty-} v^{-z} ds(v) = \lim_{B\to\infty} \int_{1-}^{B} v^{-z} ds(v). \tag{3.1}$$

If $s(\cdot)$ is nondecreasing the integral will be absolutely convergent. Assuming that $s(v)$ is equal to 0 for $v < 1$ and locally of bounded variation, one may in any case integrate by parts to obtain an ordinary Mellin transform.

Indeed, the substitution $s(e^t) = S(t)$ changes the integral (3.1) to a Laplace–Stieltjes transform $\mathcal{L}dS$ as in (1.1). Integration by parts in such integrals has been discussed in Section I.13. Always taking $S(t) = 0$ for $t < 0$, we suppose that S is locally of bounded variation on $[0, \infty)$ and such that the integral for $\mathcal{L}dS(z)$ is (at least conditionally) convergent for $\mathrm{Re}\, z = x > a \geq 0$. Under this condition $S(t) = \mathcal{O}(e^{bt})$ for every $b > a$, so that

$$\mathcal{L}dS(z) = \int_{0-}^{\infty-} e^{-zt} dS(t) = z \int_{0}^{\infty} S(t) e^{-zt} dt = z\mathcal{L}S(z) \tag{3.2}$$

whenever $x > a$; cf. Proposition I.13.1. Here the final integral is absolutely convergent.

We now state a 'complex' Abelian theorem in preparation for the Wiener–Ikehara theorem in Section 4. If $s(v) = Av$ for $v \geq 1$, the transform $f(z)$ in (3.1) is equal to $A/(z-1)$ for $x = \mathrm{Re}\, z > 1$. What can one say if $s(v) \sim Av$?

Theorem 3.1. *Suppose that $s(\cdot)$ is locally of bounded variation, $s(v) = 0$ for $v < 1$ and $s(v) \sim Av$ as $v \to \infty$. Then the analytic function $f(z)$ defined by* (3.1) *for $x = \mathrm{Re}\, z > 1$ behaves like $A/(z-1)$ at least for so-called angular approach to the point $z = 1$. There can be no pole type behavior for angular approach to any other point of the line $\{\mathrm{Re}\, z = 1\}$.*

Proof. Using integration by parts one finds that for $x = \mathrm{Re}\, z > 1$,

$$g(z) = f(z) - \frac{A}{z-1} = z \int_{1}^{\infty} \{s(v) - Av\} v^{-z-1} dv. \tag{3.3}$$

The hypothesis $s(v) \sim Av$ means that $s(v) - Av = o(v)$ as $v \to \infty$. Thus

$$\begin{aligned} \left| \int_{1}^{\infty} \{s(v) - Av\} v^{-z-1} dv \right| &\leq \int_{1}^{\infty} |s(v) - Av| v^{-x-1} dv \\ &= o\left(\int_{1}^{\infty} v^{-x} dv \right) = o\left(\frac{1}{x-1} \right) \quad \text{as } x \searrow 1. \end{aligned} \tag{3.4}$$

First suppose that $z \to 1$ in an angle given by $1 < x < 2$ and $|z-1| < C(x-1)$, so that $|z| < 1 + C$. Then by (3.3) and (3.4)

$$(z-1) g(z) = o\left(\frac{|z-1|}{x-1} \right) = o(1).$$

It follows that $(z-1) f(z) \to A$ as $z \to 1$ in the angle.

Next take z in an angle of the form $1 < x < 2$ and $|z - z_0| < C(x - 1)$, where $z_0 = 1 + iy_0$ with $y_0 \neq 0$. In that case (3.4) gives $(z - z_0)g(z) = o(1)$ as $x \searrow 1$, so that

$$(z - z_0)f(z) = (z - z_0)g(z) + A(z - z_0)/(z - 1) \to 0 \quad \text{as } z \to z_0.$$

□

Corollary 3.2. *The prime number theorem implies that $\zeta(z)$ is free of zeros on the line* $\{\operatorname{Re} z = 1\}$.

Indeed, if in Theorem 3.1 we take $s(v) = \psi(v)$, then $f(z) = -\zeta'(z)/\zeta(z)$; cf. Section 2. Now by the PNT, $\psi(v) \sim v$. Thus $f(z)$ can have no pole on the line $\{\operatorname{Re} z = 1\}$ (different from the point 1), so that $\zeta(z)$ can have no zero on that line.

4 The Wiener–Ikehara Theorem

We consider an integral form of Theorem 2.2:

Theorem 4.1. *Let $s(v)$ vanish for $v < 1$, be nondecreasing, continuous from the right and such that the Mellin–Stieltjes transform*

$$f(z) = \int_{1-}^{\infty} v^{-z} ds(v) = z \int_{1}^{\infty} s(v)v^{-z-1} dv, \quad z = x + iy, \tag{4.1}$$

exists for $\operatorname{Re} z = x > 1$. *Suppose that for some constant A, the analytic function*

$$g(z) = f(z) - \frac{A}{z - 1}, \quad x > 1, \tag{4.2}$$

has a continuous extension to the closed half-plane $\{x \geq 1\}$. Then

$$s(u)/u \to A \quad \textit{as } u \to \infty. \tag{4.3}$$

The Tauberian condition is given by the positivity of the measure ds. For the integration by parts in (4.1), cf. Section 3.

It is convenient to deal with a corresponding (slightly more general) result for Laplace–Stieltjes transforms. For the transition we set $v = e^t$, $s(e^t) = S(t)$.

Theorem 4.2. *Let $S(t)$ vanish for $t < 0$, be nondecreasing, continuous from the right and such that the Laplace–Stieltjes transform*

$$f(z) = \mathcal{L}dS(z) = \int_{0-}^{\infty} e^{-zt} dS(t) = z \int_{0}^{\infty} S(t)e^{-zt} dt, \quad z = x + iy, \tag{4.4}$$

exists for $\operatorname{Re} z = x > 1$. *Suppose that for some constant A, the analytic function*

$$g(z) = f(z) - \frac{A}{z - 1}, \quad x > 1, \tag{4.5}$$

has a boundary function $g(1+iy)$ in the following sense. For $x \searrow 1$, the function $g_x(iy) = g(x+iy)$ converges to $g(1+iy)$ uniformly or in L^1 on every finite interval $-\lambda < y < \lambda$. Then

$$e^{-t}S(t) \to A \quad as \ t \to \infty. \tag{4.6}$$

Working with Wiener, Ikehara applied his mentor's early [1928] Tauberian theory to obtain a proof [1931]. Thus their joint effort succeeded in *removing* the earlier *growth condition* on f or g at infinity which was imposed in Section 2. Most subsequent proofs and extensions have benefited from Wiener's version [1932] of the proof and from Bochner's simplification [1933a] of it. See Landau [1932a], Ingham [1935], Wiener and Pitt [1939], as well as expositions in books by Wiener [1933], Doetsch [1937], [1950], Widder [1941], Chandrasekharan [1968], and Heins [1968].

We give two proofs for the Wiener–Ikehara theorem. The proof below is along traditional lines. The elegant newer proof in Section 5 gives an optimal result for the case where there is a good boundary function $g(1+iy)$ only on *some* finite interval $-\lambda < y < \lambda$. Less precise results for that case were obtained earlier by Heilbronn and Landau [1933a], [1933c]; see also Landau [1932b].

Following Wiener [1932] and Bochner [1933a], we base our first proof of Theorem 4.2 on a nonnegative approximate identity $\{K_\lambda\}$, $0 < \lambda \to \infty$, for which the Fourier transform $\hat{K}_\lambda$ has support in $[-\lambda, \lambda]$. It is convenient to use the historical example of the 'Fejér kernel for $\mathbb{R}$' encountered in Section II.7:

$$K_\lambda(t) = \lambda K(\lambda t) = \frac{1-\cos\lambda t}{\pi\lambda t^2} = \frac{\lambda}{2\pi}\left(\frac{\sin\lambda t/2}{\lambda t/2}\right)^2,$$
$$\hat{K}_\lambda(y) = \int_{\mathbb{R}} K_\lambda(t)e^{-iyt}dt = \begin{cases} 1-|y|/\lambda & \text{for } |y| \le \lambda, \\ 0 & \text{for } |y| > \lambda. \end{cases} \tag{4.7}$$

Proposition 4.3. *Let $\sigma(t)$ vanish for $t < 0$, be nonnegative for $t \ge 0$ and such that the Laplace transform*

$$F(z) = \mathcal{L}\sigma(z) = \int_0^\infty \sigma(t)e^{-zt}dt \tag{4.8}$$

exists for $\operatorname{Re} z = x > 0$. *Suppose that for $x \searrow 0$, the function*

$$G(z) = F(z) - A/z, \quad z = x+iy, \tag{4.9}$$

converges to a boundary function $G(iy)$ in $L^1(-\lambda < y < \lambda)$. Then the integral

$$\int_{\mathbb{R}} K_\lambda(u-t)\sigma(t)dt = \int_{-\infty}^{\lambda u} \sigma(u-v/\lambda)K(v)dv$$
$$\textit{exists and tends to } A\int_{\mathbb{R}} K(v)dv = A \quad as \ u \to \infty. \tag{4.10}$$

Proof. For $x > 0$, the convolution of the L^1 functions $K_\lambda(t)$ and $\sigma(t)e^{-xt}$ has Fourier transform $\hat{K}_\lambda(y)F(x+iy)$, so that by Fourier inversion

$$\int_{\mathbb{R}} K_\lambda(u-t)\sigma(t)e^{-xt}dt = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)F(x+iy)e^{iuy}dy. \tag{4.11}$$

The special function $F_1(x+iy) = 1/(x+iy)$ is the Laplace transform of the function σ_1 which equals 1 for $t \geq 0$ and 0 for $t < 0$. Hence by (4.11) for σ_1 and F_1,

$$\int_0^\infty K_\lambda(u-t)e^{-xt}dt = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)F_1(x+iy)e^{iuy}dy. \tag{4.12}$$

One now subtracts A times this identity from (4.11) and then replaces $F - AF_1$ by G. The result may be written as

$$\begin{aligned}\int_0^\infty K_\lambda(u-t)\sigma(t)e^{-xt}dt &= A\int_0^\infty K_\lambda(u-t)e^{-xt}dt\\ &\quad+\frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)G(x+iy)e^{iuy}dy.\end{aligned} \tag{4.13}$$

Observe that K_λ is in L^1, $\hat{K}_\lambda$ is continuous, while by the hypotheses $G(x+iy)$ tends to $G(iy)$ in $L^1(-\lambda < y < \lambda)$ as $x \searrow 0$. It follows that the right-hand side of (4.13) has a finite limit as $x \searrow 0$, hence so does the left-hand side. Since the product $K_\lambda(u-t)\sigma(t)$ is nonnegative, application of the monotone convergence theorem (cf. Rudin [1966/87] or Korevaar [1968]) now shows that this product is integrable over $(0,\infty)$. Letting $x \searrow 0$, one obtains the 'basic relation' in the proofs by Wiener and Bochner,

$$\int_0^\infty K_\lambda(u-t)\sigma(t)dt = A\int_0^\infty K_\lambda(u-t)dt + \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)G(iy)e^{iuy}dy. \tag{4.14}$$

The Riemann–Lebesgue lemma finally shows that the last term tends to 0 as $u \to \infty$. Substituting $u - t = v/\lambda$ and replacing $K_\lambda(v/\lambda)$ by $\lambda K(v)$, one concludes that

$$\lim_{u\to\infty}\int_{-\infty}^{\lambda u} \sigma(u - v/\lambda)K(v)dv = A\lim_{u\to\infty}\int_{-\infty}^{\lambda u} K(v)dv = A.$$

Proof of Theorem 4.2. We set $e^{-t}S(t) = \sigma(t)$. Then for $\operatorname{Re} z = x > 0$,

$$\begin{aligned}F(z) &\stackrel{\text{def}}{=} \mathcal{L}\sigma(z) = \int_0^\infty S(t)e^{-(z+1)t}dt = \frac{f(z+1)}{z+1},\\ G(z) &\stackrel{\text{def}}{=} F(z) - \frac{A}{z} = \frac{f(z+1)}{z+1} - \frac{A}{z} = \frac{g(z+1)-A}{z+1};\end{aligned} \tag{4.15}$$

cf. (4.4), (4.5). By the hypotheses of the Theorem, σ, F and G will satisfy the conditions of Proposition 4.3, hence we have (4.10). Because $\{K_\lambda\}$ is an approximate identity, the first member of (4.10) must be close to $\sigma(u)$ if σ is well-behaved and λ is large. But is our function σ nice enough?

At this point Wiener [1932] continued with 'Wiener theory', but Bochner's method (we give Landau's version) was more direct. By the monotonicity of S, one has $\sigma(w') \geq \sigma(w)e^{w-w'}$ if $w' \geq w$. For any number $a > 0$, (4.10) thus shows that

$$
\begin{aligned}
A &= \lim_{u\to\infty} \int_{-\infty}^{\lambda u} \sigma(u - v/\lambda) K(v) dv \geq \limsup_{u\to\infty} \int_{-a}^{a} \sigma(u - v/\lambda) K(v) dv \\
&\geq \limsup_{u\to\infty} \sigma(u - a/\lambda) e^{-2a/\lambda} \int_{-a}^{a} K(v) dv.
\end{aligned}
$$

As a result,

$$
\limsup_{u\to\infty} \sigma(u) \leq \frac{e^{2a/\lambda}}{\int_{-a}^{a} K(v) dv} A. \tag{4.16}
$$

Taking $a = \sqrt{\lambda}$ one obtains an upper bound for the lim sup of the form $C(\lambda)A$, where $C(\lambda)$ is close to 1 for large λ. Under the hypotheses of Theorem 4.2 we may indeed let λ go to ∞ to conclude that

$$
\limsup_{u\to\infty} \sigma(u) \leq A. \tag{4.17}
$$

Letting λ go to ∞ or not, we know now that σ is bounded by some constant M. Taking $b > 0$ and observing that $K(v) \leq 2/(\pi v^2) < 1/v^2$, one obtains an estimate from below:

$$
\begin{aligned}
&\liminf_{u\to\infty} \sigma(u + b/\lambda) e^{2b/\lambda} \int_{-b}^{b} K(v) dv \geq \liminf_{u\to\infty} \int_{-b}^{b} \sigma(u - v/\lambda) K(v) dv \\
&\geq \lim_{u\to\infty} \int_{\mathbb{R}} \cdots - \limsup_{u\to\infty} \int_{-\infty}^{-b} \cdots - \limsup_{u\to\infty} \int_{b}^{\infty} \cdots \\
&\geq A - 2M \int_{b}^{\infty} (1/v^2) dv = A - 2M/b.
\end{aligned}
$$

This gives a lower bound for $\liminf_{u\to\infty} \sigma(u)$ which for large λ and related large b is as close to A as one wishes. Conclusion:

$$
\liminf_{u\to\infty} \sigma(u) \geq A.
$$

This completes the proof. □

For fixed λ, a more precise analysis would give a lower bound for the lim inf of the form $c(\lambda)A$, where $c(\lambda)$ is close to 1 when λ is large; cf. Landau [1932b]. A sharp 'finite form' of the Wiener–Ikehara theorem is given in the next section.

Remarks 4.4. Many authors have obtained extensions of the Wiener–Ikehara theorem and these involve a variety of boundary behavior. We mention Raikov [1938], Avakumović [1940a], [1940b], [1940c], Ingham [1941] (see Sections IV.21, IV.22 below), Agmon [1953], Delange [1954], [1955b], Subhankulov [1973] and [1976] (chapter 5), Diamond [1975], Graham and Vaaler [1981], Aramaki [1988], [1996], Tenenbaum [1995], and Čížek [1999]. Among other things, Subhankulov allowed finite sums in (4.2) of the form $\sum_{m\geq 1} A_m z/(z-1)^m$ instead of $A/(z-1)$; see also Čížek. Several of the authors obtained a form of the Wiener–Ikehara theorem with remainder. Pseudofunction boundary behavior [of $g(z) = f(z) - A/z - 1$] is referred to in Remarks 14.7 below.

In the special case of Dirichlet series (2.1) with *multiplicative* coefficients $a_n = \phi(n)$ of absolute value ≤ 1, Halász [1968] could use the behavior of $f(z) = f(x+iy)$ near the line $x = 1$ to obtain very refined results on the partial sums $s_n = \sum_{k\leq n} a_k$. He could thus extend earlier number-theoretic work on the mean values of arithmetic functions, due to Delange and Wirsing; cf. Elliott [1979].

5 Newer Approach to Wiener–Ikehara

The treatment below is based on the work of Graham and Vaaler [1981]. They were able to obtain a precise 'finite form' of the Wiener–Ikehara theorem with the aid of 'extremal majorants'. The majorants are optimal, relative to L^1 distance, among (the restrictions to $\mathbb{R}$ of) entire functions of prescribed exponential type. For some simple step functions such majorants were considered earlier by Beurling (unpublished) and Selberg; see the latter's *Collected papers* [1991] (vol. 2, pp 213–218, 225–226) and cf. Montgomery [1978] (lemma 5) and [1994]. Extending this work, Graham and Vaaler obtained extremal majorants for a larger class of functions; cf. also Vaaler [1985], Holt and Vaaler [1996].

For the present application, one has to approximate the function

$$E(t) = \begin{cases} e^{-t} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{5.1}$$

Taking $\lambda > 0$ we consider majorants and minorants for E *of type* λ. By this we mean integrable majorants $M_\lambda = M_{\lambda,E}$ and minorants $m_\lambda = m_{\lambda,E}$ on $\mathbb{R}$ which are the restrictions of entire functions of exponential type $\leq \lambda$. By the Paley–Wiener theorem [1934], the Fourier transforms $\hat{M}_\lambda$ and $\hat{m}_\lambda$ must have their support in $[-\lambda, \lambda]$.

Theorem 5.1. *Let $S(t)$ vanish for $t < 0$, be nondecreasing, continuous from the right and such that the Laplace–Stieltjes transform*

$$f(z) = \mathcal{L}dS(z) = \int_{0-}^{\infty} e^{-zt} dS(t), \quad z = x + iy, \tag{5.2}$$

exists for $\operatorname{Re} z = x > 1$. *Suppose that for some number $\lambda > 0$, there is a constant A (necessarily ≥ 0) such that the analytic function*

$$g(z) = f(z) - \frac{A}{z-1}, \quad z = x + iy, \;\; x > 1, \tag{5.3}$$

converges to a boundary function $g(1+iy)$ in $L^1(-\lambda < y < \lambda)$ as $x \searrow 1$. Then for any majorant $M_\lambda = M_{\lambda,E}$ and minorant $m_\lambda = m_{\lambda,E}$ of E of type λ,

$$A \int_{\mathbb{R}} m_\lambda(t)dt \leq \liminf_{t\to\infty} e^{-t} S(t) \leq \limsup_{t\to\infty} e^{-t} S(t) \leq A \int_{\mathbb{R}} M_\lambda(t)dt. \tag{5.4}$$

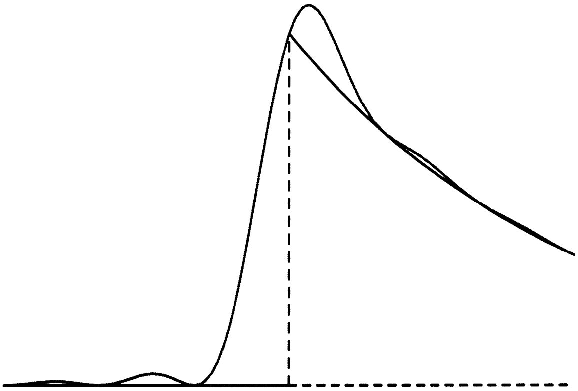

Fig. III.5. The functions E and M_λ^*

Proposition 5.2. *Among the majorants $M_\lambda = M_{\lambda,E}$ for E of type λ is the function*

$$M_\lambda^*(t) = \left(\frac{\sin \lambda t/2}{\lambda/2}\right)^2 \left\{\sum_{n=0}^{\infty} \frac{e^{-n\omega}}{(t-n\omega)^2} - \sum_{n=1}^{\infty} e^{-n\omega}\left(\frac{1}{t-n\omega} - \frac{1}{t}\right)\right\}, \tag{5.5}$$

where $\omega = 2\pi/\lambda$; see Figure III.5. One has

$$\int_{\mathbb{R}} M_\lambda^*(t)dt = \frac{\omega}{1-e^{-\omega}} = \frac{2\pi/\lambda}{1-e^{-2\pi/\lambda}}. \tag{5.6}$$

Among the minorants $m_\lambda = m_{\lambda,E}$ for E of type λ is the function

$$m_\lambda^*(t) = M_\lambda^*(t) - \left(\frac{\sin \lambda t/2}{\lambda t/2}\right)^2, \tag{5.7}$$

for which

$$\int_{\mathbb{R}} m_\lambda^*(t)dt = \frac{\omega}{e^{\omega}-1} = \frac{2\pi/\lambda}{e^{2\pi/\lambda}-1}. \tag{5.8}$$

Remarks 5.3. The proof of the Proposition will show that M_λ^* interpolates E on the sequence $\{n\omega\}$ in the 'strong' sense that

$$M_\lambda^*(n\omega) = E(n\omega), \;\; \forall n \quad \text{and} \quad (M_\lambda^*)'(n\omega) = E'(n\omega), \;\; \forall n \neq 0 \tag{5.9}$$

(Figure III.5). Similarly m_λ^* strongly interpolates the function $\tilde{E}$ which is equal to E for $t \neq 0$, but equal to 0 for $t = 0$.

For a class of upper-semicontinuous L^1 functions F, including $F(t) = E(\omega t)$, Graham and Vaaler constructed 'extremal majorants' of type 2π. The general form is

$$M_F^*(t) = \left(\frac{\sin \pi t}{\pi}\right)^2 \left\{\sum_{n=-\infty}^{\infty} \frac{F(n)}{(t-n)^2} + \sum_{n\neq 0} F'(n)\left(\frac{1}{t-n} - \frac{1}{t}\right)\right\}. \tag{5.10}$$

One may derive from their work that M_λ^* in (5.5) is the unique extremal majorant for E of type λ; it minimizes $\int_{\mathbb{R}}(M_\lambda - E)$, or equivalently, minimizes $\hat{M}_\lambda(0) = \int_{\mathbb{R}} M_\lambda$. Similarly m_λ^* is the unique extremal minorant for E of type λ; it minimizes $\int_{\mathbb{R}}(E - m_\lambda)$ or maximizes $\hat{m}_\lambda(0)$.

Theorem 5.4. *Let S, $f = \mathcal{L}dS$ and $g(z) = f(z) - A/(z-1)$ be as in Theorem 5.1. Let $\mu > 0$ be the supremum of the numbers λ such that $g(x+iy)$ converges to a boundary function $g(1+iy)$ in $L^1(-\lambda < y < \lambda)$ as $x \searrow 1$. Then*

$$\frac{2\pi/\mu}{e^{2\pi/\mu} - 1} A \le \liminf_{t\to\infty} e^{-t} S(t) \le \limsup_{t\to\infty} e^{-t} S(t) \le \frac{2\pi/\mu}{1 - e^{-2\pi/\mu}} A, \tag{5.11}$$

where the bounds should be read as A if $\mu = \infty$. The bounds in (5.11) *are sharp.*

Proof of Theorem 5.1. Take $t > 0$. Aiming for an upper bound of $e^{-t}S(t) = e^{-t}\int_{0-}^{t} dS(u)$, we initially insert a factor $e^{-(x-1)u}$ with $x > 1$ into the integral to facilitate the analysis. By the definition of E and a majorant M_λ for E of type λ,

$$\begin{aligned} e^{-t}\int_{0-}^{t} e^{-(x-1)u} dS(u) &= \int_{0-}^{t} e^{-(t-u)} e^{-xu} dS(u) \\ &= \int_{0-}^{\infty} E(t-u) e^{-xu} dS(u) \le \int_{0-}^{\infty} M_\lambda(t-u) e^{-xu} dS(u). \end{aligned} \tag{5.12}$$

[The second equality holds for all $t > 0$ for which $S(\cdot)$ is continuous, hence outside a countable set; by continuity, the final member provides an upper bound for the first member for all $t > 0$.] The Fourier transform $\hat{M}_\lambda$ is continuous and has support in $[-\lambda, \lambda]$. Since M_λ is smooth it is pointwise equal to the inverse Fourier transform of $\hat{M}_\lambda$:

$$M_\lambda(t-u) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y) e^{i(t-u)y} dy.$$

Substitution in (5.12), inversion of the order of integration and the definition $\mathcal{L}dS = f$ now give

$$\begin{aligned} \int_{0-}^{\infty} M_\lambda(t-u) e^{-xu} dS(u) &= \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y) e^{ity} dy \int_{0-}^{\infty} e^{-(x+iy)u} dS(u) \\ &= \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y) f(x+iy) e^{ity} dy. \end{aligned} \tag{5.13}$$

Observe that the 'singular part' of $f(x+iy)$ for $x \searrow 1$ and $-\lambda < y < \lambda$ is $A/(x+iy-1)$, which is equal to $\int_0^\infty e^{-(x+iy)u} Ae^u du$. In the special case where $dS(u) = Ae^u du$ for $u \ge 0$ (and 0 for $u < 0$), identity (5.13) shows that

$$\frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y) \frac{A}{x+iy-1} e^{ity} dy = \int_0^\infty M_\lambda(t-u) e^{-xu} Ae^u du. \tag{5.14}$$

Combining (5.12)–(5.14) with the relation $f = g + \textit{singular part}$, one thus obtains the inequality

$$
\begin{aligned}
e^{-t}\int_{0-}^{t} e^{-(x-1)u}dS(u) &\le \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y)f(x+iy)e^{ity}dy \\
&= \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y)g(x+iy)e^{ity}dy + A\int_{0}^{\infty} M_\lambda(t-u)e^{-(x-1)u}du.
\end{aligned}
$$

At this point one can pass to the limit as $x \searrow 1$ because $g(x+iy)$ has 'good' boundary behavior for $-\lambda < y < \lambda$ and M_λ is in L^1. The result is

$$
e^{-t}S(t) \le \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{M}_\lambda(y)g(1+iy)e^{ity}dy + A\int_{0}^{\infty} M_\lambda(t-u)du. \tag{5.15}
$$

One may finally let t go to ∞. The first integral in (5.15) involves the integrable function $\hat{M}_\lambda(y)g(1+iy)$ and the factor e^{ity}, hence by the Riemann–Lebesgue lemma, it has limit 0. The final term is equal to

$$
A\int_{-\infty}^{t} M_\lambda(v)dv, \quad \text{with limit} \quad A\int_{\mathbb{R}} M_\lambda(v)dv.
$$

Thus for $t \to \infty$, (5.15) gives the desired upper bound for the lim sup in (5.4):

$$
\limsup_{t\to\infty} e^{-t}S(t) \le A\int_{\mathbb{R}} M_\lambda(v)dv.
$$

The proof for the lower bound of the lim inf is similar. □

We will deal with Proposition 5.2 later.

Derivation of Theorem 5.4. By Theorem 5.1 applied to the majorant M_λ^* in (5.5) and by (5.6),

$$
\limsup_{t\to\infty} e^{-t}S(t) \le A\int_{\mathbb{R}} M_\lambda^*(t)dt = \frac{2\pi/\lambda}{1-e^{-2\pi/\lambda}}A. \tag{5.16}
$$

By the hypothesis this holds for every $\lambda < \mu$ and thus one obtains the (smaller) upper bound in (5.11). The proof for the lower bound of the lim inf is similar.

To show that the bounds are sharp, Graham and Vaaler used the function S corresponding to the measure dS, defined by point masses $A(2\pi/\mu)e^{2\pi k/\mu}$ at the points $2\pi k/\mu$ for $k = 1, 2, \ldots$. In this case the transform

$$
f(z) = \mathcal{L}dS(z) = \frac{2\pi A/\mu}{e^{(2\pi/\mu)(z-1)} - 1}
$$

has simple poles with residue A at the points $z = 1 + n\mu i,\ n \in \mathbb{Z}$, so that the hypotheses of Theorem 5.4 are satisfied. At the same time, the first and the last inequality in (5.11) will be equalities.

Proof of Proposition 5.2. It will suffice to consider majorants.

(i) The change of scale $t \Rightarrow \omega t$ replaces the problem of majorizing $E(t)$ by functions $M_\lambda(t)$ of type λ by the equivalent problem of majorizing $F(t) = E(\omega t)$ by functions of type $\lambda\omega = 2\pi$. We thus set out to show that

$$M^*(t) = \left(\frac{\sin \pi t}{\pi}\right)^2 Q(t), \quad \text{with}$$

$$Q(t) = \sum_{n=0}^{\infty} \frac{e^{-n\omega}}{(t-n)^2} + \sum_{n=1}^{\infty} (-\omega) e^{-n\omega} \left(\frac{1}{t-n} - \frac{1}{t}\right), \tag{5.17}$$

is a majorant for the function $E(\omega t)$ which is equal to $e^{-\omega t}$ for $t \geq 0$ and equal to 0 for $t < 0$.

(ii) To this end we introduce the auxiliary function

$$R(v) = \frac{v}{1 - e^{-v}}, \quad \text{for which } 0 < R'(v) < 1, \ \forall v \in \mathbb{R}. \tag{5.18}$$

Straightforward calculation gives the following formulas for $t < 0$:

$$\int_0^\infty R(v+\omega) e^{tv} dv = e^{-\omega t} \int_\omega^\infty R(u) e^{tu} du$$

$$= \sum_{n=0}^{\infty} e^{-n\omega} \left\{ \frac{1}{(t-n)^2} - \frac{\omega}{t-n} \right\},$$

$$\int_0^\infty R(\omega) e^{tv} dv = -\frac{R(\omega)}{t} = -\sum_{n=0}^{\infty} e^{-n\omega} \frac{\omega}{t},$$

$$\int_0^\infty \{R(v+\omega) - R(\omega)\} e^{tv} dv = Q(t);$$

cf. (5.17). By (5.18) one has $0 < R(v+\omega) - R(\omega) < v$ for $v > 0$, hence

$$0 < Q(t) < \int_0^\infty v e^{tv} dv = \frac{1}{t^2}, \quad 0 \leq M^*(t) \leq \left(\frac{\sin \pi t}{\pi t}\right)^2 \quad \text{for } t < 0. \tag{5.19}$$

Similarly for t positive (but different from an integer),

$$\int_{-\infty}^0 R(v+\omega) e^{tv} dv = e^{-\omega t} \left(\int_{-\infty}^0 + \int_0^\omega \right) R(u) e^{tu} du$$

$$= e^{-\omega t} \sum_{n=-\infty}^{\infty} \frac{1}{(t-n)^2} - \sum_{n=0}^{\infty} e^{-n\omega} \left\{ \frac{1}{(t-n)^2} - \frac{\omega}{t-n} \right\},$$

$$\int_{-\infty}^0 R(\omega) e^{tv} dv = \frac{R(\omega)}{t} = \sum_{n=0}^{\infty} e^{-n\omega} \frac{\omega}{t}.$$

Combination of these relations with the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(t-n)^2} = \left(\frac{\pi}{\sin \pi t}\right)^2$$

shows that for nonintegral $t > 0$,

$$\int_{-\infty}^0 \{R(\omega) - R(v+\omega)\} e^{tv} dv = Q(t) - e^{-\omega t} \left(\frac{\pi}{\sin \pi t}\right)^2.$$

Using the inequality $0 < R(\omega) - R(v+\omega) < |v|$ for $v < 0$; cf. (5.18), and multiplying through by $\{(\sin \pi t)/\pi\}^2$, one concludes from (5.17) that

$$0 \le M^*(t) - e^{-\omega t} \le \left(\frac{\sin \pi t}{\pi t}\right)^2 \quad \text{for nonintegral } t > 0. \tag{5.20}$$

By continuity this holds also for positive integers t.

(iii) Formulas (5.19), (5.20) show that $M^*(t)$ majorizes $E(\omega t)$ for all $t \neq 0$; at the origin both functions are equal to 1. The same formulas show that $M^*(t)$ strongly interpolates $E(\omega t)$ on the integers in the sense of (5.9) (where one now has to substitute $\lambda = 2\pi$, $\omega = 1$).

(iv) Replacing t by t/ω one concludes that $M_\lambda^*(t) = M^*(t/\omega)$ in formula (5.5) is a majorant for $E(t)$. It is easy to verify that M_λ^* is in L^1. It is the restriction of an entire function of exponential type $\le \lambda$ and hence its Fourier transform will have support in $[-\lambda, \lambda]$. The latter may be confirmed by direct computation of the transform. The value of $\int_{\mathbb{R}} M_\lambda^*(t)dt$ in (5.6) may be verified with the aid of the formulas

$$\int_{\mathbb{R}} \left(\frac{\sin \lambda t/2}{\lambda/2}\right)^2 \frac{dt}{(t-n\omega)^2} = \frac{2}{\lambda} \int_{\mathbb{R}} \frac{\sin^2 v}{v^2} dv = \frac{2\pi}{\lambda},$$

$$\int_{\mathbb{R}} \left(\frac{\sin \lambda t/2}{\lambda/2}\right)^2 \frac{dt}{t-n\omega} = 0.$$

□

6 Newman's Way to the PNT. Work of Ingham

The starting point was a theorem of Ingham [1935] for Dirichlet series. Newman [1980], also [1998], gave a simple proof for the theorem by contour integration. The Tauberian condition is given by boundedness of the coefficients:

Theorem 6.1. *Let $f(z)$ be given in the half-plane $\{\operatorname{Re} z > 1\}$ by a Dirichlet series $\sum_{n=1}^\infty a_n/n^z$ with $|a_n| \le C$ for all n. Suppose that $f(z)$ has an analytic extension to a neighborhood of every point of the line $\{\operatorname{Re} z = 1\}$. Then*

$$\textit{the series } \sum_{n=1}^\infty \frac{a_n}{n} \textit{ is convergent.} \tag{6.1}$$

If not $f(z)$ itself, but $g(z) = f(z) - A/(z-1)$ has an analytic extension to the closed half-plane $\{\operatorname{Re} z \ge 1\}$, or equivalently, if

$$h(z) = f(z) - A\zeta(z) = \sum_{n=1}^\infty \frac{a_n - A}{n^z} \tag{6.2}$$

has such an extension, the conclusion is that

$$\textit{the series} \quad \sum_{n=1}^{\infty} \frac{a_n - A}{n} \quad \textit{is convergent}; \tag{6.3}$$

cf. the weaker conclusion in Theorem 2.2 (under a weaker hypothesis).

Newman described two ways in which the PNT can be deduced from Theorem 6.1. The simplest was to consider the reciprocal of the zeta function. As we know, $\zeta(z)$ is analytic and different from 0 throughout the closed half-plane $\{\operatorname{Re} z \geq 1\}$, except that it has a simple pole at the point $z = 1$; cf. Section I.26. Thus $f(z) = 1/\zeta(z)$ is analytic for $\operatorname{Re} z \geq 1$. It can be expanded in a Dirichlet series as follows:

$$f(z) = \frac{1}{\zeta(z)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z};$$

the coefficients $\mu(n)$ define the Möbius function; cf. Section I.4. Since one has $|\mu(n)| \leq 1$ for all n, Theorem 6.1 shows that the series $\sum_{n=1}^{\infty} \mu(n)/n$ is convergent. From this fact the PNT may be obtained by appropriate manipulation, as was first shown by Landau [1912].

A somewhat more direct proof of the PNT may be obtained from an analog of Theorem 6.1 for Laplace transforms (Korevaar [1982]; cf. Zagier [1997], Lang [1999]). Set

$$s(v) = \sum_{n \leq v} a_n, \quad v = e^t, \quad e^{-t}s(e^t) = \alpha(t),$$

so that boundedness of the sequence $\{a_n\}$ implies boundedness of the function α. Furthermore, formal integration by parts suggests that convergence of the series $\sum_{n=1}^{\infty} a_n/n$ corresponds to convergence of the improper integral $\int_0^{\infty-} \alpha(t)dt$:

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \int_{1-}^{\infty-} \frac{ds(v)}{v} = \int_1^{\infty-} \frac{s(v)}{v^2} dv = \int_0^{\infty-} \alpha(t)dt.$$

[To make this rigorous one would have to show that $s(v)/v \to 0$ as $v \to \infty$.] Finally, the original Dirichlet series will correspond to a Laplace integral involving α; see Sections 7, 8 for precise results. Going back to Dirichlet series, the condition on $\{a_n\}$ in Theorem 6.1 can now be relaxed to $a_n \geq -C$ provided $s(v) = \mathcal{O}(v)$; see Theorem 8.1 below. As a special case we obtain the convergence (and the sum) of the series $\sum_{n=1}^{\infty} \{\Lambda(n) - 1\}/n$; the PNT is an easy consequence (see Section I.10). Related Tauberian results for Dirichlet series have been given by Delange [1997].

With more work, the condition $|a_n| \leq C$ in Theorem 6.1 may actually be replaced by the single condition $a_n \geq -C$, provided the Dirichlet series is known to converge for $\operatorname{Re} z > 1$; see Theorem 9.2.

After all is said and done, the way to the PNT via the Wiener–Ikehara theorem remains the most direct. It is an *open problem* if one can derive the latter theorem by complex analysis, for example, for the case of Dirichlet series with bounded coefficient sequence $\{a_n\}$!

Let us return for a moment to Theorem 6.1 with $f(z) = 1/\zeta(z)$. Translation in the vertical direction will show that the series $\sum_{n=1}^{\infty} \mu(n)/n^{1+iy}$ converges to $1/\zeta(1+iy)$

for every real y. This is interesting because the Dirichlet series $\sum_{n=1}^{\infty} 1/n^{1+iy}$ for $\zeta(1+iy)$ itself is *divergent* for every real y; cf. Section I.25. The example

$$f(z) = \zeta(z - ib) = \sum_{n=1}^{\infty} n^{ib}/n^{z}, \quad \operatorname{Re} z > 1,$$

with real $b \neq 0$ thus shows that in Theorem 6.1 it is not enough for $f(z)$ to have an analytic extension to a region $\{\operatorname{Re} z \geq 1,\ z \neq 1 + ib\}$. A pole at just one point of the line $\{\operatorname{Re} z = 1\}$ is too much, but a slightly weaker singularity would be permissible; cf. Remarks 7.2.

In [1935] Ingham proved a number of complex Tauberian theorems for Laplace–Stieltjes transforms which imply results for general Dirichlet series and ordinary Laplace transforms. Theorem 6.1 (also with the one-sided condition) is contained in Ingham's theorem III and its specialization theorem $3(l)$ for Dirichlet series. His results for Laplace transforms are essentially included in Theorems 7.1, 9.2 and 10.1 below. For his proofs, Ingham extended the method which Bochner [1933a] and Heilbronn and Landau [1933a], [1933c]; cf. Landau [1932a], [1932b], had used for the Wiener–Ikehara theorem. Besides the Fejér kernel for $\mathbb{R}$ of (4.7), Ingham used the 'Jackson kernel' for $\mathbb{R}$ (cf. Jackson [1930]), which is the (suitably normalized) square of the Fejér kernel. Cf. also Karamata [1936]. Our treatment is different, and starts with Newman's contour method.

We add the remark that Newman's complex method has recently led to refinement and extension of Ingham's results; see Sections 13, 14.

7 Laplace Transforms of Bounded Functions

We will extend Newman's contour integration method (Newman [1980]) to derive the following result; cf. Korevaar [1982], Zagier [1997]. The theorem itself is contained in Karamata [1934] (theorem B) and Ingham [1935] (theorem III).

Theorem 7.1. *Let the function* $\alpha(\cdot)$ *vanish on* $(-\infty, 0)$ *and be bounded on* $[0, \infty)$, *so that the Laplace transform*

$$F(z) = \mathcal{L}\alpha(z) = \int_0^{\infty} \alpha(t) e^{-zt} dt, \quad z = x + iy \tag{7.1}$$

is well-defined and analytic throughout the open half-plane $\{x = \operatorname{Re} z > 0\}$. *Suppose that* $F(\cdot)$ *can be continued analytically to a neighborhood of every point on the imaginary axis. Then the improper integral*

$$\int_0^{\infty-} \alpha(t) dt \quad \text{exists and equals} \quad F(0). \tag{7.2}$$

More generally, by translation in the vertical direction,

$$\int_0^{\infty-} \alpha(t)e^{-iyt}dt = F(iy) \quad \text{for every real number } y.$$

The Tauberian condition is given by the boundedness of $\alpha(\cdot)$.

Proof of Theorem 7.1. Changing α on $[0, 1]$ to $\alpha - F(0)$, the new $F = \mathcal{L}\alpha$ has $F(0) = 0$. Dividing α by a constant, one may also assume that $\sup |\alpha(t)| \leq 1$. For $0 < B < \infty$ we define

$$F_B(z) = \int_0^B \alpha(t)e^{-zt}dt. \tag{7.3}$$

It must now be shown that

$$F_B(0) = \int_0^B \alpha(t)dt \to F(0) = 0 \quad \text{as } B \to \infty. \tag{7.4}$$

(i) One begins with some simple estimates. For $x = \operatorname{Re} z > 0$,

$$|F_B(z) - F(z)| = \left| \int_B^{\infty} \alpha(t)e^{-zt}dt \right| \leq \int_B^{\infty} e^{-xt}dt = \frac{1}{x}e^{-Bx}. \tag{7.5}$$

Similarly for $x = \operatorname{Re} z < 0$,

$$|F_B(z)| = \left| \int_0^B \alpha(t)e^{-zt}dt \right| \leq \int_0^B e^{-xt}dt < \frac{1}{|x|}e^{-Bx}. \tag{7.6}$$

(ii) For given $R > 0$, let Γ be the positively oriented circle $C(0, R) = \{|z| = R\}$. We let Γ_1 be the part of Γ in the half-plane $\{x = \operatorname{Re} z > 0\}$, Γ_2 the part in the half-plane $\{x < 0\}$. Finally, let σ be the oriented segment of the imaginary axis from $+iR$ to $-iR$ (Figure III.7). Observe that for $z \in \Gamma$, one has

$$\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2}. \tag{7.7}$$

Since $F(0) = 0$, the quotient $F(z)/z$ is analytic for $x \geq 0$. Observe also that F_B is analytic everywhere. Formulas (7.5)–(7.7) motivate the following ingenious application of Cauchy's theorem and Cauchy's formula due to Newman:

$$0 = \frac{1}{2\pi i}\int_{\Gamma_1+\sigma} \frac{F(z)}{z}dz = \frac{1}{2\pi i}\int_{\Gamma_1+\sigma} F(z)e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz, \tag{7.8}$$

$$F_B(0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{F_B(z)}{z}dz = \frac{1}{2\pi i}\int_{\Gamma} F_B(z)e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz. \tag{7.9}$$

Subtracting (7.8) from (7.9) and rearranging the result, one obtains the formula

$$\begin{aligned} F_B(0) &= \frac{1}{2\pi i}\int_{\Gamma_1} \{F_B(z) - F(z)\}e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_2} F_B(z)e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz - \frac{1}{2\pi i}\int_{\sigma} F(z)e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz \\ &= T_1 + T_2 + T_3, \end{aligned} \tag{7.10}$$

say.

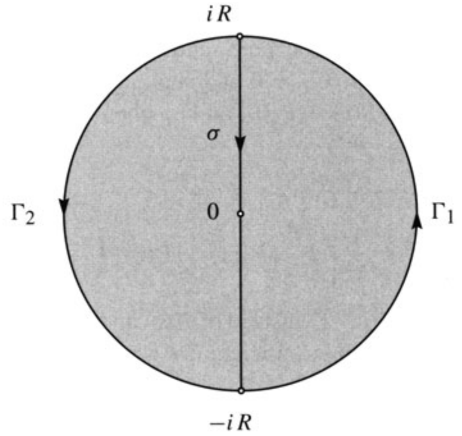

Fig. III.7. The paths of integration

(iii) By (7.5) and (7.7) for $z \in \Gamma_1$, the integrand $F^*(z)$ in T_1 can be estimated as follows:

$$|F^*(z)| = \left|\{F_B(z) - F(z)\}e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)\right| \le \frac{1}{x}e^{-Bx}e^{Bx}\frac{2x}{R^2} = \frac{2}{R^2}.$$

Thus

$$|T_1| \le \frac{1}{2\pi}\int_{\Gamma_1} |F^*(z)||dz| \le \frac{1}{2\pi}\frac{2}{R^2}\pi R = \frac{1}{R}. \tag{7.11}$$

In the same way (7.6) and (7.7) for $z \in \Gamma_2$ imply the estimate

$$|T_2| = \left|\frac{1}{2\pi i}\int_{\Gamma_2} F_B(z)e^{Bz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| < \frac{1}{R}. \tag{7.12}$$

In order to deal with

$$T_3 = \frac{1}{2\pi}\int_{-R}^{R} F(iy)\left(\frac{1}{iy} + \frac{iy}{R^2}\right)e^{iBy}dy, \tag{7.13}$$

one may apply integration by parts: $e^{iBy}dy = de^{iBy}/(iB)$, etc., or one may use the Riemann–Lebesgue lemma. Either method will show that for fixed R,

$$T_3 = T_3(R, B) \to 0 \quad \text{as} \quad B \to \infty. \tag{7.14}$$

(iv) *Conclusion.* For given $\varepsilon > 0$ one may choose $R = 1/\varepsilon$. One next determines B_0 so large that $|T_3|$ is bounded by ε for all $B \ge B_0$. Then by (7.11)–(7.14),

$$|F_B(0)| < 3\varepsilon \quad \text{for} \quad B \ge B_0.$$

In other words, $F_B(0) \to 0 = F(0)$ as $B \to \infty$. □

Remarks 7.2. In Theorem 7.1, the condition that F be analytic on the imaginary axis can be relaxed considerably. For more refined results, it is desirable to have an inequality for the case where $F(0) \neq 0$ and $\sup_{t>0} |\alpha(t)| = M \neq 1$. Estimating $|F_B(z) - F(z)|$ on Γ_1 and $|F_B(z) - F(0)|$ on Γ_2, where $|x|e^{Bx} \leq 1/(eB)$, one finds that

$$\begin{aligned} &|F_B(0) - F(0)| \\ &\leq \frac{2M}{R} + \frac{|F(0)|}{eBR} + \frac{1}{2\pi}\left| \int_{-R}^{R} \{F(iy) - F(0)\} \left(\frac{1}{iy} + \frac{iy}{R^2} \right) e^{iBy} dy \right|. \end{aligned} \tag{7.15}$$

Applying this inequality to $F(\varepsilon + z)$ instead of $F(z)$ and then letting ε go to zero, one concludes that in Theorem 7.1, it is sufficient if $F(x)$ tends to a limit $F(0)$ as $x \searrow 0$ and in addition, the quotient

$$Q(x + iy) = \frac{F(x + iy) - F(x)}{iy}$$

satisfies one of the following three conditions. (i) It can be extended continuously to the closed half-plane $x \geq 0$; (ii) the quotient tends to a limit function $Q(iy)$ as $x \searrow 0$ in $L^1(-R < y < R)$ for every $R > 0$; (iii) $Q(x + iy)$ converges distributionally on every interval $\{-R < y < R\}$ to a so-called pseudofunction $Q(iy) = Q_R(iy)$ as $x \searrow 0$. For case (iii), cf. Theorem 14.6 below. If one has convergence of $Q(x + iy)$ only on *some* interval $\{-R < y < R\}$, analysis will give a 'finite form' of Theorem 7.1:

$$\limsup_{B\to\infty} \left| \int_0^B \alpha(t)dt - F(0) \right| \leq \frac{2M}{R}; \tag{7.16}$$

cf. Korevaar [2003] for the distributional case. In Theorem 7.1, the boundedness of $\alpha(t)$ for $t \geq 0$ may be relaxed to local integrability on $\mathbb{R}^+$ and boundedness for $t \geq t_0$. For real α, the boundedness condition can be relaxed further to one-sided boundedness, provided the Laplace transform $F(z) = \mathcal{L}\alpha(z)$ is known to exist for $\operatorname{Re} z > 0$; cf. Section 9.

8 Application to Dirichlet Series and the PNT

Theorem 7.1 implies the following result for classical Dirichlet series.

Theorem 8.1. *Let* $f(z)$ *be given for* $\operatorname{Re} z > 1$ *by a convergent Dirichlet series* $\sum_{n=1}^{\infty} a_n/n^z$ *with* $a_n \geq -C$ *and* $s(v) = \sum_{n\leq v} a_n = \mathcal{O}(v)$. *Suppose that*

$$g(z) = f(z) - \frac{A}{z-1} \quad or \quad h(z) = f(z) - A\zeta(z)$$

has an analytic extension to the closed half-plane $\{\operatorname{Re} z \geq 1\}$. *Then*

$$\frac{1}{u}s(u) \to A \ \ as\ \ u \to \infty \quad and \quad \sum_{n=1}^{\infty} \frac{a_n - A}{n} = h(1). \tag{8.1}$$

The Tauberian condition is essentially a positivity condition with a little extra. The extra condition $s(v) = \mathcal{O}(v)$ may be dropped, but this requires a more complicated proof; cf. Theorem 9.2 below.

Proof of Theorem 8.1. Define

$$\alpha(t) = \frac{s(e^t) - A[e^t]}{e^t} = e^{-t} \sum_{1 \le n \le e^t} (a_n - A). \tag{8.2}$$

Then $\alpha(t) = 0$ for $t < 0$ and α is bounded for $t \ge 0$. For $\operatorname{Re} z > 0$ the Laplace transform is equal to

$$\begin{aligned} F(z) = \mathcal{L}\alpha(z) &= \int_0^\infty \{s(e^t) - A[e^t]\} e^{-(z+1)t} dt \\ &= \int_1^\infty \{s(v) - A[v]\} v^{-z-2} dv = \frac{1}{z+1} \int_{1-}^\infty v^{-z-1} d\{s(v) - A[v]\} \\ &= \frac{1}{z+1} \sum_{n=1}^\infty \frac{a_n - A}{n^{z+1}} = \frac{f(z+1) - A\zeta(z+1)}{z+1} = \frac{h(z+1)}{z+1}. \end{aligned} \tag{8.3}$$

The hypotheses imply that the transform has an analytic extension to the closed half-plane $\operatorname{Re} z \ge 0$. Thus by Theorem 7.1

$$\begin{aligned} \int_0^{\infty-} \alpha(t) dt &= \int_0^{\infty-} \frac{s(e^t) - A[e^t]}{e^t} dt \\ &= \int_1^{\infty-} \frac{s(v) - A[v]}{v^2} dv = F(0) = h(1). \end{aligned} \tag{8.4}$$

(i) We will verify that the condition $a_n \ge -C$ and the convergence of the integrals in (8.4) imply $s(u) \sim A[u] \sim Au$.

Adding a suitable constant C' to the a_n and A, we make $a_n > 0$, $\forall n$ and $A > 0$. This adjustment makes $s(u)$ nondecreasing. It increases $s(u)$ by $C'[u]$ but does not change the integrals in (8.4). Suppose now that $\limsup s(u)/u$ is greater than A. Then there is a number $\delta > 0$ such that for some sequence of u's tending to ∞, one has $s(u) > (A + 2\delta)u$. As a result,

$$s(v) \ge s(u) > (A + 2\delta)u > (A + \delta)v \quad \text{for} \quad u < v < \rho u,$$

where $\rho = (A + 2\delta)/(A + \delta)$. But then for these numbers $u \to \infty$,

$$\int_u^{\rho u} \frac{s(v) - A[v]}{v^2} dv > \int_u^{\rho u} \frac{\delta}{v} dv = \delta \log \rho,$$

and this would contradict the convergence of the third integral in (8.4). Thus $\limsup s(u)/u \le A$.

One similarly shows that one cannot have $\liminf s(u)/u < A$.

(ii) To deal with the sum in (8.1) we use a Stieltjes integral and integration by parts, together with (8.4) and the relation $s(u) \sim Au$:

$$\sum_{n=1}^{N} \frac{a_n - A}{n} = \int_{1-}^{N} \frac{1}{v} d\{s(v) - A[v]\}$$

$$= \frac{s(N) - AN}{N} + \int_1^N \frac{s(v) - A[v]}{v^2} dv \to h(1) \quad \text{as } N \to \infty.$$

Corollary 8.2. *As usual, let $\psi(u)$ be Chebyshev's function $\sum_{n \le u} \Lambda(n)$ and let γ be Euler's constant. Then*

$$\psi(u) \sim u \quad \textit{as } u \to \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n} = -2\gamma. \tag{8.5}$$

Recall that the first relation (8.5) is equivalent to the *prime number theorem*,

$$\pi(u) \sim \frac{u}{\log u};$$

cf. Section I.10.

Proof of the Corollary. We apply Theorem 8.1 with $a_n = \Lambda(n)$, so that $f(z) = -\zeta'(z)/\zeta(z)$ and $s(u) = \psi(u)$; cf. Section 2 or I.4. The boundedness of $s(u)/u$ follows from Chebyshev's classical estimate $\psi(u) = \mathcal{O}(u)$; cf. Section II.17 or Landau [1909]. Since $f(z)$ and $\zeta(z)$ both have a first order pole at $z = 1$ with residue 1, the difference

$$h(z) = f(z) - \zeta(z) = -\frac{\zeta'(z)}{\zeta(z)} - \zeta(z)$$

has an analytic extension to the closed half-plane $\{\operatorname{Re} z \ge 1\}$; cf. Section 2. Thus we may apply Theorem 8.1 with $A = 1$. Formula (8.1) now implies the first relation (8.5) and the convergence of the series. The sum $h(1)$ of the series must be equal to its Lambert sum, which was found to be -2γ in Section I.4. Alternatively, the value of the sum may be obtained from the expansion of the zeta function around the point $z = 1$:

$$\zeta(z) = \frac{1}{z-1} + \gamma + c_1(z-1) + \cdots, \qquad \zeta'(z) = -1/(z-1)^2 + c_1 + \cdots;$$

cf. Section I.26 or Landau (loc. cit.). One thus finds once again that

$$h(1) = -\lim_{z \to 1} \left\{ \frac{\zeta'(z)}{\zeta(z)} + \zeta(z) \right\} = -2\gamma.$$

9 Laplace Transforms of Functions Bounded From Below

We begin with a simple addition to Proposition 4.3.

Lemma 9.1. *Let $\sigma \geq 0$, $F = \mathcal{L}\sigma$, $G(z) = F(z) - A/z$ and $\lambda > 0$ be as in Proposition 4.3, so that in particular $G(x + iy) \to G(iy)$ in $L^1(-\lambda < y < \lambda)$ as $x \searrow 0$. Then for every number $h \geq 2\pi/\lambda$,*

$$\sigma_h(u) = \frac{1}{h} \int_u^{u+h} \sigma(t)dt \leq CA + o(1) \quad \text{as } u \to \infty, \tag{9.1}$$

with an absolute constant $C < 3$.

If λ can be taken arbitrarily large, then actually $\sigma_h(u) \to A$ as $u \to \infty$ for every $h > 0$; cf. Wiener and Pitt [1939], or Pitt [1958] (section 6.1) and Diamond [1972].

Proof of the Lemma. By Proposition 4.3 where $K(v) = (1 - \cos v)/(\pi v^2)$,

$$\int_{-\infty}^{\lambda u} \sigma(u - v/\lambda) K(v) dv = A + o(1) \quad \text{as } u \to \infty. \tag{9.2}$$

Now $K \geq 0$, and for $|v| \leq \pi$ one has $K(v) \geq K(\pi) > 0$, hence by (9.2)

$$A + o(1) \geq K(\pi) \int_{-\pi}^{\pi} \sigma(u - v/\lambda) dv = 2\pi K(\pi) \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} \sigma(u + w) dw.$$

This inequality implies (9.1) with $h = 2\pi/\lambda$ and $C = 1/\{2\pi K(\pi)\} < 3$. Since the hypotheses of the Lemma are *a fortiori* satisfied if we replace λ by a smaller number, conclusion (9.1) holds also for $h > 2\pi/\lambda$.

We can now prove an extension of Theorem 7.1 involving one-sided boundedness. A different proof may be obtained from Ingham [1935] (theorem III).

Theorem 9.2. *Let $\alpha(t)$ vanish for $t < 0$, be bounded from below for $t \geq 0$ and such that the Laplace transform $G(z) = \mathcal{L}\alpha(z)$, $z = x + iy$, exists for $x > 0$. Suppose that $G(\cdot)$ can be continued analytically to the closed half-plane $\{x = \operatorname{Re} z \geq 0\}$. Then the improper integral $\int_0^{\infty-} \alpha(t)dt$ exists and equals $G(0)$.*

The proof and Remarks 7.2 will show that one does not need analyticity of G on the line $\{x = 0\}$. It is enough to know that for some constant $G(0)$ and every $\lambda > 0$, the quotient $\{G(x + iy) - G(0)\}/(x + iy)$ converges in $L^1(-\lambda < y < \lambda)$ to a limit function $H(iy)$ as $x \searrow 0$.

Proof of Theorem 9.2. Shifting α one may assume that $\alpha(t) = 0$ for $t < 1$; such a shift does not affect the value of $G(0)$ or $\int_0^{\infty-} \alpha$. Now choose $A > 0$ such that $\alpha \geq -A$ and set $\sigma = \alpha + A$ on $\mathbb{R}^+$, $\sigma = 0$ on $\mathbb{R}^-$. Then σ, $F = \mathcal{L}\sigma$ and $G(z) = \mathcal{L}\alpha(z) = F(z) - A/z$ satisfy the conditions of Lemma 9.1 for every number $\lambda > 0$.

For $0 < h \leq 1$ we consider the average

$$\alpha_h(u) = \frac{1}{h} \int_u^{u+h} \alpha(t)dt. \tag{9.3}$$

Then $\alpha_h(u) = \sigma_h(u) - A$ with σ_h as in (9.1) when $u > 0$ and $\alpha_h(u) = 0$ for $u < 0$. Also [because $\alpha(t) = 0$ for $t < 1$],

$$G_h(z) = \mathcal{L}\alpha_h(z) = \frac{1}{h}\int_0^\infty e^{-zu}du \int_u^{u+h} \alpha(t)dt$$
$$= \frac{1}{h}\int_0^\infty \alpha(t)dt \int_{t-h}^{t} e^{-zu}du = \frac{e^{hz}-1}{hz}G(z).$$

Thus G_h can be continued analytically to the closed half-plane $\{\mathrm{Re}\, z \geq 0\}$ just like G and $G_h(0) = G(0)$. By Lemma 9.1, σ_h is bounded for every fixed h, hence the same is true for α_h. We can therefore apply Theorem 7.1 to α_h and its Laplace transform to conclude that

$$\lim_{B\to\infty} \int_0^B \alpha_h(u)du = G(0). \tag{9.4}$$

Taking $B \geq 1$ we next compare $\int_0^B \alpha_h$ with $\int_0^B \alpha$. By a short calculation

$$R(h, B) = \int_0^B \alpha_h(u)du - \int_0^B \alpha(t)dt = \int_0^B \sigma_h(u)du - \int_0^B \sigma(t)dt$$
$$= -\int_0^h \sigma(t)\frac{h-t}{h}dt + \int_B^{B+h} \sigma(t)\frac{B+h-t}{h}dt. \tag{9.5}$$

Now $\sigma(t) = A$ on $(0, h)$, hence by Lemma 9.1

$$-\frac{1}{2}Ah \leq R(h, B) \leq \int_B^{B+h} \sigma(t)dt \leq 3Ah + o(1) \quad \text{as} \quad B \to \infty.$$

Thus by (9.4) and (9.5),

$$\limsup_{B\to\infty} \left| \int_0^B \alpha(t)dt - G(0) \right| \leq \limsup_{B\to\infty} |R(h, B)| \leq 3Ah.$$

Since h can be taken arbitrarily small this completes the proof of the Theorem.

10 Tauberian Conditions Other Than Boundedness

In this section we use ideas from the proof of Proposition 4.3 to obtain a convergence theorem involving more general Tauberian conditions.

Theorem 10.1. *Let σ be real, equal to* 0 *on* $\mathbb{R}^-$ *and such that the Laplace transform* $F(z) = \mathcal{L}\sigma(z)$ *exists for* $\mathrm{Re}\, z = x > 0$. *Suppose that for* $x \searrow 0$, $F(x+iy)$ *converges to a boundary function* $F(iy)$ *uniformly or in* L^1 *on some interval* $-\lambda \leq y \leq \lambda$ *with* $\lambda > 0$. *Then each of the following Tauberian conditions implies that* $\sigma(u)$ *tends to* 0 *as* $u \to \infty$:

(i) σ *is slowly decreasing on* $\mathbb{R}$ *(Definition II.2.3): for any given* $\varepsilon > 0$, *there are numbers* $\delta > 0$ *and* B *such that*

$$\sigma(t) \geq \sigma(u) - \varepsilon \quad \textit{whenever} \;\; u + 2\delta \geq t \geq u \geq B, \tag{10.1}$$

AND λ *may be taken arbitrarily large;*

(i′) σ is 'very slowly decreasing' on $\mathbb{R}$ in the sense that

$$\liminf \{\sigma(t) - \sigma(u)\} \geq 0 \quad for \;\; u + 1 \geq t \geq u \to \infty \tag{10.2}$$

(while λ is fixed);

(ii) $\sigma(t)$ is piecewise constant, its intervals of constancy have length at least $2\delta - o(1)$ as $t \to \infty$ and $\lambda\delta > b$, where b is the constant determined by equation (10.9) *below. (Additional work will give the optimal constant $b = \pi/2$, see Section 11 and Example 10.3.)*

In the case of complex σ one may apply the Tauberian conditions to $\sigma^{(1)} = \operatorname{Re}\sigma$ and $\sigma^{(2)} = \operatorname{Im}\sigma$ separately. The transforms $F_j = \mathcal{L}\sigma^{(j)}$ both inherit the good behavior of F.

Convergence results under conditions related to (i) and (i′) may be found in Karamata [1934] and Ingham [1935]. In Ingham [1936] there is a result on Dirichlet series corresponding to the step function condition (ii); cf. condition (ii) in Theorem 12.2 below. See also Pitt [1958] (section 6.1).

For the proof of the Theorem we first establish *boundedness*.

Proposition 10.2. *Let σ and $F = \mathcal{L}\sigma$ be as in the first four lines of the Theorem. Then each of the following conditions implies that σ is bounded:*

(i) σ is slowly decreasing or at least, (10.1) *holds for* SOME *ε, δ, B (no condition on λ required);*

(ii) σ and λ are as in part (ii) of the Theorem.

Proof of the Proposition. (i) Suppose for simplicity that inequality (10.1) can be satisfied for arbitrarily small $\varepsilon > 0$, although this is not necessary for the proof that σ is bounded. For convenience, set

$$\sigma_x(t) = e^{-xt}\sigma(t) \qquad (x > 0).$$

[Not to be confused with σ_h in (9.1).] We verify first that $\sigma_x(t)$ is bounded (and in fact, tends to 0) as $t \to \infty$ for every $x > 0$. Indeed, suppose $\sigma_x(u) \geq 2\varepsilon$ for a sequence of $u \to \infty$. Then for $u + 2\delta > t \geq u \geq B$, since also $\sigma(u) \geq 2\varepsilon$,

$$\begin{aligned}\sigma_x(t) \geq e^{-xt}\{\sigma(u) - \varepsilon\} &> e^{-2x\delta}e^{-xu}\sigma(u)/2 \\ &= e^{-2x\delta}\sigma_x(u)/2 \geq e^{-2x\delta}\varepsilon.\end{aligned} \tag{10.3}$$

But then $\int_u^{u+2\delta} \sigma_x(t)dt$ would fail to tend to 0 as $u \to \infty$, contradicting the convergence of $\mathcal{L}\sigma(x)$. Similarly, one cannot have $\sigma_x(u) \leq -2\varepsilon$ for a sequence of $u \to \infty$.

Changing $\sigma(\cdot)$ on a finite interval if necessary, we may assume that σ is locally bounded. Then

$$\beta_x = \sup_{t>0} |\sigma_x(t)| < \infty, \qquad \forall x > 0. \tag{10.4}$$

Let $K_\lambda(v) = \lambda K(\lambda v)$ again be the Fejér kernel of (4.7). By the hypothesis that the Laplace transform $F(x + iy)$ of σ has good boundary behavior for $-\lambda \leq y \leq \lambda$, there is a constant M such that

$$\left| \int_{\mathbb{R}} K_\lambda(v)\sigma_x(u + v)dv \right| = \left| \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)F(x + iy)e^{iuy}dy \right| \leq M \tag{10.5}$$

for all $x > 0$; cf. formula (4.11). We may assume that $\beta_x = \sup_t \sigma_x(t)$. If σ is unbounded (as we may suppose for the moment) and x is small, then β_x is large and hence the values of t for which $\sigma_x(t)$ is close to its supremum must be large. For given $\varepsilon' > 0$ we now choose $u - \delta \geq B$ such that $\sigma_x(u-\delta) \geq \beta_x - \varepsilon'$ and we take $u + \delta > t \geq u - \delta$. Then by (10.1); cf. (10.3),

$$\sigma_x(t) \geq e^{-xt}\{\sigma(u-\delta) - \varepsilon\} \geq e^{-2x\delta}\sigma_x(u-\delta) - \varepsilon \geq e^{-2x\delta}\beta_x - \varepsilon - \varepsilon'.$$

Thus by (10.5)

$$\begin{aligned} M &\geq \int_{\mathbb{R}} K_\lambda(v)\sigma_x(u+v)dv \\ &\geq \int_{-\delta}^{\delta} K_\lambda(v)\sigma_x(u+v)dv - \beta_x \int_{|v|>\delta} K_\lambda(v)dv \\ &\geq \{e^{-2x\delta}\beta_x - \varepsilon - \varepsilon'\} \int_{-\delta}^{\delta} K_\lambda(v)dv - \beta_x\left(1 - \int_{-\delta}^{\delta} K_\lambda(v)dv\right) \\ &> \left((1 + e^{-2x\delta}) \int_{-\delta}^{\delta} K_\lambda(v)dv - 1\right)\beta_x - \varepsilon - \varepsilon'. \end{aligned} \tag{10.6}$$

For fixed δ we may take $x = \eta/(2\delta)$, where η is small so that $e^{-\eta}$ is close to 1. We may then rewrite (10.6) as

$$M > \left((1 + e^{-\eta}) \int_{-\lambda\delta}^{\lambda\delta} K(w)dw - \int_{\mathbb{R}} K(w)dw\right)\beta_x - \varepsilon - \varepsilon'. \tag{10.7}$$

Now by (10.1) we may take δ as large as we like provided we enlarge ε correspondingly. For large δ and associated x the coefficient of β_x in (10.7) will be positive and independent of x.

Conclusion: for small $x > 0$, the absolute value of $\sigma_x(t) = e^{-xt}\sigma(t)$ has upper bound $\beta_x = \beta$ independent of x, hence $\sigma(t)$ is bounded.

(ii) For piecewise constant σ as described, the proof of (10.4) is easy; cf. the lines following (10.3). Supposing σ unbounded, we continue roughly as before. Decreasing the original δ just a little (if necessary), we can change σ on a finite interval and at isolated points in such a way that all the intervals of constancy have the form $u - \delta \leq t < u'$ with $u' \geq u + \delta$. Choosing such an interval $[u - \delta, u')$ for which $\sigma_x(u - \delta) \geq \beta_x - \varepsilon'$, we again get (10.6) and (10.7) (now with $\varepsilon = 0$). For small x or η, the coefficient of β_x in (10.7) can be made positive if and only if

$$2\int_{-\lambda\delta}^{\lambda\delta} K(w)dw > \int_{\mathbb{R}} K(w)dw. \tag{10.8}$$

This condition requires (for the new or the original δ) that $\lambda\delta$ be larger than the constant b for which

$$2\int_{-b}^{b} K(w)dw = \int_{\mathbb{R}} K(w)dw. \tag{10.9}$$

Since for our kernel

$$2\int_{-\pi/2}^{\pi/2} K(w)dw = \int_{-\pi/2}^{\pi/2} \frac{4\sin^2(w/2)}{\pi w^2}dw < 1,$$

the constant b determined by (10.9) will be larger than $\pi/2$. However, one can use another kernel to show that in the Theorem, it is sufficient to have $\lambda\delta > \pi/2$; see Section 11.

Proof of Theorem 10.1. By Proposition 10.2, $\beta = \sup|\sigma(u)|$ is finite; we have to show that $\beta^* = \limsup|\sigma(u)|$ as $u \to \infty$ is equal to 0. For given $\varepsilon' > 0$ we can change σ on a finite interval $[0, B]$ in such a way that $\beta < \beta^* + \varepsilon'$ for the new σ while condition (i), (i′) or (ii) is preserved. The new σ and $F = \mathcal{L}\sigma$ will still satisfy the other conditions of the Theorem. Since σ is bounded and $F(x+iy)$ has good boundary behavior, one can let x go to 0 in (10.5) to obtain the relation

$$\left|\int_{\mathbb{R}} K_\lambda(v)\sigma(u+v)dv\right| = \left|\frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)F(iy)e^{iuy}dy\right|.$$

Application of the Riemann–Lebesgue lemma then shows that

$$M_u = \left|\int_{\mathbb{R}} K_\lambda(v)\sigma(u+v)dv\right| = o(1) \quad \text{as } u \to \infty.$$

Proceeding as before we may now use (10.6) and (10.7) with $x = 0$, replacing M by M_u, β_x by β and $e^{-2x\delta} = e^{-\eta}$ by 1. Under each of the conditions (i), (i′) and (ii) we can ensure (without undue increase in ε's) that $\lambda\delta$ is large enough for (10.8) to be satisfied. Taking u large, one concludes that β is small and, finally, that $\beta^* = 0$.

Example 10.3. We will show that part (ii) in the Proposition and Theorem fails if $\lambda\delta < \pi/2$. For an example we may change the scale to make $2\delta = 1$, after which we choose $\lambda < \pi$. Now consider the series

$$\frac{1-e^{-z}}{(1+e^{-z})^2} = \sum_{n=0}^{\infty} a_n e^{-nz} = 1 - 3e^{-z} + 5e^{-2z} - 7e^{-3z} + \cdots.$$

Then $\sigma(t) = \sum_{n\le t} a_n$ is piecewise constant and the intervals of constancy have length $2\delta = 1$. For $z = x + iy$ with $x > 0$, the sum of the series is equal to

$$\int_{0-}^{\infty} e^{-zt}d\sigma(t) = z\int_0^{\infty} \sigma(t)e^{-zt}dt.$$

Thus

$$F(z) = \mathcal{L}\sigma(z) = \frac{1-e^{-z}}{z}\frac{1}{(1+e^{-z})^2};$$

for $x \searrow 0$ and $-\lambda \le y \le \lambda$, the functions $F_x(iy) = F(x+iy)$ are uniformly convergent to a boundary function $F(iy)$. Hence, if part (ii) of Proposition 10.2 would hold for our value of $\lambda\delta$, the function σ would have to be bounded – but it is not!

11 An Optimal Constant in Theorem 10.1

We will show that one may take $b = \pi/2$ in part (ii) of Theorem 10.1. For this we deal with the following *extremal problem:*

Problem 11.1. Determine the infimum b^* of the constants $b > 0$ for which there is an L^1 kernel K on $\mathbb{R}$ with the following properties: K is positive on a neighborhood of $[-b, b]$,

$$2\int_{-b}^{b} K(t)dt = \int_{\mathbb{R}} |K(t)|dt, \tag{11.1}$$

and the Fourier transform $\hat{K}$ has its support in $[-1, 1]$. For convenience we also impose the normalization $\hat{K}(0) = 1$.

Proposition 11.2. *For $b = \pi/2$ the conditions in Problem 11.1 are satisfied by the kernel K, given by*

$$K(t) = \frac{2\cos t}{\pi^2 - 4t^2}, \quad \hat{K}(y) = \begin{cases} \cos(\pi y/2) & \textit{for } |y| \le 1, \\ 0 & \textit{for } |y| > 1. \end{cases} \tag{11.2}$$

No smaller constant will work, hence $b^* = \pi/2$.

That the function $\hat{K}$ in (11.2) is the Fourier transform of K may be verified by Fourier inversion. With different scaling the kernel K occurs in an article by Ingham [1936] on trigonometric inequalities and Dirichlet series. A closely related extremal problem has been treated by Johansson [1993]. His work implies that the extremal kernel is unique.

Proof of the Proposition. (i) Let K be as in (11.2). Then $K > 0$ on the interval $(-3\pi/2, 3\pi/2)$ and $\hat{K}$ has support $[-1, 1]$. We next verify (11.1) with $b = \pi/2$. For the following computation, cf. Pitt [1958] (section 4.3). Consider the difference between the first and the second member of (11.1),

$$\begin{aligned} \Delta_K &= 2\int_{-\pi/2}^{\pi/2} K(t)dt - \int_{\mathbb{R}} |K(t)|dt \\ &= 2\int_0^{\pi/2} K(t)dt - 2\int_{\pi/2}^{\infty} |K(t)|dt = 2I_0 - 2\sum_{n=0}^{\infty} J_n, \end{aligned}$$

say, where the integration in J_n will be from $(4n + 1)\pi/2$ to $(4n + 5)\pi/2$. Then

$$\begin{aligned} I_0 &= \int_0^{\pi/2} K(t)dt = \frac{1}{\pi}\int_0^1 \frac{\cos(\pi v/2)}{1 - v^2}dv \\ &= \frac{1}{2\pi}\int_0^1 \cos(\pi v/2)\left(\frac{1}{1+v} + \frac{1}{1-v}\right)dv = \frac{1}{2\pi}\int_{-1}^1 \frac{\cos(\pi w/2)}{w+1}dw, \\ J_n &= \int_{(4n+1)\pi/2}^{(4n+5)\pi/2} |K(t)|dt = \frac{1}{\pi}\int_{4n+1}^{4n+5} \frac{|\cos(\pi v/2)|}{v^2 - 1}dv \end{aligned}$$

$$= \frac{1}{2\pi}\left(\int_{4n+1}^{4n+3} + \int_{4n+3}^{4n+5}\right) |\cos(\pi v/2)| \left(\frac{1}{v-1} - \frac{1}{v+1}\right) dv$$
$$= \frac{1}{2\pi}\int_{-1}^{1} \cos(\pi w/2)\left(\frac{1}{w+4n+1} - \frac{1}{w+4n+5}\right) dw.$$

It follows that Δ_K is equal to the following expression:

$$\frac{1}{\pi}\int_{-1}^{1} \cos(\pi w/2)\left\{\frac{1}{w+1} - \sum_{n=0}^{\infty}\left(\frac{1}{w+4n+1} - \frac{1}{w+4n+5}\right)\right\} dw.$$

Thus via a telescoping series under the integral sign, $\Delta_K = 0$. This proves (11.1) for the kernel K of (11.2) and $b = \pi/2$. With this we have established the first part of Proposition 11.2.

(ii) We will verify that no constant $b < \pi/2$ can work in Problem 11.1 after we make an application of the first part of Proposition 11.2.

Proposition 11.3. *In part (ii) of Theorem 10.1 and Proposition 10.2 it is sufficient to take* $\lambda\delta > \pi/2$.

Proof. It will be enough to consider the case of Proposition 10.2. Observe that in its proof, the Fejér kernel $K_\lambda(v)$ may be replaced by any other kernel $K_\lambda(v) = \lambda K(\lambda v)$ such that K has the properties listed in Problem 11.1. Indeed, $\hat{K}_\lambda$ will then have its support in $[-\lambda, \lambda]$ so that (10.5) is valid for K_λ. We next fix $\lambda\delta$ just a little larger than b so that $K_\lambda(v) > 0$ on $(-\delta, \delta)$. In (10.6) one now replaces

$$\int_{|v|>\delta} K_\lambda \quad \text{by} \quad \int_{|v|>\delta} |K_\lambda| = \int_{\mathbb{R}} |K_\lambda| - \int_{-\delta}^{\delta} K_\lambda.$$

Thus (10.7) becomes

$$M > \left\{(1+e^{-\eta})\int_{-\lambda\delta}^{\lambda\delta} K - \int_{\mathbb{R}} |K|\right\}\beta_x - \varepsilon - \varepsilon'.$$

For sufficiently small $\eta = 2\delta x$, the coefficient of β_x will then be positive by (11.1). In this way one obtains an upper bound for β_x independent of (our small) x, and this bound will also work for $|\sigma|$. Hence, *whenever b is admissible* in Problem 11.1, the piecewise constancy condition in Proposition 10.2 works as soon as $\lambda\delta > b$. [One can always decrease λ or δ so that $\lambda\delta$ is only a little larger than b.]

We know that $b = \pi/2$ works for the kernel K of (11.2), so that the final part of Proposition 10.2 is valid for $\lambda\delta > \pi/2$.

Proof of the Second Part of Proposition 11.2. Suppose that there would be a kernel K which satisfies the conditions in Problem 11.1 with $b < \pi/2$. In Example 10.3 (where $2\delta = 1$) we could then take $2b < \lambda < \pi$ (so that $\lambda\delta > b$). The conclusion from the proof above would be that the function σ in that example must be bounded – but it is not!

12 Fatou and Riesz. General Dirichlet Series

Fatou's original theorem [1906] (p. 389) may be stated as follows:

Theorem 12.1. *Let $f(z) = \sum_0^\infty a_n z^n$, where the coefficients satisfy the 'Tauberian' condition $a_n \to 0$, so that in particular f is analytic in the unit disc. Suppose also that f is analytic at the point $z_0 = e^{it_0} \in C(0,1)$, or more precisely, that f has an analytic continuation (also called f) to a neighborhood of z_0. Then the series $\sum_0^\infty a_n z_0^n$ converges to $f(z_0)$.*

Proofs. Marcel Riesz gave several proofs [1909], [1911], [1916] for Fatou's theorem; cf. Zeller and Beekmann [1958/70] (p. 93). The simplest proof goes as follows. One may take $z_0 = 1$ and choose $\lambda > 0$, $R > 1$ such that f is analytic on the closed circular sector $V : \{0 \le |z| \le R,\ -\lambda \le \arg z \le \lambda\}$. Setting $s_n(z) = \sum_{k=0}^n a_k z^k$, one considers the analytic functions

$$g_n(z) = \frac{f(z) - s_n(z)}{z^{n+1}}(z - e^{i\lambda})(z - e^{-i\lambda}), \quad z \in V. \tag{12.1}$$

If one can show that $g_n(z) \to 0$ uniformly on $\Gamma = \partial V$ as $n \to \infty$, the maximum principle will imply that $g_n(1) \to 0$ and hence $s_n(1) \to f(1)$. To prove the uniform convergence $g_n(z) \to 0$ on Γ, one considers different parts of Γ separately. On the segments $z = re^{\pm i\lambda}$ with $0 \le r < 1$, the smallness of the numbers a_k for $k > n$ together with one of the factors $z - e^{\pm i\lambda}$ gives the desired result. On the segments $z = re^{\pm i\lambda}$ with $1 \le r \le R$ and on the circular arc in Γ, one combines suitable bounds on the numbers a_k for $k \le n$ with the effect of the denominator z^{n+1} and the factors $z - e^{\pm i\lambda}$. More details may be found in Landau and Gaier [1986] (section 18); cf. also Section 13 below.

Fatou himself used Riemann's localization principle for trigonometric series $\mathcal{S}$ of the form $\sum_{-\infty}^\infty a_n e^{int}$ with $a_n \to 0$ as $|n| \to \infty$. Let T be the generalized or distributional sum of the series:

$$T = a_0 + D^2\Phi, \quad \Phi(t) = -\sum_{n \ne 0} \frac{a_n}{n^2} e^{int}. \tag{12.2}$$

[T is a so-called pseudofunction; cf. Section 14.] Suppose now that $T = g$ on (α, β), where g is a periodic integrable function which is pointwise equal to the sum of its Fourier series $\sum_{-\infty}^\infty b_n e^{int}$. If one represents g as $b_0 + D^2 G$ with continuous periodic G, the equality $T = g$ on (α, β) means that $a_0 = b_0$ while $\Phi - G$ is linear on (α, β). By Riemann's theorem, the trigonometric series $\mathcal{S}$ converges to g pointwise on (α, β); cf. Riemann [1892] (pp 227–271), Zygmund [1959] (chapter 9, (5.7)). To the function f of Theorem 12.1 one can construct a periodic C^1 function g such that $f(e^{it}) = g(t)$ for t in a neighborhood of the point t_0. □

Another proof, based on W.H. Young's work [1918], may be found in Titchmarsh [1939] (section 7.31). It is of Fourier series type and (like Fatou's proof) can be used to obtain a more general result.

Fatou's theorem has been extended in many ways. Riesz considered general Dirichlet series and formulated conditions weaker than analyticity which imply convergence under appropriate conditions on the coefficients. Theorem 12.2 below contains the more notable classical extensions, including the striking part (ii) on lacunary series due to Ingham; cf. Remarks 12.3. Other extensions, by Riesz and by Gaier, may be found in Riesz [1924], Gaier [1953a], and Landau and Gaier [1986] (section 12 and comments). Halász [1969] obtained convergence under the function-theoretic condition that f map the unit disc one to one onto a starlike domain. Hayman [1970] considered related local conditions. More recent results will be discussed in Sections 13, 14.

Theorem 10.1 implies the following complex Tauberian theorem for Dirichlet series.

Theorem 12.2. *Let* $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ *with* $\mu_n \to \infty$ *and let the series*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{-\mu_n z}, \quad z = x + iy, \tag{12.3}$$

be convergent for $x > 0$. *Suppose that for some constant (appropriately called)* $f(0)$ *and some number* $\lambda > 0$, *the quotient*

$$F(z) = \frac{f(z) - f(0)}{z}, \quad z = x + iy, \quad x > 0, \tag{12.4}$$

converges uniformly or in L^1 *on* $-\lambda \le y \le \lambda$ *to a boundary function* $F(iy)$ *as* $x \searrow 0$. *Then each of the following Tauberian conditions is sufficient for convergence of* $\sum_0^\infty a_n$ *to* $f(0)$:

(i) the partial sums s_n *are 'very slowly decreasing' (cf. Theorem 10.1):*

$$\liminf \sum_{u<\mu_n\le t} a_n \ge 0 \quad \text{for } u + 1 \ge t \ge u \to \infty \tag{12.5}$$

(no special condition required on λ*);*

(ii) $\mu_{n+1} - \mu_n \ge 2\delta > 0$ *for* $n \ge n_0$ *and* $\lambda > \pi/(2\delta)$ *(no special condition required on the coefficients* a_n*).*

Proof. By changing a_0 one may assume $f(0) = 0$. Define $\sigma(t) = \sum_{\mu_n \le t} a_n$ and take $x > 0$. By the convergence of the series in (12.3) one may write

$$f(z) = \int_{0-}^{\infty-} e^{-zt} d\sigma(t) = z \int_0^\infty \sigma(t) e^{-zt} dt = zF(z),$$

say (cf. Section 3). The hypotheses imply that the functions σ and $F = \mathcal{L}\sigma$ satisfy the conditions of Theorem 10.1, including one of the conditions (i′) and (ii) for σ. Thus $\sigma(u) \to 0$ and this completes the proof.

Remarks 12.3. We first comment on part (i). There are related results in Ingham [1935] (theorem 3(l)). The present result which involves a one-sided Tauberian condition is an extension of the classical theorems of Fatou and Riesz. As observed already by Riesz, for convergence of $\sum_0^\infty a_n$, the sum function $f(z)$ in (12.3) need not be analytic at $z = 0$; weak regularity as in the Theorem is sufficient. Fatou's theorem for power series corresponds to the case $\mu_n = n$. In his extension to Dirichlet series (12.3), Riesz [1916] gave a condition on the coefficients which implies 'very slow oscillation' of the partial sums s_n as $n \to \infty$:

$$\lim \sum_{u<\mu_n\le t} a_n = 0 \quad \text{for } u+1 \ge t \ge u \to \infty.$$

He also noted the sufficiency of the pair of conditions given by $a_n = o(1)$ plus $a_n = o(\mu_n - \mu_{n-1})$. For the case of 'classical' Dirichlet series $\sum_1^\infty a_n/n^z$, Riesz explicitly mentioned the sufficient condition $na_n = o(1)$. Theorem 12.2 implies the sufficiency of the one-sided condition $\liminf a_n \ge 0$ in Fatou's theorem for power series, and of the one-sided condition $\liminf na_n \ge 0$ in Riesz's theorem for classical Dirichlet series; cf. Korevaar [1954b] and Section 17 below. The one-sided condition for Fatou's theorem was also discussed in Postnikov [1980] (section 17).

Part (ii) is in Ingham [1936]. It is a gap theorem for Dirichlet series which reminds one of the Hardy–Littlewood high-indices theorem (Section I.23): there is *no order condition* on the coefficients a_n. In a later article [1950], Ingham observed that the inequality $\lambda > \pi/(2\delta)$ may be relaxed to $\lambda \ge \pi/(2\delta)$.

13 Newer Extensions of Fatou–Riesz

Some newer complex Tauberian theorems were motivated by operator theory; cf. Section 15. In that context Katznelson and Tzafriri [1986] obtained results for power series which contain Propositions 13.3 and 14.4. Proposition 13.3 was extended by Allan, O'Farrell and Ransford [1987] to Theorem 13.4 and by Arendt and Batty [1988] to Theorem 13.5. The formulations below are slightly more general; cf. also Theorem 14.5.

In this section f stands for an analytic function on the unit disc:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } |z| < 1. \tag{13.1}$$

In connection with Fatou's theorem we focus on conditions under which the coefficients a_n tend to zero as $n \to \infty$. It is sufficient if f is in the *Hardy class* H^1. This condition can be expressed by saying that $f(re^{it})$ converges to a boundary function $f(e^{it})$ in L^1:

$$\int_{-\pi}^{\pi} |f(re^{it}) - f(e^{it})|dt \to 0 \quad \text{as } r \nearrow 1;$$

see for example Duren [1970]. In this case Cauchy's formula and the Riemann–Lebesgue lemma imply that

$$a_n = \frac{1}{2\pi i}\int_{C(0,r)} f(z)z^{-n-1}dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-int}dt \to 0 \quad \text{as } n \to \infty. \tag{13.2}$$

It is convenient to introduce a *local H^1 condition.*

Definition 13.1. We say that the function f in (13.1) has H^1 boundary behavior at the point $z_0 = e^{it_0}$ if there is a number $\lambda > 0$ such that $f(re^{it})$ converges in $L^1(t_0 - \lambda, t_0 + \lambda)$ to a boundary function $f(e^{it})$ as $r \nearrow 1$.

Proposition 13.2. *Let $f(z) = \sum_0^\infty a_n z^n$ be analytic and of class H^1 on the unit disc. Suppose that the quotient*

$$q(z) = \frac{f(z) - A}{1 - z} = \sum_{n=0}^{\infty} (s_n - A)z^n, \quad \textit{with } s_n = \sum_{k=0}^{n} a_k, \tag{13.3}$$

has H^1 boundary behavior at the point $z = 1$. Then the series $\sum_0^\infty a_n$ converges to A.

Proof. Since f is in H^1, the quotient q has H^1 boundary behavior on arcs of the form $\{z = e^{it},\ \varepsilon \le |t| \le \pi\}$ with $\varepsilon > 0$, on which $1 - z$ stays away from 0. By the hypothesis q also has H^1 boundary behavior on some arc with $-\lambda < t < \lambda$, hence q is of class H^1 on the unit disc. By (13.2) applied to q instead of f, the coefficients $s_n - A$ in the power series for q tend to 0 as $n \to \infty$, so that $s_n \to A$.

Proposition 13.3. *Let f as in* (13.1) *have H^1 boundary behavior everywhere on the circle $C(0, 1)$ except at the point $z = 1$. Suppose that the sequence of partial sums $\{s_n = \sum_0^n a_k\}$ is bounded. Then $a_n \to 0$ as $n \to \infty$.*

Boundedness of the sequence $\{s_n\}$ appears to be essential here; cf. Tomilov and Zemánek [2001] (example 4.6).

Proof of the Proposition. The proof uses contour integration *à la Newman*. One may assume that $|s_n| \le 1$, $\forall n$. Setting $s_n(z) = \sum_{k=0}^n a_k z^k$ and taking $0 < \rho < 1$, $\lambda > 0$, one has

$$a_n = \frac{1}{2\pi i}\int_{C(0,\rho)} \frac{f(z) - s_{n-1}(z)}{z^{n+1}} \frac{(z - e^{i\lambda})(z - e^{-i\lambda})}{(z-1)^2} dz. \tag{13.4}$$

Indeed, on the circle $C(0, \rho)$, the integrand $I(z)$ has the uniformly convergent expansion

$$I(z) = (a_n + a_{n+1}z + \cdots)(1 - e^{-i\lambda}z)(1 - e^{i\lambda}z)(1 + 2z + \cdots)/z. \tag{13.5}$$

By Cauchy's theorem, the circle of integration may be changed to a new path Γ^ε consisting of a short arc of the original circle from a point $\rho e^{-i\lambda}$ to $\rho e^{i\lambda}$, radial segments extending from the end points of this arc to a short distance ε from $C(0, 1)$, and a long arc of the circle $C(0, 1 - \varepsilon)$. Let $\rho = 1 - \lambda$ with small $\lambda \in (0, 1/2)$. We will verify that one may actually let ε go to 0 and replace Γ^ε by the limit path $\Gamma = \sum_1^4 \Gamma_j$ (Figure III.13), where

$$\begin{cases} \Gamma_1 \text{ is the arc } z = \rho e^{it} \text{ with } -\lambda \le t \le \lambda, \\ \Gamma_2 \text{ is the segment } z = re^{i\lambda} \text{ with } \rho < r < 1, \\ \Gamma_3 \text{ is the arc } z = e^{it} \text{ with } \lambda \le t \le 2\pi - \lambda, \\ \Gamma_4 \text{ is the segment } z = re^{-i\lambda} \text{ with } 1 > r > \rho. \end{cases}$$

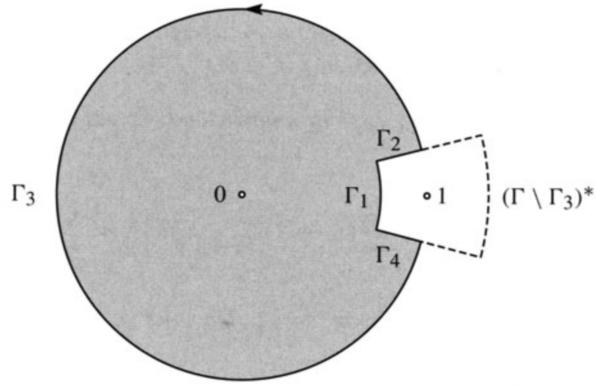

Fig. III.13. The paths of integration

For $z = re^{it}$ with $r < 1$, partial summation and the hypothesis $|s_k| \le 1$ show that

$$\left| \sum_{k=n}^{\infty} a_k z^k \right| = \left| \sum_{k=n}^{\infty} s_k z^k (1 - z) - s_{n-1} z^n \right| \le 2 \frac{r^n}{1 - r} |1 - z|. \tag{13.6}$$

On the part $\Gamma \setminus \Gamma_3$ of Γ inside the unit circle, geometric considerations now give an upper bound for $|I(z)|$ independent of n and our λ:

$$|I(z)| \le \frac{2}{r} \frac{|re^{it} - e^{i\lambda}||re^{it} - e^{-i\lambda}|}{(1 - r)|1 - re^{it}|} < \frac{4}{r} < 8. \tag{13.7}$$

Because of this bound and the hypothesis that f is in H^1 away from the point $z = 1$, the auxiliary path of integration Γ^ε may indeed be changed to Γ. It also follows that

$$\left| \int_{\Gamma \setminus \Gamma_3} I(z) dz \right| < 8L(\Gamma \setminus \Gamma_3) < 32\lambda. \tag{13.8}$$

In the integral over Γ_3 we treat f and s_{n-1} separately. For the case of f one obtains the integral

$$\int_{\lambda}^{2\pi - \lambda} f(e^{it})(e^{it} - e^{i\lambda})(e^{it} - e^{-i\lambda})(e^{it} - 1)^{-2} e^{-int} i dt; \tag{13.9}$$

by the Riemann–Lebesgue lemma it tends to 0 as $n \to \infty$. To deal with the integral containing s_{n-1}, we observe that for $|z| > 1$ the integrand can be written as

$$J(z) = \frac{s_{n-1}(z)}{z^{n+1}} \frac{(z - e^{i\lambda})(z - e^{-i\lambda})}{(z-1)^2} =$$
$$\left(a_{n-1} + \frac{a_{n-2}}{z} + \cdots + \frac{a_0}{z^{n-1}}\right)\left(1 - \frac{e^{i\lambda}}{z}\right)\left(1 - \frac{e^{-i\lambda}}{z}\right)\left(1 + \frac{2}{z} + \cdots\right)\frac{1}{z^2}.$$

Thus since J is analytic outside the unit circle, the integral of J over any circle $C(0, R)$ with $R > 1$ is equal to 0. Taking $R = 1/\rho$ we may change the circle of integration $C(0, R)$ here to the path Γ^* which is obtained from Γ by reflection in the unit circle. Γ^* consists of Γ_3 plus the reflection $(\Gamma \setminus \Gamma_3)^*$ of $\Gamma \setminus \Gamma_3$; cf. Figure III.13. Thus the desired integral of J over Γ_3 is equal to minus the integral of J over $(\Gamma \setminus \Gamma_3)^*$. Now for $|z| = r > 1$ one may use partial summation similar to (13.6) to obtain

$$\left|\sum_{k=0}^{n-1} a_k z^k\right| \leq 2\frac{r^{n-1}}{r-1}|z-1|.$$

For z lying on $(\Gamma \setminus \Gamma_3)^*$, so that $1/z \in (\Gamma \setminus \Gamma_3)$, comparison with (13.7) shows that $|J(z)| < 4$. By our definition of ρ as $1 - \lambda$, the length of $(\Gamma \setminus \Gamma_3)^*$ is less than 8λ. The end result is that the integral of J over $(\Gamma \setminus \Gamma_3)^*$, and hence the integral of J over Γ_3, is majorized by 32λ.

For any given number $\lambda \in (0, 1/2)$ one may take n_0 so large that the integral (13.9) involving f on Γ_3 is bounded by λ for all $n \geq n_0$. Putting everything together one concludes that $|\int_{\Gamma_3} I(z)dz| < 33\lambda$, and

$$|a_n| = \left|\frac{1}{2\pi i}\int_\Gamma I(z)dz\right| < \frac{65}{2\pi}\lambda \quad \text{for all } n \geq n_0.$$

Since $\lambda > 0$ may be taken arbitrarily small, $a_n \to 0$ as $n \to \infty$. □

Theorem 13.4. *Let $f(z) = \sum_0^\infty a_n z^n$ be analytic for $|z| < 1$ and let E be the set of points $\zeta \in C(0, 1)$ where f is singular, in the sense that f does not have H^1 boundary behavior at the points ζ. Suppose that E has linear measure* 0 *and that*

$$\sup_{\zeta \in E} \sup_{n \geq 0} \left|\sum_{k=0}^{n} a_k \zeta^k\right| = M < \infty. \tag{13.10}$$

Then $a_n \to 0$ as $n \to \infty$. As a result, the series $\sum_0^\infty a_n z_0^n$ converges to $f(z_0)$ at every point $z_0 \in C(0, 1)$ where f is weakly regular, in the sense that the quotient

$$q(z) = \frac{f(z) - f(z_0)}{z - z_0}, \quad |z| < 1,$$

has H^1 boundary behavior at the point z_0.

Proof. (*Outline*) The crucial observation is that the (closed) 'singularity set' E in $C(0, 1)$ can be enclosed in the union of a finite number of disjoint open arcs of arbitrarily small total length. One can then extend the complex method used for

Proposition 13.3 (where $E = \{1\}$) to show that $a_n \to 0$. Knowing this, one may appeal to Theorem 12.2 for the convergence of the series $\sum_0^\infty a_n z_0^n$. For details, cf. the direct proof for the convergence of such series by Allan, O'Farrell and Ransford (loc. cit.). In Section 14 we will give a simpler proof with the aid of pseudofunctions, but the complex method is important for applications.

Arendt and Batty (loc. cit.) have proved an analog to Theorem 13.4 for Laplace transforms, which extends Theorem 7.1. In the formulation below we use the following terminology. Let F be analytic throughout the half-plane $\{x = \operatorname{Re} z > 0\}$. Then F has H^1 boundary behavior at $z_0 = iy_0$ if there is a number $\lambda > 0$ such that $F(x+iy)$ converges to a boundary function $F(iy)$ in $L^1(y_0 - \lambda < y < y_0 + \lambda)$ as $x \searrow 0$.

Theorem 13.5. *Let $\alpha(\cdot)$ be defined and bounded on $[0, \infty)$, so that the Laplace transform $F(z) = \mathcal{L}\alpha(z)$, $z = x + iy$, is analytic for $x = \operatorname{Re} z > 0$. Let iE denote the set of all 'singular points' $i\eta$ of F on $i\mathbb{R}$, in the sense that F does not have H^1 boundary behavior at the points $i\eta$. Suppose that iE has linear measure zero and that*

$$\sup_{\eta \in E} \sup_{B > 0} \left| \int_0^B \alpha(t) e^{-i\eta t} dt \right| = M < \infty.$$

Then

$$\int_0^{\infty -} \alpha(t) e^{-z_0 t} dt = F(z_0)$$

at all points $z_0 = iy_0$ where F is weakly regular, in the sense that the quotient

$$Q(z) = \frac{F(z) - F(z_0)}{z - z_0}, \quad \operatorname{Re} z > 0,$$

has H^1 boundary behavior at the point z_0.

Remarks 13.6. The original work of Fatou and Riesz was refined by Ingham [1935]; cf. Remarks 12.3. Newman's contour integration method, as described in Section 7, has played a role in recent developments. There are now extensions, related results and applications to operator theory by many authors. Besides the papers referred to already one may mention Ransford [1988], Batty [1990], [1994a], Arendt and Prüss [1992], Arendt and Batty [1995], Batty, van Neerven and Räbiger [1998], and Chill [1998]. The book by Arendt, Batty, Hieber and Neubrander [2001] contains numerous results on the subject.

14 Pseudofunction Boundary Behavior

The results in Section 13 will be refined with the aid of the distributional approach initiated by Katznelson and Tzafriri [1986]. As before, let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } |z| < 1. \tag{14.1}$$

We continue our discussion of conditions under which $a_n \to 0$. Instead of convergence in L^1 we consider *distributional convergence* of $f(re^{it})$ to a boundary distribution $T = f(e^{it})$. That is,

$$\int_{-\pi}^{\pi} f(re^{it})\phi(t)dt \to < T, \phi > = \text{`}\int_{-\pi}^{\pi} f(e^{it})\phi(t)dt\text{'} \quad \text{as } r \nearrow 1 \tag{14.2}$$

for all C^∞ functions ϕ of period 2π, the *testing functions* for periodic distributions.

PERIODIC DISTRIBUTIONS. In the class $C^\infty_{2\pi}$ of 2π-periodic testing functions, convergence $\phi_j \to \phi$ is defined by the set of relations

$$\begin{aligned}&\phi_j \to \phi \ \text{ uniformly}, \quad \phi'_j \to \phi' \ \text{ uniformly},\\ &\phi''_j \to \phi'' \ \text{ uniformly}, \quad \cdots \ \text{ on } \mathbb{R} \text{ or } [-\pi, \pi].\end{aligned}$$

With this definition $C^\infty_{2\pi}$ becomes the *testing space* $\mathcal{D}_{2\pi}$. Observe that the Fourier series $\sum_{n\in\mathbb{Z}} c_n[\phi]e^{int}$ of a testing function ϕ converges to ϕ in $\mathcal{D}_{2\pi}$:

$$s_N[\phi](t) = \sum_{n=-N}^{N} c_n[\phi]e^{int} \to \phi(t) \ \text{ uniformly as } \ N \to \infty,$$

$$s'_N[\phi](t) = \sum_{n=-N}^{N} inc_n[\phi]e^{int} = \sum_{n=-N}^{N} c_n[\phi']e^{int} \to \phi'(t) \ \text{ uniformly, etc.}$$

By definition, a distribution T of period 2π is a *continuous linear functional* on $C^\infty_{2\pi}$, or rather, $\mathcal{D}_{2\pi}$:

$$< T, \phi_j > \to < T, \phi > \quad \text{whenever } \ \phi_j \to \phi \ \text{ in } \ \mathcal{D}_{2\pi}.$$

The periodic distributions form the dual space $\mathcal{D}'_{2\pi}$ of $\mathcal{D}_{2\pi}$, with convergence $T_k \to T$ defined by the relation

$$< T_k, \phi > \to < T, \phi > \quad \text{for all } \ \phi \in C^\infty_{2\pi}.$$

A periodic integrable function g is represented in $\mathcal{D}'_{2\pi}$ via the rule $< g, \phi > = \int_{-\pi}^{\pi} g(t)\phi(t)dt$. Working modulo 2π, one will say that $T_1 = T_2$ on (a, b) if $< T_1, \phi > = < T_2, \phi >$ for all testing functions ϕ with support in (a, b). From here on through Theorem 14.5 we deal only with testing functions and distributions of period 2π, without saying so every time.

The FOURIER SERIES of a periodic distribution T is defined as

$$\sum_{n\in\mathbb{Z}} c_n[T]e^{int} \quad \text{with } \ c_n[T] = \frac{1}{2\pi} < T, e^{-int} > .$$

It converges to T in $\mathcal{D}'_{2\pi}$:

$$< s_N[T], \phi > = \sum_{n=-N}^{N} c_n[T] < e^{int}, \phi > = 2\pi \sum_{n=-N}^{N} c_n[T] c_{-n}[\phi]$$
$$= \sum_{n=-N}^{N} c_{-n}[\phi] \; < T, e^{-int} > = < T, s_N[\phi] > \to < T, \phi >,$$

because $s_N[\phi] \to \phi$ in $\mathcal{D}_{2\pi}$ and T is continuous.

In the class of 2π-periodic integrable functions, testing functions ϕ may be characterized by the fact that their Fourier coefficients $c_n[\phi]$ are $\mathcal{O}(|n|^{-p})$ as $|n| \to \infty$ for every integer p. It follows that a trigonometric series $\sum_{-\infty}^{\infty} b_n e^{int}$ is convergent in $\mathcal{D}'_{2\pi}$ if [and only if] $b_n = \mathcal{O}(|n|^q)$ as $|n| \to \infty$ for some constant q. More generally, distributions T_k converge to a distribution T as $k \to \infty$ if [and only if]

$$c_n[T_k] \to c_n[T], \quad \forall n,$$

and there are constants C and q such that

$$|c_n[T_k]| \le C(|n| + 1)^q, \quad \forall n, \, k.$$

Distributions T are *multiplied* by testing functions ω according to the rule

$$< T\omega, \phi > = < \omega T, \phi > = < T, \omega\phi >, \quad \forall \phi \in C_{2\pi}^{\infty}.$$

For the Fourier coefficients this means that

$$\begin{aligned} 2\pi c_n[T\omega] &= < T\omega, e^{-int} > = < e^{-int}T, \omega > \\ &= < e^{-int}T, \sum_k c_k[\omega] e^{ikt} > = 2\pi \sum_k c_k[\omega] c_{n-k}[T]. \end{aligned} \tag{14.3}$$

Definition 14.1. A periodic distribution T whose Fourier coefficients $b_n = c_n[T]$ form a bounded sequence is called a *pseudomeasure*. If $b_n \to 0$ as $n \to \pm\infty$, one speaks of a *pseudofunction*.

There are corresponding notions of pseudomeasures and pseudofunctions in the class of tempered distributions on $\mathbb{R}$; cf. Katznelson [1968/76] (section 6.4). A typical pseudomeasure on $\mathbb{R}$ is the distribution

$$\frac{1}{t + i0} = \lim_{\varepsilon \searrow 0} \frac{1}{t + i\varepsilon} = \lim_{\varepsilon \searrow 0} (-i) \int_0^{\infty} e^{-\varepsilon x} e^{itx} dx,$$

whose Fourier transform is bounded: it is equal to $(-2\pi i)$ times the unit step function or Heaviside function, $1_+(x)$. Other examples are the Dirac measure and the principal value distribution, p.v. $(1/t)$. First order poles correspond to pseudomeasures, slightly milder singularities to pseudofunctions.

The *product* of a pseudomeasure or pseudofunction T and a *testing function* ω is again a pseudomeasure or pseudofunction, respectively. This follows from (14.3):

$$|c_n[T\omega]| \le \sum_{|k| \le B} |c_k[\omega]| \sup_{|j| \ge |n| - B} |c_j[T]| + \sup_j |c_j[T]| \sum_{|k| > B} |c_k[\omega]|;$$

if $c_n[T] \to 0$ as $n \to \pm\infty$, then also $c_n[T\omega] \to 0$. These results are also true for certain functions ω that are less smooth than testing functions. Indeed, one may use formula (14.3) to *define* the product $T\omega$. In the case of pseudomeasures and pseudofunctions it is then enough to require that ω have an *absolutely convergent Fourier series*. Useful functions of the latter kind are the periodic *trapezoidal* functions τ_λ with $\lambda \le \pi/2$, which for $|t| \le \pi$ are given by

$$\tau_\lambda(t) = \begin{cases} 1 & \text{for } |t| \le \lambda, \\ 0 & \text{for } 2\lambda \le |t| \le \pi, \\ 2 - |t|/\lambda & \text{for } \lambda \le |t| \le 2\lambda. \end{cases} \tag{14.4}$$

By a short calculation

$$c_n[\tau_\lambda] = \frac{\cos \lambda n - \cos 2\lambda n}{\pi \lambda n^2}; \tag{14.5}$$

cf. formula (II.9.7). One sometimes needs other trapezoidal functions. We will speak of a *trapezoidal testing function* τ_λ if τ_λ is in $C^\infty_{2\pi}$, equal to 1 for $|t| \le \lambda$ and equal to 0 for $2\lambda \le |t| \le \pi$. [One may, but need not, require that $0 \le \tau_\lambda(t) \le 1$ for all values of t.]

If the analytic function $f(z) = \sum_0^\infty a_n z^n$ on the unit disc has a boundary distribution $T = f(e^{it})$, it follows from (14.2) with $\phi(t) = e^{-int}$ that $f(e^{it})$ has Fourier coefficients equal to a_n for $n \ge 0$ and equal to 0 for $n < 0$.

We now turn to *local* boundary behavior.

Definition 14.2. We will say that the function f in (14.1) has pseudofunction boundary behavior at the point $z_0 = e^{it_0}$ if $f(re^{it})$ has a boundary distribution $f(e^{it})$ on $C(0, 1)$ which on some interval $(t_0 - \lambda, t_0 + \lambda)$ coincides with a pseudofunction $g(t)$.

If f has pseudofunction boundary behavior at every point of the circle $C(0, 1)$, then $f(e^{it})$ *is* a pseudofunction. This may be proved with the aid of a suitable 'partition of unity': one may represent the constant function 1 as a sum of trapezoidal functions with small support.

Proposition 14.3. *Let $f(z) = \sum_0^\infty a_n z^n$ be analytic in the unit disc and have a pseudofunction boundary distribution $F(t) = f(e^{it})$ on the circle $C(0, 1)$. Suppose that the quotient*

$$q(z) = \frac{f(z) - A}{1 - z} = \sum_{n=0}^{\infty} (s_n - A) z^n, \quad \textit{with } s_n = \sum_{k=0}^{n} a_k, \tag{14.6}$$

has pseudofunction boundary behavior at the point $z = 1$. Then the series $\sum_{n=0}^\infty a_n$ converges to A.

Proof. The hypotheses imply that $a_n = o(1)$, hence $s_n - A = o(n)$ as $n \to \infty$. It follows that $q(re^{it}) = \sum_0^\infty (s_n - A) r^n e^{int}$ tends to the boundary distribution $Q(t) = q(e^{it}) = \sum_0^\infty (s_n - A) e^{int}$ as $r \nearrow 1$. By the hypotheses Q is equal to a pseudofunction G on some interval $(-\mu, \mu)$. As a result one has

$$Q\tau_\lambda = G\tau_\lambda, \quad \text{a pseudofunction,} \tag{14.7}$$

whenever τ_λ is a trapezoidal testing function with support $[-\lambda, \lambda] \subset (-\mu, \mu)$ [we work modulo 2π]. Indeed, for any testing function ϕ,

$$< Q\tau_\lambda, \phi > = < Q, \tau_\lambda\phi > = < G, \tau_\lambda\phi > = < G\tau_\lambda, \phi > .$$

It remains to show that Q is also equal to a pseudofunction away from the point $t = 0$. To that end we observe that for $r \nearrow 1$, on the one hand,

$$\frac{f(e^{it}) - A}{1 - re^{it}} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k r^{n-k} - Ar^n \right) e^{int} \to Q(t)$$

in distributional sense, while on the other hand,

$$\frac{1 - \tau_\lambda(t)}{1 - re^{it}} \to \frac{1 - \tau_\lambda(t)}{1 - e^{it}}$$

in the sense of $\mathcal{D}_{2\pi}$. Hence

$$Q(t)\{1 - \tau_\lambda(t)\} = \lim_{r\nearrow 1} \frac{f(e^{it}) - A}{1 - re^{it}}\{1 - \tau_\lambda(t)\}$$
$$= \lim_{r\nearrow 1} \{f(e^{it}) - A\}\frac{1 - \tau_\lambda(t)}{1 - re^{it}} = \{f(e^{it}) - A\}\frac{1 - \tau_\lambda(t)}{1 - e^{it}}. \tag{14.8}$$

The final product, of a pseudofunction and a testing function, is a pseudofunction.
Combining (14.7) and (14.8), one finds that for small $\lambda > 0$,

$$Q = Q\tau_\lambda + Q(1 - \tau_\lambda)$$

is a pseudofunction, hence the Fourier coefficients $s_n - A$ of $Q(t) = q(e^{it})$ tend to 0 as $n \to \infty$. □

Proposition 14.4. *Let $f(z) = \sum_0^\infty a_n z^n$ as in (14.1) have pseudofunction boundary behavior everywhere on the circle $C(0, 1)$ except perhaps at the point $z = 1$. Suppose that the sequence of partial sums $s_n = \sum_{k=0}^n a_k$ is bounded. Then $a_n \to 0$ as $n \to \infty$.*

Proof. Since $|s_n| \le M < \infty$ for all n, the quotient

$$q(z) = \frac{f(z)}{1 - z} = \sum_0^\infty s_n z^n$$

has boundary pseudomeasure $Q(t) = \sum_0^\infty s_n e^{int}$ on $C(0, 1)$. The sequence $\{a_n\} = \{s_n - s_{n-1}\}$ is also bounded, hence $f(z)$ has boundary pseudomeasure $F(t) = \sum_0^\infty a_n e^{int}$. By the hypotheses F is equal to a pseudofunction G away from the point $t = 0$. Thus for any trapezoidal function τ_λ as in (14.4),

$$F(1 - \tau_\lambda) = G(1 - \tau_\lambda) = H_\lambda, \quad \text{a pseudofunction.} \tag{14.9}$$

On the other hand $f(z) = (1 - z)q(z)$, so that $F(t) = (1 - e^{it})Q(t)$ and

$$F(t)\tau_\lambda(t) = Q(t)(1-e^{it})\sum_k c_k[\tau_\lambda]e^{ikt} = Q(t)\sum_k\{c_k[\tau_\lambda]-c_{k-1}[\tau_\lambda]\}e^{ikt}. \tag{14.10}$$

We proceed to estimate the Fourier coefficients of the pseudomeasure $F\tau_\lambda$. By (14.10), (14.3) and the fact that $|c_j[Q]| = |s_j| \le M$,

$$|c_n[F\tau_\lambda]| \le \sum_k |c_{n-k}[Q]|\,|c_k[\tau_\lambda]-c_{k-1}[\tau_\lambda]| \le M\sum_k |c_k[\tau_\lambda]-c_{k-1}[\tau_\lambda]|. \tag{14.11}$$

Formula (14.5) shows that the final sum is majorized by

$$\begin{aligned}
&\sum_k \frac{\lambda}{\pi}\left|\frac{\cos k\lambda - \cos 2k\lambda}{k^2\lambda^2} - \frac{\cos(k-1)\lambda - \cos 2(k-1)\lambda}{(k-1)^2\lambda^2}\right| \\
&= \frac{\lambda}{\pi}\sum_k\left|\int_{(k-1)\lambda}^{k\lambda}\frac{d}{dx}\frac{\cos x - \cos 2x}{x^2}dx\right| \\
&\le \frac{\lambda}{\pi}\int_{\mathbb{R}}\left|\frac{d}{dx}\frac{\cos x - \cos 2x}{x^2}\right|dx = C\lambda.
\end{aligned} \tag{14.12}$$

Combining (14.9)–(14.12) one obtains the inequality

$$|a_n| = |c_n[F]| = |c_n[F(1-\tau_\lambda)] + c_n[F\tau_\lambda]| \le |c_n[H_\lambda]| + MC\lambda. \tag{14.13}$$

Now H_λ is a pseudofunction, hence $|a_n| \le (1+MC)\lambda$ for $n \ge n_0(\lambda)$. Since $\lambda > 0$ may be taken arbitrarily small, $a_n \to 0$ as $n \to \infty$. □

We can now prove the Fatou–Riesz extension in Theorem 13.4 under weaker conditions.

Theorem 14.5. *Let $f(z) = \sum_0^\infty a_n z^n$ be analytic for $|z| < 1$ and have pseudofunction boundary behavior at all points $\zeta \in C(0,1)$ outside a closed subset E of $C(0,1)$. Suppose that E has linear measure zero and that*

$$\sup_{\zeta\in E}\sup_{n\ge 0}\left|\sum_{k=0}^n a_k\zeta^k\right| = M < \infty. \tag{14.14}$$

Then $a_n \to 0$ as $n \to \infty$ (so that f has pseudofunction boundary behavior everywhere). As a result, the series $\sum_0^\infty a_n z_0^n$ converges to $f(z_0)$ at every point $z_0 \in C(0,1)$ where f is weakly regular, in the sense that the quotient

$$q(z) = \frac{f(z)-f(z_0)}{z-z_0}, \quad |z| < 1,$$

has pseudofunction boundary behavior at the point z_0.

Proof. In view of Proposition 14.3 it is enough to show that $a_n \to 0$. If E is empty, $f(z)$ has boundary pseudofunction $f(e^{it})$ on $C(0,1)$ so that $a_n \to 0$. We may thus assume that E is not empty or there is nothing to prove. Then it follows from (14.14) that the sequence $\{a_n\}$ is bounded, so that $f(z)$ has boundary pseudomeasure $F(t) = f(e^{it})$.

The complement of the compact set E on the unit circumference has measure 2π. It can be represented as the union of a countable family of maximal open arcs. One may choose a finite number of these arcs of total length very close to 2π. On each of these arcs F is equal to a pseudofunction. The finitely many complementary closed arcs jointly cover E. It is convenient to replace degenerate arcs among them by closed arcs of very small positive length. Let l be the minimal length of the arcs in our covering. To prepare for an argument similar to that used for Proposition 14.4, we extend each of the closed arcs by a closed arc of length l at both ends. Combining arcs with nonempty intersection we obtain a covering of E by a finite set of disjoint closed arcs $J_1, \cdots, J_p$, each having length $\geq 3l$. It may be assumed that their total length is bounded by a preassigned number $\varepsilon \in (0, 2\pi)$.

Let us focus on one of the arcs $I = J_k$ and identify the point e^{it} of the circle with $t \pmod{2\pi}$. Then I becomes an interval $a - l \leq t \leq b + l$ such that $[a, b]$ intersects E while $[a - l, a)$ and $(b, b + l]$ do not. We now introduce a partition of unity $\mathcal{P}$ on 'the circle' $(-\pi, \pi]$, consisting of N translates τ_λ^* of an elementary trapezoidal function τ_λ as in (14.4). Observe that the support of τ_λ^* has length 4λ, that the sum of two consecutive functions τ_λ^* has support of length 7λ, and that 2π must be N times 3λ. We will take $\lambda < l/7$, so that every open interval of length l contains the support of one of the functions in $\mathcal{P}$. Taking $N = 2\pi/(3\lambda) > 14\pi/(3l)$ minimal one finds that $\lambda \geq l/8$. A picture indicates that the number of functions τ_λ^* whose support intersects $[a, b]$ is majorized by $(b - a)/\lambda$. Thus the total number of $\tau_\lambda^* \in \mathcal{P}$ whose support meets E is majorized by $\sum_k L(J_k)/\lambda \leq \varepsilon/\lambda$.

For a function τ_λ^* whose support meets E, a computation similar to the one in (14.10)–(14.12) will show that the Fourier coefficients $c_n[F\tau_\lambda^*]$ are majorized by $MC'\lambda$, where M is as in (14.14) and C' is an absolute constant. We sketch the details. In the earlier proof, the trapezoidal function τ_λ was centered at the singular point $t = 0$. In the present case, the 'center' of τ_λ^* may have distance μ to E, where $\mu \leq 2\lambda$. This corresponds to the situation in the earlier proof in which τ_λ would be shifted over a distance μ. The effect would be that the Fourier coefficients $c_n = c_n[\tau_\lambda]$ are multiplied by $e^{in\mu}$ or $e^{-in\mu}$, say the former. In this situation

$$\sum_k |c_k - c_{k-1}| \leq \frac{\lambda}{\pi} \int_{\mathbb{R}} \left| \frac{d}{dx} \left(\frac{\cos x - \cos 2x}{x^2} e^{i(\mu/\lambda)x} \right) \right| dx$$

$$\leq \frac{\lambda}{\pi} \int_{\mathbb{R}} \left\{ \frac{\mu}{\lambda} \left| \frac{\cos x - \cos 2x}{x^2} \right| + \left| \frac{d}{dx} \frac{\cos x - \cos 2x}{x^2} \right| \right\} dx \leq C'\lambda.$$

[Here C' corresponds to the worst case $|\mu/\lambda| = 2$.] Thus $|c_n[F\tau_\lambda^*]| \leq MC'\lambda$ whenever the support of τ_λ^* meets E.

We now consider all functions $\tau_\lambda^* \in \mathcal{P}$ whose support meets E. Multiplying their number by the above bound on $|c_n|$, we find that the sum of the Fourier coefficients $c_n[F\tau_\lambda^*]$ with $\tau_\lambda^* \in \mathcal{P}$ has absolute value $\leq (\varepsilon/\lambda)MC'\lambda = MC'\varepsilon$. For the finitely many $\tau_\lambda^* \in \mathcal{P}$ whose support does not meet E, the product $F\tau_\lambda^*$ is a pseudofunction; the sum of the corresponding coefficients c_n tends to 0 as $|n|$ goes to ∞. The final conclusion is that $a_n \to 0$; cf. (14.13).

The second part of the Theorem now follows from Proposition 14.3.

We do not formulate the corresponding general result for Laplace transforms, but describe how one can strengthen Theorem 7.1 by the introduction of pseudofunction boundary behavior.

Theorem 14.6. *Let $\alpha(\cdot)$ vanish on $(-\infty, 0)$ and be bounded on $[0, \infty)$, so that the Laplace transform $F(z) = \mathcal{L}\alpha(z)$, $z = x + iy$ is analytic for $x = \operatorname{Re} z > 0$. Suppose that $F(x)$ tends to a limit $F(0)$ as $x \searrow 0$ and that the quotient*

$$Q(x+iy) = \frac{F(x+iy) - F(x)}{iy}, \quad x > 0,$$

converges distributionally on every finite interval $\{-R < y < R\}$ to a pseudofunction $Q(iy) = Q_R(iy)$ as $x \searrow 0$. Then

$$\int_0^{\infty-} \alpha(t)dt = F(0).$$

Locally integrable functions $G_x(y) = G(x+iy)$ on $-\infty < y < \infty$ converge distributionally to $G(iy)$ as $x \searrow 0$ if

$$\int_{\mathbb{R}} G(x+iy)\phi(y)dy \to < G(iy), \phi(y) >$$

for all testing functions ϕ, usually the C^∞ functions with compact support. The limit distribution $G(iy)$ is locally equal to a pseudofunction if the Fourier coefficients c_n of the products $G(iy)\phi(y)$ on an appropriate interval tend to zero as $|n| \to \infty$. Replacing $\phi(y)$ by products $\phi(y)e^{i\delta y}$, one may verify the equivalent statement that $< G(iy), \phi(y)e^{iBy} > \to 0$ as $|B| \to \infty$.

Proof of Theorem 14.6. Denoting $\sup_{t>0} |\alpha(t)|$ by M and taking $\varepsilon > 0$, we apply formula (7.15) to $F(\varepsilon + z)$ instead of $F(z)$. Thus we obtain the inequality

$$\begin{aligned} &\left| \int_0^B \alpha(t)e^{-\varepsilon t}dt - F(\varepsilon) \right| \qquad (14.15) \\ &\le \frac{2M}{R} + \frac{|F(\varepsilon)|}{eBR} + \frac{1}{2\pi}\left| \int_{-R}^R \{F(\varepsilon+iy) - F(\varepsilon)\}\left(\frac{1}{iy} + \frac{iy}{R^2}\right)e^{iBy}dy \right|. \end{aligned}$$

To treat the final integral we set

$$G(\varepsilon + iy) = \{F(\varepsilon+iy) - F(\varepsilon)\}\left(\frac{1}{iy} + \frac{iy}{R^2}\right). \qquad (14.16)$$

Let χ_R denote the characteristic function of the interval $[-R, R]$ and let τ_λ denote a trapezoidal testing function which is equal to 1 on $[-\lambda, \lambda]$ and equal to 0 outside $(-2\lambda, 2\lambda)$. The last integral in (14.15) may then be written in distributional notation as

$$I(B, \varepsilon) = < G(\varepsilon + iy)\tau_R(y)\chi_R(y), e^{iBy}\tau_R(y) > . \qquad (14.17)$$

By the hypothesis $G(\varepsilon + iy)\tau_\lambda(y)$ tends to a pseudofunction $G(iy)\tau_\lambda(y)$ for any $\lambda > 0$ as $\varepsilon \searrow 0$, but will $I(B, \varepsilon)$ tend to the formal limit $I(B, 0)$? Multiplication by the cut-off function $\chi_R(y)$ may cause problems!

A solution may be obtained by splitting of the integral $I(B, \varepsilon)$. Taking $R \geq 2$ we first consider the relation

$$< G(\varepsilon + iy)\tau_1(y), e^{iBy}\tau_R(y) > \to < G(iy)\tau_1(y), e^{iBy}\tau_R(y) > \quad \text{as } \varepsilon \searrow 0;$$

here the final expression tends to zero as $B \to \infty$. It remains to consider

$$< G(\varepsilon + iy)\tau_R(y)\{1 - \tau_1(y)\}\chi_R(y), e^{iBy}\tau_R(y) > . \tag{14.18}$$

Since the term involving $F(\varepsilon)$ converges to an ordinary trigonometric integral, we focus on the constituent of the first factor which involves $F(\varepsilon + iy)$:

$$F(\varepsilon + iy)\tau_R(y) \cdot \left(\frac{1}{iy} + \frac{iy}{R^2}\right)\{1 - \tau_1(y)\}\chi_R(y). \tag{14.19}$$

The hypotheses imply that $F(\varepsilon + iy)\tau_R(y)$ tends to a pseudofunction as $\varepsilon \searrow 0$. Observe now that the functions $F(\varepsilon + iy)$, with $0 < \varepsilon < 1$, are the Fourier transforms of the functions $\alpha(t)e^{-\varepsilon t}$ which are all bounded by M. It follows that the Fourier coefficients of $F(\varepsilon + iy)\tau_R(y)$ for the interval $[-2R, 2R]$ form a uniformly bounded family. On the other hand, the Fourier coefficients c_n of the factor

$$\left(\frac{1}{iy} + \frac{iy}{R^2}\right)\{1 - \tau_1(y)\}\chi_R(y)$$

are $\mathcal{O}\{1/(n^2 + 1)\}$. [Use integration by parts and observe that the factor vanishes for $|y| \leq 1$ and for $|y| \geq R$.] Thus the limit of the functions in (14.19) is a pseudomeasure, and in fact, a pseudofunction. The same will be true for the limit

$$G(iy)\tau_R(y)\{1 - \tau_1(y)\}\chi_R(y) = \lim_{\varepsilon \searrow 0} G(\varepsilon + iy)\tau_R(y)\{1 - \tau_1(y)\}\chi_R(y)$$

of the functions in (14.18). Combining the results, one concludes that the limit $I(B, 0)$ of $I(B, \varepsilon)$ can be written as an inner product

$$I(B, 0) = < H(y), e^{iBy}\tau_R(y) >$$

involving a pseudofunction H, so that $I(B, 0) \to 0$ as $B \to \infty$.

To complete the proof of the Theorem we return to (14.15). First letting ε go to zero and then B to ∞, one obtains inequality (7.16). Finally let R go to ∞. □

Remarks 14.7. A proof of inequality (7.16) for the case where one has convergence to a pseudofunction only on a *fixed interval* $\{-R < y < R\}$ may be found in Korevaar [2003]. One may also introduce pseudofunction boundary behavior in the statements of the Wiener–Ikehara theorem and Proposition 4.3. Furthermore, the method of Proposition 14.3 can be used to derive a 'pseudofunction form' of Riemann's localization principle.

15 Applications to Operator Theory

The results in Sections 13, 14 have analogs for analytic functions with values in a Banach space, including a space of bounded linear operators. The proofs are similar to those for the scalar case. For details we refer to the papers and the book quoted in Section 13.

For the discussion of applications we recall some definitions. Let L be a bounded linear operator on a (complex) Banach space X. The resolvent set of L is the (open) set of complex numbers λ for which $L - \lambda I$ has a bounded inverse. The *spectrum* of L, $\mathrm{sp}(L)$, is defined as the complement of the resolvent set. There are names for special subsets of $\mathrm{sp}(L)$. The numbers λ for which $L - \lambda I$ fails to be one to one form the point spectrum. The continuous spectrum consists of those λ for which $L - \lambda I$ is one to one onto a dense subspace of X. If the range of $L - \lambda I$ fails to be dense in X, one says that λ belongs to the *residual spectrum* of L. It coincides with the point spectrum of the adjoint L^* and may overlap the point spectrum of L.

Here we consider so-called *power bounded operators* L on X, that is,

$$\sup_{n\geq 0} \|L^n\| = M < \infty. \tag{15.1}$$

In this case $I - L/\lambda$ has a bounded inverse whenever $|\lambda| > 1$, given by the usual geometric series. Hence the spectrum belongs to the closed unit disc. For the following we need only the 'peripheral spectrum', the part of $\mathrm{sp}(L)$ on the circumference $C(0, 1)$.

Work of Esterle [1983] (section 9) and Katznelson and Tzafriri [1986] implies

Proposition 15.1. *Let L be a power bounded linear operator on X whose spectrum meets $C(0, 1)$ only in the point $z = 1$. Then*

$$\|L^n - L^{n+1}\| \to 0 \quad \textit{as } n \to \infty. \tag{15.2}$$

Proof. The resolvent set of L contains the exterior of the unit disc and the circle $C(0, 1)$ except for the point $z = 1$. Thus the operator $I - zL$ is invertible for $|z| \leq 1$ except when $z = 1$. For $|z| < 1$ its inverse is given by the operator-valued function

$$g(z) = \sum_0^\infty s_n z^n \stackrel{\text{def}}{=} \sum_0^\infty L^n z^n.$$

This function is analytic for $|z| < 1$ as well as in a neighborhood of each point $\zeta \in C(0, 1) \setminus 1$. The same is true for

$$f(z) = (1 - z)g(z) = \sum_{n=0}^\infty a_n z^n = I + \sum_{n=1}^\infty (L^n - L^{n-1})z^n.$$

Now the sequence $\{s_n\}$ is bounded, hence by the analog of Proposition 13.3 for operator-valued functions, $\|a_n\| = \|L^n - L^{n-1}\| \to 0$. This proves the result.

As a corollary one obtains a classical result of Gel'fand [1941b]:

Corollary 15.2. *If L is an isometry, or more generally, if*

$$\sup_{n\in\mathbb{Z}} \|L^n\| = M < \infty,$$

and if $\mathrm{sp}(L) = \{1\}$, *then L is the identity.*

Indeed, for given $\varepsilon > 0$ and large $n > 0$, $\|L^n - L^{n+1}\| < \varepsilon$ by (15.2), hence

$$\|I - L\| = \|L^{-n}(L^n - L^{n+1})\| \leq M\varepsilon.$$

The method of Proposition 15.1 also gives

Proposition 15.3. *Let L be a power bounded operator on X as in* (15.1) *such that the intersection* $\mathrm{sp}(L) \cap C(0, 1)$ *is finite and does not contain a point of the residual spectrum. Then $L^n x \to 0$ for every $x \in X$.*

Proof. Denote the points of $\mathrm{sp}(L) \cap C(0, 1)$ by λ_j, $j = 1, \cdots, q$, and set

$$p(L) = (L - \lambda_1 I) \cdots (L - \lambda_q I).$$

This time $I - zL = -z(L - I/z)$ will be invertible at every point $z = \zeta \in C(0, 1)$ different from the points $\zeta_j = 1/\lambda_j$. Thus the function $\sum_0^\infty L^n z^n$ has an analytic continuation to a neighborhood of those points ζ. The same will be true for

$$f(z) = \sum_0^\infty a_n z^n \stackrel{\text{def}}{=} \sum_{n=0}^\infty L^n p(L) z^n. \tag{15.3}$$

We will verify that for the singular points ζ_j, the partial sums

$$s_n(\zeta_j) = \sum_{k=0}^n L^k p(L) \zeta_j^k$$

form bounded sequences. Indeed, by the hypotheses

$$s_n(\zeta_j) = p(L) \frac{(\zeta_j L)^{n+1} - I}{\zeta_j L - I} = \lambda_j \prod_{m\neq j} (L - \lambda_m I) \cdot \{(\zeta_j L)^{n+1} - I\}$$

is in norm bounded by $(\|L\| + 1)^{q-1}(M + 1)$.

Thus by an operator analog of Theorem 13.4 or Theorem 14.5 for the simple case of a finite singularity set E, one has $L^n p(L)x \to 0$ for every $x \in X$. Since the points λ_j are not in the residual spectrum of L, the ranges of the operators $L - \lambda_j I$ are dense in X. It now follows by approximation that the elements $p(L)x$ are dense in X, so that also $L^n y \to 0$ for every $y \in X$.

The following extension of Proposition 15.3 to the case of countably infinite 'peripheral spectrum' is more difficult.

Theorem 15.4. *Let L be a power bounded linear operator on X. Suppose that* $\mathrm{sp}(L)$ *meets* $C(0, 1)$ *only in a countable set and that* $C(0, 1)$ *contains no point of the residual spectrum of L (no eigenvalue of L if X is reflexive). Then*

$$L^n x \to 0 \quad \text{as } n \to \infty, \ \forall x \in X.$$

Remarks 15.5. For the case of Hilbert space there is a related result, due to Sz.-Nagy and Foiaş [1970] (p. 85), which involves so-called completely nonunitary contractions.

Theorem 15.4 has a companion for continuous semigroups of operators. Let $\{L(t),\ t \geq 0\}$ be such a semigroup with generator A. Suppose that it satisfies the following conditions: $\sup_{t\geq 0} \|L(t)\| < \infty$, $L(t)x$ is continuous for every $x \in X$, $\mathrm{sp}(A) \cap i\mathbb{R}$ is countable and the residual spectrum of A does not meet $i\mathbb{R}$. Then $L(t)x \to 0$ as $t \to \infty$ for every $x \in X$. Arendt and Batty [1988] obtained this result and Theorem 15.4 with the aid of a precise estimate in the proof of Theorem 13.4 by Allan, O'Farrell and Ransford [1987]. Another proof is contained in the work of Lyubich and Vu [1988]. A third proof was given by Esterle, Strouse and Zouakia [1990]. Related results on semigroups of operators may be found in Batty [1994b], Batty and Yeates [2000], and in the book by Arendt, Batty, Hieber and Neubrander [2001].

16 Complex Remainder Theory

There is an extensive body of Tauberian remainder theory in which the transforms of series or functions are subject to conditions in the complex domain; cf. the books by Ganelius [1971] and Postnikov [1980], and especially the elaborate treatment by Subhankulov [1976] (Russian). Here we restrict ourselves to some striking examples of the strong results provided by complex remainder theory.

We begin with a remainder theorem associated with Fatou's theorem. The result was originally stated for Laplace integrals and derived by real approximation (Korevaar [1954b]). The proof in Section 17 below is more direct.

In Section 18 we use a method of Postnikov and Subhankulov to derive a complex remainder theorem for power series which corresponds to the Hardy–Littlewood theorem mentioned in Section 1. More general remainder theorems for Dirichlet series and Laplace integrals can be found in the book by Subhankulov.

Finally, in Section 19, we discuss a complex remainder estimate for the Stieltjes transform, due to Malliavin [1962] and Pleijel [1963]. This result has been used to estimate the counting function for the eigenvalues in certain elliptic problems; cf. Agmon and Kannai [1967].

For the case of power series we will use the following auxiliary result.

Proposition 16.1. *Let $p \in \mathbb{N}$ be fixed and let* $0 < r < 1$. *Then for λ running over* $(0, 1]$ *and for m running over $\mathbb{Z}$,*

$$I_m(\lambda) = I_m(\lambda, p, r) = \int_{-\lambda}^{\lambda} \frac{(e^{it} - e^{i\lambda})^p (e^{it} - e^{-i\lambda})^p}{1 - re^{it}} e^{imt} dt \tag{16.1}$$

$$= \begin{cases} \mathcal{O}\left(\dfrac{\lambda^{2p}}{\lambda^p |m|^p + 1}\right) & \textit{for } m > -2p, \\ 2\pi r^{-2p-m}(1 - 2r\cos\lambda + r^2)^p + \mathcal{O}\left(\dfrac{\lambda^{2p}}{\lambda^p |m|^p + 1}\right) & \textit{for } m \le -2p. \end{cases}$$

Here the constants in the $\mathcal{O}$-terms may be taken independent of r if one requires that $1 - r \le \lambda/4$.

In Section 17 we need only the case of fixed λ and corresponding error term $\mathcal{O}\{1/(|m|^p + 1)\}$. For this case the result is due to Postnikov, who used real analysis, in [1953] for $p = 2$, in [1980] for general p. The more precise result which allows $\lambda \to 0$ will be used in Section 18 for a remainder estimate involving power series. A result of similar character was used by Subhankulov [1960] for remainder estimates involving Dirichlet series.

Proof of Proposition 16.1. Setting $e^{it} = z$ one has

$$I_m(\lambda) = \frac{1}{i} \int_\Gamma \frac{(z - e^{i\lambda})^p (z - e^{-i\lambda})^p}{1 - rz} z^{m-1} dz, \tag{16.2}$$

where Γ is the arc of the (positively oriented) unit circle from $e^{-i\lambda}$ to $e^{i\lambda}$. To estimate the integral we will change the path of integration.

For $m \ge 1$ we replace Γ by the arc Γ_1 of the circle $|z - 1| = |e^{i\lambda} - 1|$ from $e^{-i\lambda}$ to $e^{i\lambda}$ on which $|z| \le 1$. Since $|1 - rz| \ge c\lambda$ on Γ_1, straightforward estimation shows that $I_m(\lambda) = \mathcal{O}(\lambda^{2p})$. Starting with $z^{m-1}dz = dz^m/m$ one can also integrate by parts p times on Γ_1. This results in the estimate that $I_m(\lambda) = \mathcal{O}(\lambda^p/m^p)$. It is convenient to combine the two estimates into the $\mathcal{O}$-term of the Proposition.

Next suppose that $m \le -2p$. Assuming now that $1 - r \le \lambda/4$ we replace the path of integration Γ by the arc Γ_2 of the circle $|z - 1| = |e^{i\lambda} - 1|$ from $e^{-i\lambda}$ to $e^{i\lambda}$ on which $|z| \ge 1$. In doing so we pick up a residue at the pole of the integrand:

$$\begin{aligned} \int_\Gamma - \int_{\Gamma_2} &= -2\pi i \times \text{ residue of the integrand at } z = 1/r \\ &= 2\pi i (1 - re^{i\lambda})^p (1 - re^{-i\lambda})^p r^{-2p-m}. \end{aligned}$$

Divided by i, this result gives the principal term in the last line of (16.1). To estimate the remainder from $\int_{\Gamma_2}$ one may proceed as in the case of Γ_1.

For $-2p < m < 1$ direct estimation on Γ_1 gives $I_m(\lambda) = \mathcal{O}(\lambda^{2p})$, which implies the desired estimate for these restricted values of m.

Corollary 16.2. *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ for $r = |z| < 1$. Then for $\lambda \in (0, 1]$, $1 - r \le \lambda/4$ and $n \ge 2p$,*

$$\int_{-\lambda}^{\lambda} f(re^{it}) \frac{(e^{it} - e^{i\lambda})^p (e^{it} - e^{-i\lambda})^p}{1 - re^{it}} e^{-int} dt = \sum_{k=0}^{\infty} a_k r^k I_{k-n}(\lambda, p, r) =$$

$$2\pi r^{n-2p}(1 - 2r\cos\lambda + r^2)^p \sum_{k=0}^{n-2p} a_k + \mathcal{O}\left(\sum_{k=0}^{\infty} \frac{|a_k| r^k \lambda^{2p}}{\lambda^p |k-n|^p + 1}\right). \tag{16.3}$$

For the proof one applies the Proposition to each term in the series $f(re^{it}) = \sum_{k=0}^{\infty} a_k r^k e^{ikt}$, taking $m = k - n$.

17 The Remainder in Fatou's Theorem

The comparison functions below involve *slowly varying functions* in the sense of Karamata [1930b], [1933b]: positive (measurable) functions L on $[0, \infty)$ such that $L(\lambda t)/L(t) \to 1$ as $t \to \infty$ for every number $\lambda > 0$. We state the basic integral representation which will be derived in Chapter IV:

$$L(t) = c(t) \exp\left\{\int_0^t \varepsilon(v) \frac{dv}{v}\right\}, \tag{17.1}$$

where $c(\cdot) > 0$ (is measurable and) tends to a limit $c > 0$ at ∞, while $\varepsilon(\cdot)$ is bounded and tends to 0 at ∞. Here we need L only on the nonnegative integers and we may therefore assume that L and $c(\cdot)$ are continuous. Then for any number $\delta > 0$, there are positive constants b and B such that

$$b\left(\frac{u+1}{t+1}\right)^{-\delta} \le \frac{L(u)}{L(t)} \le B\left(\frac{u+1}{t+1}\right)^{\delta} \quad \text{whenever } 0 \le t \le u < \infty. \tag{17.2}$$

Theorem 17.1. *Let $\sum_0^\infty a_n z^n$ converge for $|z| < 1$ and let the sum function $f(z)$ be analytic at the point $z = 1$. Suppose that the numbers a_n are real and that*

$$a_n \ge -\phi(n) = -(n+1)^\alpha L(n), \quad \forall n, \tag{17.3}$$

where α is real and $L(\cdot)$ is slowly varying. Then there is a constant C (depending on f and ϕ) such that for $s_n = \sum_{k=0}^n a_k$,

$$|s_n - f(1)| \le C\phi(n), \quad \forall n. \tag{17.4}$$

The result is contained in Korevaar [1954b] (theorem 4.1). The present proof extends a method of Heilbronn and Landau [1933b] for the case $\phi(n) = c$ and of Postnikov [1980] for the case $\phi(n) = c(n+1)^\alpha$ with $\alpha > -1$. It will use Proposition 16.1 and the auxiliary results below. The proof also uses the fact that

$$\phi(n) = \mathcal{O}\{(n+1)^{\alpha+\delta}\}, \quad (n+1)^{\alpha-\delta} = \mathcal{O}\{\phi(n)\} \tag{17.5}$$

for every $\delta > 0$; cf. (17.2).

We need higher-order kernels related to the Fejér kernel for $\mathbb{R}$. In the following $\mathcal{B}^s$ denotes the class of the indefinite integrals of order s of bounded functions.

Lemma 17.2. *For $p \in \mathbb{N}$ and $\lambda > 0$, let*

$$H_p(x) = \frac{1}{2\pi}\left(\frac{\sin x/p}{x/p}\right)^p, \quad H_{p,\lambda}(x) = \lambda H_p(\lambda x). \tag{17.6}$$

Then the Fourier transform $\hat{H}_{p,\lambda}$ has support in $[-\lambda, \lambda]$ and is of class $\mathcal{B}^{p-1}$. If $p \geq 2$, the derivatives of $\hat{H}_{p,\lambda}$ of order $\leq p-2$ will vanish at $\pm\lambda$.

Cf. the Fourier pair in (4.7) which corresponds to the case $p = 2$. For the proof one may change the scale and consider the p-th power $M_p(x)$ of the simple kernel $M(x) = (\sin x)/(\pi x)$. The latter has Fourier transform $\hat{M}(t)$ equal to 1 for $|t| < 1$ and equal to 0 for $|t| > 1$. Thus $\hat{M}_p$ is the p-th convolution power of $\hat{M}$. Hence it has support $[-p, p]$ and is of class $\mathcal{B}^{p-1}$. In particular its derivatives of order $\leq p-2$ vanish at $\pm p$. The Lemma readily follows.

Observe that by Fourier inversion,

$$\int_{-\lambda}^{\lambda} \hat{H}_{p,\lambda}(t)e^{ixt}dt = 2\pi H_{p,\lambda}(x) = \lambda\left(\frac{\sin \lambda x/p}{\lambda x/p}\right)^p. \tag{17.7}$$

Proposition 17.3. *Under the hypotheses of Theorem 17.1 there is a constant C_1 (depending on f and ϕ) such that*

$$|a_n| \leq C_1\phi(n), \quad \forall n. \tag{17.8}$$

Proof. For $f(re^{it}) = \sum_{k=0}^{\infty} a_k r^k e^{ikt}$, $0 \leq r < 1$ and $n \geq 0$, formula (17.7) gives

$$\begin{aligned}\int_{-\lambda}^{\lambda} \hat{H}_{p,\lambda}(t)f(re^{it})e^{-int}dt &= \sum_{k=0}^{\infty} a_k r^k \lambda\left(\frac{\sin \lambda(k-n)/p}{\lambda(k-n)/p}\right)^p \\ &= \lambda a_n r^n + \lambda^{1-p}p^p \sum_{k\geq 0,\, k\neq n} a_k r^k \frac{\sin^p \lambda(k-n)/p}{(k-n)^p}.\end{aligned} \tag{17.9}$$

We continue with *even* $p = 2q$ and observe that the integral is real, hence (17.3) and (17.9) imply the inequality

$$\lambda a_n r^n \leq \int_{-\lambda}^{\lambda} \hat{H}_{2q,\lambda}(t)f(re^{it})e^{-int}dt + \lambda^{1-2q}(2q)^{2q} \sum_{k\geq 0,\, k\neq n} \frac{\phi(k)}{(k-n)^{2q}} r^k. \tag{17.10}$$

Now fix $\lambda > 0$ so small that f is analytic on the arc $\{|z| = 1, \ |\arg z| \leq \lambda\}$ and take $2q > |\alpha| + 1$. In view of (17.5) (with k instead of n) we may then pass to the limit as $r \nearrow 1$ and henceforth consider (17.10) with $r = 1$.

To prove (17.8) one may take $n \geq 1$, write $e^{-int}dt = -de^{-int}/(in)$ and integrate by parts $2q - 1$ times. This will show that the integral with $r = 1$ is $\mathcal{O}(n^{-2q+1}) = \mathcal{O}\{n^\alpha L(n)\} = \mathcal{O}\{\phi(n)\}$. [The integrated terms drop out.] To verify that the sum with $r = 1$ is also $\mathcal{O}\{\phi(n)\}$ one first estimates the numbers $\phi(k)$ in terms of $\phi(n)$ with the aid of (17.2). The parts of the resulting sum where $k > 2n$ or $k < n/2$ may be

compared with integrals; the part where $n/2 \le k \le 2n$ is easy. The conclusion is that indeed $a_n = \mathcal{O}\{\phi(n)\}$.

Proof of Theorem 17.1. By changing the constant term in the power series for f we may assume that $f(1) = 0$. Observe that by Proposition 17.3 and (17.5) one has $a_k = \mathcal{O}\{\phi(k)\} = \mathcal{O}\{(k+1)^{\alpha+\delta}\}$ for every $\delta > 0$. We now apply Corollary 16.2. Here we fix $\lambda \in (0, 1]$ so small that $f(z)$, hence also $f(z)/(1-z)$, is analytic on the arc $\{|z| = 1,\ |\arg z| \le \lambda\}$ and we take $p > |\alpha| + 1$. Under our hypotheses we may then let $r \nearrow 1$.

From here on we use formula (16.3) with $r = 1$: for $n \ge 2p$,

$$\int_{-\lambda}^{\lambda} \frac{f(e^{it})}{1-e^{it}}(e^{it} - e^{i\lambda})^p (e^{it} - e^{-i\lambda})^p e^{-int} dt$$
$$= 2\pi(2 - 2\cos\lambda)^p s_{n-2p} + \mathcal{O}\left(\sum_{k=0}^{\infty} \frac{\phi(k)}{|k-n|^p + 1}\right). \tag{17.11}$$

By the analyticity of $f(z)/(1-z)$ we may integrate by parts p times in the integral to show that it is $\mathcal{O}(n^{-p}) = \mathcal{O}\{\phi(n)\}$. By the argument used in the proof of Proposition 17.3, the sum is likewise $\mathcal{O}\{\phi(n)\}$. It follows that $s_{n-2p} = \mathcal{O}\{\phi(n)\}$ and this completes the proof of (17.4).

Remarks 17.4. The proof does not require analyticity of f at the point $z = 1$, but one does need appropriate smoothness in a neighborhood of 1 in the closed disc $\{|z| \le 1\}$, depending on α. Cf. also Kraĭnova [1984].

Further inequalities may be derived by applying Theorem 17.1 to, for example,

$$\sum_0^{\infty}\{s_n - f(1)\}z^n = -\frac{f(z) - f(1)}{z-1}.$$

Since $|s_n - f(1)| \le C\phi(n)$ we can use Theorem 17.1 with a_n replaced by $s_n - f(1)$. The result is

$$\left|\frac{s_0 + \cdots + s_n}{n+1} - f(1) + \frac{f'(1)}{n+1}\right| \le C'\frac{\phi(n)}{n+1};$$

cf. Korevaar [1954b]. Related inequalities have been obtained by T.M. Safarov [1974a], [1974b].

18 Remainders in Hardy–Littlewood Theorems Involving Power Series

Let the power series

$$f(x) = \sum_0^{\infty} a_n x^n \tag{18.1}$$

converge for $|x| < 1$ and let $a_n \ge -C$. Under the hypothesis

$$(1-x)f(x) \to A \quad \text{as } x \nearrow 1,$$

Hardy and Littlewood [1914a] found that $s_n = \sum_0^n a_k \sim An$ as $n \to \infty$. (For the simple proof by Karamata [1930a], see Section I.11.) Under the stronger hypothesis that

$$f(x) - \frac{A}{1-x} = \mathcal{O}\left\{\frac{1}{(1-x)^\alpha}\right\} \quad \text{for } 0 \le x < 1, \tag{18.2}$$

with $\alpha < 1$, the real remainder theory of Freud [1951], [1952/53], [1954] and the author [1951], [1953], [1954a] gives the estimate $s_n - An = \mathcal{O}(n/\log n)$; cf. Chapter VII. Here the 'disappointingly large' remainder $\mathcal{O}(n/\log n)$ is optimal! In fact, real conditions give smaller remainders $s_n - An$ only if the deviation $f(x) - A/(1-x)$ tends to zero extremely fast. For example, a deviation of the form $\mathcal{O}(\exp\{-c/(1-x)\})$ in (18.2) gives (optimal) remainder $s_n - An = \mathcal{O}(n^{1/2})$; see Korevaar [1954a] and Section VII.2.

In this section we treat the following complex remainder theorem which is contained in work of Subhankulov [1960] for general Dirichlet series; see also his book [1976] (chapter 2).

Theorem 18.1. *Let the power series*

$$f(z) = \sum_0^\infty a_n z^n, \quad z = re^{it}, \tag{18.3}$$

with $a_n \ge -C$, converge for $|z| < 1$. Let $0 \le \alpha,\ \beta < 1,\ c > 0$ and suppose that

$$f(z) - \frac{A}{1-z} = \mathcal{O}\left\{\frac{1}{|1-z|^\alpha}\right\} \quad \textit{for } 0 \le r < 1,\ |t| \le c(1-r)^\beta \tag{18.4}$$

(which describes a region tangent to the unit circle at the point $z = 1$). Then the partial sums s_n satisfy the estimate

$$s_n - An = \mathcal{O}(n^\alpha) + \mathcal{O}(n^\beta) \quad \textit{as } n \to \infty. \tag{18.5}$$

Here n^α has to be replaced by $\log n$ *if $\alpha = \beta = 0$.*

Proof. (First Part) Subtracting A from the numbers a_n it may henceforth be assumed that $A = 0$. One then has to prove that $s_n = \mathcal{O}(n^\alpha) + \mathcal{O}(n^\beta)$ or $\mathcal{O}(\log n)$ for $n \ge n_0$. To this end we will apply Corollary 16.2 with $p = 2$ and $\lambda = c(1-r)^\beta$; it may of course be assumed that $\lambda \le 1$ and $1 - r \le \lambda/4$. The result is that for $n \ge 4$,

$$\begin{aligned} &\int_{-\lambda}^{\lambda} f(re^{it}) \frac{(e^{it} - e^{i\lambda})^2 (e^{it} - e^{-i\lambda})^2}{1 - re^{it}} e^{-int} dt \\ &= 2\pi r^{n-4}(1 - 2r\cos\lambda + r^2)^2 s_{n-4} + \lambda^4 \mathcal{O}\left(\sum_{k=0}^\infty \frac{|a_k| r^k}{\lambda^2 (k-n)^2 + 1}\right). \end{aligned} \tag{18.6}$$

Here we take $1 - r = 1/n$, so that the coefficient of s_{n-4} is at least equal to a positive constant times λ^4. By (18.4) with $A = 0$, the integral in (18.6) is

$$\mathcal{O}\left(\lambda^4 \int_{-\lambda}^{\lambda} \frac{dt}{|1-re^{it}|^{\alpha+1}}\right) = \lambda^4\, \mathcal{O}\left(\int_0^1 \frac{dt}{(1-r+t)^{\alpha+1}}\right) = \lambda^4\, \mathcal{O}\{(1-r)^{-\alpha}\}$$

for $\alpha > 0$ and equal to $\lambda^4\mathcal{O}(\log\{1/(1-r)\})$ if $\alpha = 0$. For our r the result is $\lambda^4\, \mathcal{O}(n^\alpha)$ if $\alpha > 0$ and $\lambda^4\, \mathcal{O}(\log n)$ if $\alpha = 0$. We now complete the proof for

THE CASE $|a_k| \le C$. To estimate the sum in (18.6) we split it. The sum over the terms with $|k-n| \le 1/\lambda$ is obviously $\mathcal{O}(1/\lambda)$, which for our λ and r becomes $\mathcal{O}(n^\beta)$. Likewise, the sum with $|k-n| > 1/\lambda$ is

$$\mathcal{O}\left(\frac{1}{\lambda^2} \sum_{|k-n|>1/\lambda} \frac{1}{(k-n)^2}\right) = \mathcal{O}(1/\lambda) = \mathcal{O}(n^\beta).$$

This is also true if $\beta = 0$. Inserting the estimates for integral and sum into (18.6) and dividing by λ^4, one obtains the desired result for the partial sums s_{n-4} when $|a_k| \le C$.

For the GENERAL CASE of Theorem 18.1, we show first that under the one-sided condition $a_k \ge -C$, where we take $C > 0$, certain running averages of the numbers $|a_k|$ are bounded.

Lemma 18.2. *Let f satisfy the conditions in Theorem 18.1 and suppose that* $0 < \lambda = c(1-r)^\beta \le 1$ *and* $1 - r = 1/n \le \lambda/4$ *as in the proof above. Then*

$$\sum_{|k-m|\le 1/\lambda} |a_k| \le \sum_{|k-m|\le 1/\lambda} (a_k + 2C) = \mathcal{O}(1/\lambda) = \mathcal{O}(n^\beta), \qquad (18.7)$$

uniformly for m satisfying $|m-n| \le \sqrt{n/\lambda}$.

Proof. By formula (17.9) with $p = 2$ and m instead of n,

$$\sum_{k=0}^{\infty}(a_k + 2C)r^k \left(\frac{\sin\lambda(k-m)/2}{\lambda(k-m)/2}\right)^2 \qquad (18.8)$$
$$= \frac{1}{\lambda}\int_{-\lambda}^{\lambda}\left(1 - \frac{|t|}{\lambda}\right) f(re^{it})e^{-imt}dt + 2C\sum_{k=0}^{\infty} r^k \left(\frac{\sin\lambda(k-m)/2}{\lambda(k-m)/2}\right)^2;$$

cf. (4.7). Since in the integral one has $f(re^{it}) = \mathcal{O}(1/|t|^\alpha)$ with $\alpha < 1$, the first term on the right will be $\mathcal{O}(1/\lambda^\alpha)$. By the computation in the preceding proof the final sum is $\mathcal{O}(1/\lambda)$. Now for $|k-m| \le 1/\lambda$, $|m-n| \le \sqrt{n/\lambda}$ and $r = 1 - 1/n$,

$$r^k \ge r^{n+\sqrt{n/\lambda}+1/\lambda} \ge c_1 > 0, \qquad \left(\frac{\sin\lambda(k-m)/2}{\lambda(k-m)/2}\right)^2 \ge c_2 > 0.$$

Thus (18.8) implies (18.7).

Proof of Theorem 18.1. (SECOND PART) To complete the proof of the Theorem, we need a good estimate for the final sum in (18.6) under the one-sided condition $a_k \ge -C$. This time we split the sum in a different manner. First let $|k-n| > \sqrt{n/\lambda}$:

$$\sum_{|k-n|>\sqrt{n/\lambda}} \frac{|a_k|r^k}{\lambda^2(k-n)^2+1}$$
$$\leq \frac{1}{\lambda n}\sum_{k=0}^{\infty}(a_k+2C)r^k = \frac{1}{\lambda n}\left(f(r)+\frac{2C}{1-r}\right) = \mathcal{O}(1/\lambda). \tag{18.9}$$

By Lemma 18.2 the sum with $|k-n| < 1/\lambda$ is also $\mathcal{O}(1/\lambda)$. It remains to consider the sum with $1/\lambda \leq |k-n| \leq \sqrt{n/\lambda}$, which we majorize by a sum over parts in which $|k-m| \leq 1/\lambda$ for suitable values of m. Decreasing c as necessary one may assume that $1/\lambda = n^{\beta}/c$ is an integer. Now for $m_s = n + 2s/\lambda$ with $s = 1, 2, \ldots$ and $|k - m_s| \leq 1/\lambda$, one has

$$\frac{1}{\lambda^2(k-n)^2+1} < \frac{1}{(2s-1)^2}.$$

Thus by Lemma 18.2,

$$\sum_{1/\lambda\leq k-n\leq\sqrt{n/\lambda}} \frac{|a_k|r^k}{\lambda^2(k-n)^2+1} \leq \sum_{1\leq s\leq\frac{1}{2}\sqrt{\lambda n}} \;\sum_{|k-m_s|\leq 1/\lambda} \frac{|a_k|}{\lambda^2(k-n)^2+1}$$
$$\leq \sum_{1\leq s\leq\frac{1}{2}\sqrt{\lambda n}} \frac{1}{(2s-1)^2}\sum_{|k-m_s|\leq 1/\lambda}|a_k| = \mathcal{O}(1/\lambda). \tag{18.10}$$

Also considering $m_s' = n - 2s/\lambda$ one obtains the same estimate for the sum with $-\sqrt{n/\lambda} \leq k - n \leq -1/\lambda$. Combining the estimates one finds that the final sum in (18.6) is $\mathcal{O}(1/\lambda) = \mathcal{O}(n^{\beta})$. This completes the proof of the Theorem.

Remark 18.3. Postnikov [1953] had treated the important case $\alpha = \beta = 0$ of Theorem 18.1 which does not require letting λ go to 0. In his monograph [1980], he also showed that in this case, the remainder estimate $s_n - An = \mathcal{O}(\log n)$ in (18.5) is essentially optimal. However, if $g(z) = f(z) - A/(1-z)$ in Theorem 18.1 is *analytic* at $z = 1$, it follows from Theorem 17.1 applied to g that $s_n - An = \mathcal{O}(1)$.

Remarks 18.4. Subhankulov [1964] also obtained remainder estimates for *Littlewood's theorem* based on complex information. Just like Postnikov in his monograph (loc. cit. section 19), we restrict ourselves here to Littlewood's original Tauberian condition

$$|na_n| \leq C. \tag{18.11}$$

If $f(x) = \sum_0^{\infty} a_n x^n$ tends to A as $x \nearrow 1$, then $s_n = \sum_0^n a_k \to A$ (Littlewood [1911]; cf. Section I.7). In the case $f(x) - A = \mathcal{O}\{(1-x)^{\alpha}\}$ for $0 \leq x < 1$ with $\alpha > 0$, the real remainder theory of the author [1953] and Freud [1954] gives the optimal estimate $s_n - A = \mathcal{O}(1/\log n)$; see Section VII.2.

Setting $z = re^{it}$, we now impose the complex condition

$$f(z) - A = \mathcal{O}(|1-z|^{\alpha}) \quad \text{for } 0 \leq r < 1,\ |t| \leq c(1-r)^{\beta}, \tag{18.12}$$

with $0 < \alpha \leq 1,\ 0 < \beta < 1$. Under the Tauberian condition (18.11) Subhankulov then found that

$$s_n - A = \mathcal{O}(n^{\beta-1}) + \mathcal{O}(n^{-\alpha\beta}). \tag{18.13}$$

Observe that there must be a better result for small β. Indeed, take $\alpha = 1$. Then the estimate $s_n - A = \mathcal{O}(n^{-1/2})$ for $\beta = 1/2$ must also hold for $\beta = 0$; see (18.12).

Open Problem. The question of an optimal result in Littlewood's theorem under the complex condition (18.12) seems to be open. In his book, Subhankulov [1976] (section 2.6) found, among other things, that condition (18.12), together with the condition $f'(z) = \mathcal{O}\{(1-r)^{\alpha-1}\}$ in the region of (18.12), implies

$$s_n - A = \mathcal{O}(n^{\beta-1}) + \mathcal{O}(n^{-\alpha}).$$

In all of this, (18.11) may be replaced by the one-sided condition $na_n \geq -C$ of Hardy and Littlewood (loc. cit.), provided $\sum a_n z^n$ is known to converge for $|z| < 1$.

19 A Remainder for the Stieltjes Transform

Let $s(t)$ vanish for $t \leq 0$, be nondecreasing, continuous from the right and such that the Stieltjes transform

$$g(x) = \int_0^\infty \frac{ds(t)}{t+x}$$

exists for $x > 0$. Suppose that

$$g(x) = Ax^{\alpha-1} + o(x^{\alpha-1}) \quad \text{as } x \to \infty,$$

with $0 < \alpha < 1$ and $A \geq 0$. Then by a Tauberian theorem of Hardy and Littlewood [1929],

$$s(t) = A't^\alpha + o(t^\alpha) \quad \text{as } t \to \infty, \qquad \text{where } A' = A(\sin\alpha\pi)/(\alpha\pi).$$

A more direct proof by Karamata [1931] may be found in Section I.21. As in the case of power series, real remainder formulas for the transform give only weak remainder estimates for $s(t)$. In the real case one again needs an extremely small deviation $g(x) - Ax^{\alpha-1}$ in order to get a small remainder $s(t) - A't^\alpha$. See Avakumović [1950a] and Subhankulov's book [1976] (chapter 3); more on this in Section VII.18.

Using complex asymptotics for the Stieltjes transform, Malliavin [1962] and Pleijel [1963] obtained the following useful remainder theorem.

Theorem 19.1. *Let $s(\cdot)$, α, A and A' be as above, $0 < \beta < \alpha < 1$ and $0 \leq \gamma < 1$. Suppose that*

$$g(z) = \int_0^\infty \frac{ds(t)}{t+z} = Az^{\alpha-1} + \mathcal{O}(|z|^{\beta-1}) \tag{19.1}$$

on the two curves $z = x \pm i|x|^\gamma$, $-\infty < x \leq -1$. Here $z^{\alpha-1}$ denotes the principal value. Then

$$s(t) - A't^\alpha = \mathcal{O}(t^\beta) + \mathcal{O}(t^{\alpha-1+\gamma}) \quad \textit{as } t \to \infty. \tag{19.2}$$

The proof below, after Pleijel, starts with an important approximate *inversion formula* for the Stieltjes transform.

Lemma 19.2. *Let $w = u + iv$ with $u < 0,\ v > 0$ and let $L(w)$ be any rectifiable arc from $\overline{w}$ to w which does not meet the negative real axis $\{y = 0,\ x \le 0\}$. Define*

$$I(w) = \frac{1}{2\pi i} \int_{L(w)} g(z)dz \tag{19.3}$$

and set $g = g_1 + ig_2$ with real g_j. Then

$$\left| s(|u|) - I(w) + \frac{1}{\pi} v g_1(w) \right| \le \frac{1}{2} v |g_2(w)|. \tag{19.4}$$

Proof. Let $\phi = \phi(t, w)$ be the angle between the positive real direction at the point $z = -t$ and the direction from $z = -t$ to $z = w$. Then by (19.1)

$$\begin{aligned} vg_1(w) &= \int_0^\infty \frac{(t+u)v}{(t+u)^2 + v^2} ds(t) = \int_0^\infty \cos\phi \sin\phi \, ds(t), \\ vg_2(w) &= \int_0^\infty \frac{-v^2}{(t+u)^2 + v^2} ds(t) = -\int_0^\infty \sin^2\phi \, ds(t). \end{aligned} \tag{19.5}$$

Introducing the definition of g into (19.3) and inverting the order of integration, one obtains

$$I(w) = \frac{1}{2\pi i} \int_0^\infty ds(t) \int_{L(w)} \frac{dz}{t+z} = \frac{1}{\pi} \int_0^\infty \phi(t, w) ds(t). \tag{19.6}$$

Hence by (19.5)

$$\begin{aligned} &s(|u|) - I(w) + \frac{1}{\pi} v g_1(w) \\ &= \frac{1}{\pi} \int_0^{|u|} (\pi - \phi + \frac{1}{2} \sin 2\phi) ds(t) + \frac{1}{\pi} \int_{|u|}^\infty (-\phi + \frac{1}{2} \sin 2\phi) ds(t). \end{aligned} \tag{19.7}$$

A drawing helps to verify that $0 \le t \le |u|$ implies $\pi/2 \le \phi \le \pi$, while $t \ge |u|$ implies $0 \le \phi \le \pi/2$. These cases lead to the respective inequalities

$$\left| \pi - \phi + \frac{1}{2} \sin 2\phi \right| \le \frac{1}{2} \pi \sin^2\phi, \qquad \left| -\phi + \frac{1}{2} \sin 2\phi \right| \le \frac{1}{2} \pi \sin^2\phi.$$

Thus (19.4) follows from (19.7) and (19.5).

Proof of Theorem 19.1. Let L be a positively oriented curve which consists of the arcs $\{y = \pm |x|^\gamma\}$ $(x \le -1)$ and a connecting rectifiable arc in the half-plane $\{x \ge -1\}$ which does not meet the half-line $\{y = 0,\ x \le 0\}$. Taking $w = u + iv$ on L with $u < -1$ and $v > 0$, we let $L(w)$ be the part of L from $\overline{w}$ to w and $I(w)$ the corresponding integral (19.3). We split $I(w)$ as follows:

$$I(w) = \frac{1}{2\pi i} \int_{L(w)} \{g(z) - Az^{\alpha-1}\} dz + \frac{A}{2\pi i} \int_{L(w)} z^{\alpha-1} dz. \tag{19.8}$$

By (19.1) the first term on the right is $\mathcal{O}(|w|^{\beta}) = \mathcal{O}(|u|^{\beta})$ as $w \to \infty$ along L. The final term is equal to

$$\frac{A}{2\pi i}\frac{w^{\alpha} - (\overline{w})^{\alpha}}{\alpha} = \frac{A}{\alpha\pi}|w|^{\alpha}\sin\alpha\psi, \qquad \psi = \arg w.$$

Now for $w \to \infty$ along L,

$$\begin{aligned}|w|^{\alpha} &= |u|^{\alpha} + \mathcal{O}(|u|^{\alpha-1}v) = |u|^{\alpha} + \mathcal{O}(|u|^{\alpha-1+\gamma}),\\ \sin\alpha\psi &= \sin\{\alpha\pi - \alpha(\pi-\psi)\} = \sin\alpha\pi + \mathcal{O}(|v|/|w|) = \sin\alpha\pi + \mathcal{O}(|u|^{\gamma-1}).\end{aligned}$$

Putting it all together we obtain from (19.8) that

$$I(w) = \mathcal{O}(|u|^{\beta}) + \frac{A\sin\alpha\pi}{\alpha\pi}|u|^{\alpha} + \mathcal{O}(|u|^{\alpha-1+\gamma}). \tag{19.9}$$

Combination with (19.4) and (19.1) gives (19.2).

Remarks 19.3. Theorem 19.1 was extended to other Stieltjes-type transforms and considerably generalized by Subhankulov et al. See Subhankulov and An [1974] and Subhankulov's book (loc. cit. chapter 4). A different generalization was obtained by Ganelius [1971] (chapter 7).

Applications 19.4. Beginning with Valiron [1914] and Titchmarsh [1927], methods of Tauberian character for Stieltjes-type transforms have been used extensively to study the counting function for the zeros of entire functions. The simplest case is that of an entire function f of order $\rho \in (0, 1)$ whose zeros are negative real. If $f(0) = 1$, then for the branch of the logarithm which is real on $\mathbb{R}^+$,

$$\log f(z) = \int_0^{\infty} \log\left(1 + \frac{z}{t}\right) dn(t) = z\int_0^{\infty} \frac{n(t)}{t(t+z)}dt,$$

where $n(t)$ is the number of zeros of f of absolute value $\le t$. Thus growth estimates for f on suitable curves give estimates for a Stieltjes-type transform which can be used to obtain estimates for $n(t)$. See the books by Boas [1954], Levin [1964], Bingham, Goldie and Teugels [1987] (chapter 7) for asymptotic results, and Subhankulov's book for remainder theorems.

Applications to eigenvalues go back to Carleman [1934]; cf. also Titchmarsh [1958] (chapters 17, 22) and Agmon [1965] (section 14). Remainder theorems have been used for precise results on the number $N(t)$ of eigenvalues λ_j with absolute value $\le t$ in elliptic boundary value problems. Under appropriate conditions one may use resolvents or spectral densities to derive a formula of the type

$$\sum_j \frac{1}{\lambda_j^{2k} + z} = \int_0^{\infty} \frac{dN(t^{1/(2k)})}{t+z} = Az^{q/(2k)-1} + \mathcal{O}(z^{q/(2k)-1-\delta}), \quad \delta > 0,$$

valid on curves as in Theorem 19.1. The Theorem then gives an estimate for $N(t^{1/(2k)})$ which leads to a conclusion of the form

$$N(t) = A't^{q} + \mathcal{O}(t^{q-\eta}) \quad \text{with } \eta > 0.$$

See for example Agmon and Kannai [1967].

For later results on eigenvalues in elliptic problems (independent of complex remainder theory), see Hörmander [1983–85] (chapters 17, 29). In this connection we also refer to the (older) work of Levitan, Marčenko and others on so-called Fourier Tauberian theorems. Here one estimates the difference $S - K_\lambda * S$, for nondecreasing S and a suitable approximate identity K_λ, under appropriate conditions on the Fourier transform of $K_\lambda * S$. See Levitan [1953], Marčenko [1955], the book by Levitan and Sargsjan [1975], and more recent publications including Levitan [1995] and Yu. Safarov [2001].

IV

Karamata's Heritage: Regular Variation

1 Introduction

As we know, the prime counting function satisfies the asymptotic relation $\pi(x) \sim x/\log x$, so that the n-th prime p_n is asymptotic to $n \log n$. In order to describe the asymptotic behavior of sequences and functions, one quickly recognized a need for standards of 'regular growth'. Landau [1911] studied sums, which involve regularly increasing sequences $\{q_n\}$ such as the primes, by comparing them to integrals. Pólya [1917], [1923] extended the results and investigated the counting function for the zeros of entire functions. For the comparison of sums and integrals, counting functions such as $N(u) = \sum_{q_n \le u} 1$ were required to satisfy an asymptotic relation

$$N(u) \sim u^\alpha L(u) \quad \text{for some number } \alpha > 0, \tag{1.1}$$

where L is positive, continuous, monotonic and such that

$$\frac{L(\lambda u)}{L(u)} \to 1 \quad \text{as } u \to \infty \text{ for every number } \lambda > 0. \tag{1.2}$$

(Because of the monotonicity, it is enough to require (1.2) for $\lambda = 2$.) Here Pólya spoke of slowly increasing ('langsam wachsende') and slowly decreasing functions L. Among other things he showed that for Riemann integrable functions f on $[0, 1]$ and 'regular' sequences $\{q_n\}$,

$$\lim_{u\to\infty} \frac{1}{N(u)} \sum_{q_n \le u} f\left(\frac{q_n}{u}\right) = \int_0^1 f(t)dt^\alpha. \tag{1.3}$$

For the proof it is sufficient to consider piecewise constant functions, or just 'characteristic functions' of intervals $[0, c]$; cf. Pólya and Szegő [1925/78] (chapter 2, problems 147–161).

Early in their Tauberian work, Hardy and Littlewood [1914a] introduced comparisons involving special 'logarithmico-exponential functions' of the form

$$\phi(u) = u^\alpha L(u), \quad \text{with } L(u) = (\log u)^{\alpha_1}(\log\log u)^{\alpha_2} \cdots \quad (\alpha_j \text{ real}). \tag{1.4}$$

(For general logarithmico-exponential functions and 'orders of infinity', see Hardy [1911a].) Lasker and Pringsheim had used comparison functions (1.4) in Abelian theorems; see Pringsheim [1904]. One of the Hardy–Littlewood Tauberian theorems went as follows; cf. Sections I.7 and I.15.

Theorem 1.1. *Let* $\sum_{n=0}^{\infty} a_n e^{-ny}$ *converge for* $y > 0$. *Suppose that*

$$F(1/u) = \sum_{n=0}^{\infty} a_n e^{-n/u} \sim A\phi(u) \quad \text{as } u \to \infty, \tag{1.5}$$

where $\phi(u) = u^\alpha L(u)$ *with* $\alpha > 0$, *and that*

$$na_n \geq -C\phi(n) \quad \text{for } n \geq n_0. \tag{1.6}$$

Then

$$s_n = \sum_{k=0}^{n} a_k \sim A\phi(n)/\Gamma(\alpha + 1) \quad \text{as } n \to \infty. \tag{1.7}$$

R. Schmidt [1925a] used somewhat more general comparison functions. However, the concept of *regularly varying functions* $\phi(u) = u^\alpha L(u)$, with *slowly varying* L defined just by (1.2), is due to Karamata [1930b], [1933b]. The slow and regular variation are preserved under summation and integration. It turns out that for a large class of functions $k(\cdot)$,

$$\int_0^\infty k(v)L(uv)dv \sim L(u)\int_0^\infty k(v)dv, \tag{1.8}$$

$$\int_0^\infty k(v)\phi(uv)dv \sim \phi(u)\int_0^\infty k(v)v^\alpha dv \quad \text{as } u \to \infty. \tag{1.9}$$

Although Karamata became well-known through his simple proof of Theorem 1.1 with $\phi(u) = u$ (Karamata [1930a]; cf. Section I.11), his theory of regularly varying functions has had much greater impact. Such functions play a role in many parts of mathematics: number theory, complex analysis, Tauberian theory, differential equations, and especially, since Feller's work, probability theory (see Feller's book [1966/71]).

The topics in this chapter fall roughly into three groups. In the first, comprising Sections 1–6, we develop the theory of regular variation. The basic results all go back to Karamata. He showed that pointwise convergence in (1.2) implies uniform convergence even for non-monotonic functions L, obtained a characteristic integral representation for slowly varying functions, and characterized these functions also by their behavior under integration. The theory was refined and extended by members of 'Karamata's school' in (and outside) Yugoslavia and by many other authors; cf. the subsequent sections. Several major contributors were motivated by probability theory. After Feller we mention de Haan and collaborators, and Bingham with his numerous coauthors. The book 'Regular Variation' by Bingham, Goldie and Teugels [1987] has become a landmark. We frequently refer to it simply as BGT.

The second part of the chapter, consisting of Sections 7–14, deals with a variety of Abelian and Tauberian theorems involving regular variation. We list the highlights:
— the basic theorems for Laplace and Stieltjes transforms, which were obtained by Karamata [1931];
— the elegant theory of Bingham and Teugels for general-kernel Stieltjes integrals, as presented in *BGT*;
— the ratio theorem, which goes back to Feller [1963];
— the higher-order Abel–Tauber theory initiated by de Haan [1976];
— the refined Mercerian theorems of Drasin [1968], Shea [1969] and others, following initial work by Edrei and Fuchs [1966].

The third part of the chapter is devoted mainly to Abelian and Tauberian theory for Laplace transforms of rapid growth. Again, there have been (and are) many contributors; see the introduction in Section 15. There is an extensive logarithmic theory which started with the work of Hardy and Ramanujan [1917] on partitions. Passing through many stages – for the moment, we only mention Kohlbecker [1958] and Kasahara [1978] – the theory culminated in refined results of Geluk, de Haan and Stadtmüller [1986]. There are also logarithm-free results, going back to Martin and Wiener [1938], which involve running averages of the object function. A high point was reached with the strong results of Ingham [1941] (Sections 21, 22), which belong to the area of complex Tauberian theory.

We finally mention some recent work involving transforms of less rapid growth which play an important role in probability theory. Key names are Feigin and Yashchin [1983], and Balkema and coauthors ([2003], [2002] and earlier papers). Our book does not deal with multidimensional theory, although that too has become important for applications; see for example Yakymiv [2002].

Results involving regular variation have been applied in number theory, notably to partitions (cf. Section 23), in analysis (growth and zeros of entire functions), and especially in probability theory; see chapters 6–8 of *BGT*. For entire functions, see also papers such as Edrei and Fuchs (loc. cit.), Baernstein [1969], and Hellerstein, Shea and Williamson [1970]. For other parts of analysis we mention Aljančić, Bojanić and Tomić [1974], and (for differential equations) Marić [2000].

Biographical information on Karamata can be found in Tomić [2001] and Marić [2002]. 'Selected Papers' of Karamata ([2004?]) have been prepared for publication in Yugoslavia.

2 Slow and Regular Variation

Karamata initially focused on the properties (1.8), (1.9); cf. Section 5, but the limit relation (1.2) soon became the standard definition for slow growth ('croissance lente'), slow oscillation or, as one says today, *slow variation*. In order to avoid pathology (Section 4), it is customary to add the requirement of measurability.

Definition 2.1. A function L defined on some interval (a, ∞) or $[a, \infty)$ with $a \geq 0$ is called SLOWLY VARYING (at infinity) if it is measurable, eventually positive, and

such that

$$\frac{L(\lambda u)}{L(u)} \to 1 \quad \text{as } u \to \infty \text{ for every number } \lambda > 0. \tag{2.1}$$

Functions $\phi(u)$ of the form $u^\alpha L(u)$ with real α are then called REGULARLY VARYING of *index* α. Occasional notation: $\phi \in RV_\alpha$. Such functions are characterized by the relation

$$\frac{\phi(\lambda u)}{\phi(u)} \to \lambda^\alpha \quad \text{as } u \to \infty \text{ for every number } \lambda > 0. \tag{2.2}$$

Besides the functions in (1.4), there are such examples of slowly varying functions as

$$L(u) = \exp\{(\log u)^{\alpha_1}(\log\log u)^{\alpha_2}\cdots\} \quad \text{with } \alpha_j \text{ real}, \ \alpha_1 < 1. \tag{2.3}$$

The following BASIC PROPERTIES will be verified in Section 3.

Theorem 2.2. *(i) If L is slowly varying, the convergence in* (2.1) *is uniform on every interval $b \le \lambda \le B$ with $0 < b < B < \infty$, and*

(ii) both L and $1/L$ are bounded on every finite interval far enough to the right.

(iii) A function L on $[a, \infty)$ with $a > 0$ is slowly varying if and only if it has an integral representation of the form

$$L(u) = c(u)\exp\left\{\int_a^u \varepsilon(v)\frac{dv}{v}\right\}, \tag{2.4}$$

where $c(\cdot)$ is measurable and tends to a limit $c > 0$ at ∞, while $\varepsilon(\cdot)$ is bounded and tends to 0 at ∞. The function $\varepsilon(\cdot)$ may actually be taken continuous.

We list a number of consequences. If L is slowly varying, then for any number $\delta > 0$ and any constant $C > 1$,

$$|L(u)/L(t)| \le C\max\{(u/t)^\delta, (u/t)^{-\delta}\} \quad \text{provided } t,\ u > B(\delta, C). \tag{2.5}$$

In applications to asymptotics at ∞, one may often replace $c(u)$ by its limit c and set $\varepsilon(v) = 0$ for $0 \le v \le a$, so that L and $1/L$ become positive and continuous for $u \ge 0$. One then has an inequality (2.5) with $C = C(\delta)$ *for all* $t,\ u > 0$. One can also say that for any number $\delta > 0$, there now are positive constants b and B such that

$$b\left(\frac{u+1}{t+1}\right)^{-\delta} \le \left|\frac{L(u)}{L(t)}\right| \le B\left(\frac{u+1}{t+1}\right)^{\delta} \quad \text{whenever } 0 \le t \le u < \infty. \tag{2.6}$$

(Such an inequality was used already in Section III.17.)

For functions ϕ of regular variation on $[0, \infty)$ with index $\alpha \ne 0$, there are *monotonic equivalents*: monotonic functions of regular variation with the same asymptotic behavior. Thus if $\alpha > 0$ and L has been adjusted as before, then for $u \to \infty$

$$\phi(u) \sim \phi_1(u) = \sup_{0 \le t \le u} \phi(t), \qquad \phi(u) \sim \phi_2(u) = \inf_{t \ge u} \phi(t).$$

(If $\alpha < 0$ one would interchange sup and inf.) To prove the first result one may verify that for large u,

$$\phi_1(u) = \sup_{\frac{1}{2}u \le t \le u} \phi(t).$$

Indeed, over intervals $\frac{1}{2}v \le t \le v$ with large v, t^α increases by a factor 2^α, while $L(t)$ decreases at most by a factor 2^δ, say. Now

$$\frac{\phi(\lambda u)}{\phi(u)} = \lambda^\alpha \frac{L(\lambda u)}{L(u)} \to \lambda^\alpha \quad \text{uniformly for } \frac{1}{2} \le \lambda \le 1,$$

hence

$$\frac{\phi_1(u)}{\phi(u)} = \sup_{\frac{1}{2} \le \lambda \le 1} \frac{\phi(\lambda u)}{\phi(u)} \to 1 \quad \text{as } u \to \infty.$$

There are also smooth equivalents $u^\alpha L(u)$, with $c(\cdot) = c$ and $\varepsilon(\cdot)$ in (2.4) of class C^k; cf. Seneta [1976]. An 'analytic equivalent' of a slowly varying function is discussed by Simić [2001].

For regular variation, the requirements in Definition 2.1 can be relaxed considerably. Here one has the following important result; cf. Section 4.

Theorem 2.3. *Let Φ on $\mathbb{R}^+$ be measurable, eventually positive, and such that*

$$\frac{\Phi(\lambda u)}{\Phi(u)} \quad \textit{tends to a limit function} \quad \psi(\lambda) > 0 \quad \textit{as} \quad u \to \infty \tag{2.7}$$

for every number $\lambda > 0$ (or at least for every λ in a subset of $\mathbb{R}^+$ of positive measure). Then $\psi(\lambda) = \lambda^\alpha$ for some number $\alpha \in \mathbb{R}$ (and the limit relation holds for all $\lambda > 0$), so that Φ is regularly varying.

Remarks 2.4. The uniform convergence and the integral representation in Theorem 2.2 are due to Karamata [1930b], [1933b], at least for the case of continuous functions L. For measurable L the uniform convergence was proved by Korevaar, van Aardenne–Ehrenfest and de Bruijn [1949] and also by Delange [1955a]. The integral representation without additional hypothesis on L is due to de Bruijn [1959]. Since then, there have been additional refinements of theorems and proofs. We mention work by Karamata [1962], Bojanic and Karamata [1963a], [1963b], Matuszewska [1964/65], Feller [1966/71], de Haan [1970], Bojanic and Seneta [1971], Seneta [1976], Bingham, Goldie and Teugels [1987] (BGT). For relatively recent contributions, see Drasin and Seneta [1986], Ostrogorski [1998], and Seneta [2002]. Theorem 2.3 is due to Feller [1966/71] (section 8.8); cf. BGT.

For certain applications it is useful to relax the condition of positivity in Definition 2.1 and to allow complex-valued functions L which eventually avoid the value 0; cf. the end of Section 3.

3 Proof of the Basic Properties

For the proof of Theorem 2.2, it is convenient to consider an additive problem rather than a multiplicative one. In (2.1), set $u = e^x$, $L(u) = m(x)$ and $\lambda = e^y$, so that $L(\lambda u)$ becomes $m(x + y)$. We henceforth take L *positive* (except at the end of the present section), and work with the real function $f(x) = \log m(x)$. Then (2.1) becomes

$$f(x + y) - f(x) \to 0 \quad \text{as } x \to \infty \text{ for every real number } y. \tag{3.1}$$

The basic properties in Theorem 2.2 will readily follow from the results below.

Theorem 3.1. *Let f on (a', ∞) be finite, measurable and such that* (3.1) *is satisfied. Then the convergence in* (3.1) *is uniform on every finite y-interval.*

Proof. It will be enough to prove the uniform convergence for one y-interval, for which we take $\{0 \le y \le 1\}$. If E is a (measurable) subset of $\mathbb{R}$, we write $|E|$ for its measure, and χ_E for its characteristic function: $\chi_E(t) = 1$ for $t \in E$ and $\chi_E(t) = 0$ for $t \notin E$. Let $\varepsilon > 0$ be given. For any given x, let E_x denote the set of those points $z = x + t$ with $t \in [-1, 1]$ for which

$$|f(z) - f(x)| = |f(x+t) - f(x)| \ge \varepsilon/2. \tag{3.2}$$

The corresponding set of points $t \in [-1, 1]$, a translate of the set E_x, will be denoted by T_x. By (3.2) and (3.1), a given number t cannot belong to T_x when x is sufficiently large. Equivalently, $\chi_{T_x}(t) \to 0$ as $x \to \infty$ for every t, hence by bounded convergence, $|T_x| = \int_{-1}^{1} \chi_{T_x}(t)dt \to 0$. It follows that

$$|E_x| = |T_x| < 1/2 \quad \text{for } x > x_0.$$

For any $y \in [0, 1]$ we consider, more generally, the set E_{x+y} of those points $z = x + y + t$ with $t \in [-1, 1]$, for which

$$|f(z) - f(x+y)| = |f(x+y+t) - f(x+y)| \ge \varepsilon/2. \tag{3.3}$$

In our notation the corrresponding set of points $t \in [-1, 1]$ is T_{x+y}; one will have $|E_{x+y}| = |T_{x+y}| < 1/2$ for $x + y > x_0$, hence *a fortiori* for $x > x_0$.

Observe now that E_x is a subset of the interval $I_x = [x - 1, x + 1]$, while E_{x+y} is a subset of $I_{x+y} = [x + y - 1, x + y + 1]$. The intersection $I_x \cap I_{x+y}$ contains the interval $[x, x + 1]$, hence it must contain a point z outside the union $E_x \cup E_{x+y}$ whenever $x > x_0$, no matter which point $y \in [0, 1]$ we started with. For such a point z one has

$$\textit{both} \quad |f(z) - f(x)| < \varepsilon/2 \quad \textit{and} \quad |f(z) - f(x+y)| < \varepsilon/2. \tag{3.4}$$

Conclusion:

$$|f(x+y) - f(x)| < \varepsilon \quad \text{for } x > x_0, \ \forall\, y \in [0, 1]. \tag{3.5}$$

Corollary 3.2. (Bojanic and Seneta [1971]) *If f is measurable on (a', ∞) and satisfies relation* (3.1)*, then f is bounded (and hence integrable) on every finite interval far enough to the right.*

Indeed, there will be a constant B such that $|f(x + y) - f(x)| < \varepsilon = 1$ for all $y \in [0, 1]$ when $x \ge B$. Thus $|f(z) - f(B)| < 1$ for $B \le z \le B + 1$, etc.

Theorem 3.3. *A measurable function f on (a', ∞) satisfies relation* (3.1) *if and only if it can be represented in the form*

$$f(x) = \gamma(x) + \int_{a'}^{x} \eta(z)dz, \tag{3.6}$$

where $\gamma(\cdot)$ is measurable and tends to a finite limit γ at ∞, while $\eta(z)$ is bounded for $z \geq a'$ and tends to 0 at ∞. By modification of $\gamma(\cdot)$ the function $\eta(\cdot)$ can be made continuous.

Proof. The 'if' part is obvious: an integral representation (3.6) for f, with $\gamma(\cdot)$ and $\eta(\cdot)$ as described, implies (3.1).

Let us now start with (3.1). Then by the preceding f is bounded on every finite interval starting at B, say. Thus if we take $x \geq B$, the uniform convergence in (3.1) shows that

$$\delta(x) \stackrel{\text{def}}{=} \int_x^{x+1} f(z)dz - f(x) = \int_0^1 \{f(x+y) - f(x)\}dy \to 0 \quad \text{as } x \to \infty. \tag{3.7}$$

Defining $F(x) = \int_x^{x+1} f(z)dz$, one has

$$F(x) = \int_B^{B+1} f(z)dz + \int_B^x \{f(z+1) - f(z)\}dz. \tag{3.8}$$

Hence if we put

$$\gamma \stackrel{\text{def}}{=} \int_B^{B+1} f(z)dz, \quad \eta(z) \stackrel{\text{def}}{=} f(z+1) - f(z),$$

then $\eta(z) \to 0$ as $z \to \infty$ and by (3.8),

$$F(x) = \gamma + \int_B^x \eta(z)dz.$$

Combining this relation with (3.7) one obtains

$$f(x) = F(x) - \delta(x) = \{\gamma - \delta(x)\} + \int_B^x \eta(z)dz. \tag{3.9}$$

This is a representation (3.6) for $x \geq B$, but with B instead of a'. To complete the proof one may simply define $\eta(z) = 0$ for $z < B$ and $\gamma(x) = f(x)$ for $x < B$. Unruly initial behavior of f is incorporated in $\gamma(\cdot)$.

Representation with continuous $\eta(\cdot)$. Following Seneta [1976], we write (3.9) in the form

$$f(x) = \gamma(x) + f_1(x), \quad f_1(x) = \int_B^x \{f(z+1) - f(z)\}dz, \quad x \geq B, \tag{3.10}$$

and show that the continuous function f_1 satisfies (3.1). Indeed, for any given real number y,

$$\begin{aligned} f_1(x+y) - f_1(x) &= \int_x^{x+y} \{f(z+1) - f(z)\}dz \\ &= \int_0^y \{f(x+t+1) - f(x+t)\}dt \\ &= \int_0^y [\{f(x+t+1) - f(x)\} - \{f(x+t) - f(x)\}]dt. \end{aligned} \tag{3.11}$$

Now let x go to infinity. Then the final member of (3.11) tends to zero since $f(x+u) - f(x) \to 0$ uniformly on every finite u-interval.

Application of the first part of Theorem 3.3 to f_1 gives a representation of the form

$$f_1(x) = \gamma_1(x) + \int_{B_1}^{x} \eta_1(z)dz \quad \text{when } x \geq B_1. \tag{3.12}$$

Here $B_1 \geq B$, $\gamma_1(x)$ tends to a limit γ_1 as $x \to \infty$ and $\eta_1(z) = f_1(z+1) - f_1(z)$ is continuous. To complete the proof one uses (3.10) and chooses a suitable new function $\gamma(x)$ for $x < B_1$.

The Complex Case. For (measurable) complex-valued functions L which are never zero and satisfy relation (2.1), the method that was used for Theorem 3.1 may be applied directly to $m(x) = L(e^x)$ to show that

$$\frac{m(x+y)}{m(x)} - 1 = \frac{L(e^{x+y})}{L(e^x)} - 1 \to 0,$$

uniformly on every finite y-interval, cf. Elliott [1979]. In particular

$$\left| \frac{m(x+y)}{m(x)} - 1 \right| < 1, \quad \forall y \in [0, 1]$$

when $x \geq B$, say. Thus for $x \geq B$, one can unambiguously define a branch $f(x)$ of $\log m(x)$, proceeding step by step over intervals of length 1. Indeed, for $0 \leq y \leq 1$ one would take $|\text{Im} \log m(B+y) - \text{Im} \log m(B)| < \pi/2$, etc. For the resulting function f one then obtains (3.1), with uniform convergence on finite y-intervals, and an integral representation (3.6) when $x \geq B$. It can be converted to an integral representation for L on $[a, \infty)$.

Remark 3.4. The method used for Theorem 3.1 can be adapted to prove the following extension. Let f on (a', ∞) be finite and measurable. Let g be nonnegative but $\not\equiv 0$, nondecreasing, with derivative $g'(x) = o(1)$ as $x \to \infty$ and such that

$$\frac{f\{x + yg(x)\} - f(x)}{g(x)} \to 0, \quad \text{or} \quad \to Ay, \quad \text{as } x \to \infty, \quad \forall y.$$

Then the convergence is uniform on every finite y-interval; cf. Tenenbaum [1980] for the case $g(x) = \sqrt{x}$. The case $g = f$ corresponds to 'Beurling slow variation'; see Section 11.

4 Possible Pathology

Examples show that some condition of good behavior (such as measurability) is required to obtain uniform convergence from condition (3.1); see Korevaar, van Aardenne–Ehrenfest and de Bruijn [1949] and cf. Bingham, Goldie and Teugels [1987] (BGT). The beautiful counterexample below may be obtained from a paper by Ash, Erdős and Rubel [1974].

Proposition 4.1. *Let B be a Hamel basis for the linear space of the reals over the rationals. For every real number x, let $n = n(x)$ be the number of terms in the unique representation of x as a linear combination of distinct basis elements:*

$$x = \sum_{k=1}^{n} r_j b_j, \quad b_j \in B, \ \ r_j \ \textit{rational and} \ \neq 0.$$

Then the function

$$f(x) \stackrel{\text{def}}{=} \begin{cases} \log\{x + n(x)\} & \textit{for } x > 0, \\ 0 & \textit{for } x \leq 0 \end{cases} \tag{4.1}$$

satisfies relation (3.1), *but the convergence is not uniform on any y-interval.*

Proof. Clearly $|n(x+y) - n(x)| \leq n(y)$, hence for $x, y > 0$,

$$\begin{aligned} |f(x+y) - f(x)| &= \left| \int_{x+n(x)}^{x+y+n(x+y)} \frac{dt}{t} \right| \leq \frac{1}{x} |y + n(x+y) - n(x)| \\ &\leq \frac{1}{x}\{y + n(y)\} \to 0 \quad \text{as } x \to \infty. \end{aligned}$$

However, the convergence fails to be uniform for $y \in (0, 1)$, say. Indeed, for any fixed $x > 0$, there is a number $y \in (0, 1)$ such that $n(x + y)$, and hence also $f(x + y) = \log\{x + y + n(x + y)\}$, is as large as we please. This is so because for every fixed n and every n-tuple of basis elements $\{b_1, \cdots, b_n\}$, the combinations $\sum_1^n r_j b_j$ with nonzero rational r_j lie dense in $\mathbb{R}$.

The function f in Proposition 4.1 is unbounded. For a bounded counterexample one may take $f_1 = \sin f$.

CAUCHY'S FUNCTIONAL EQUATION. There is related pathology in the case of Cauchy's equation,

$$k(y + z) = k(y) + k(z), \quad \forall y, z \in \mathbb{R}. \tag{4.2}$$

However, a solution $k(\cdot)$ which is continuous or measurable, or bounded above on a set of positive measure, must have the form $k(x) = Ax$ for some constant A; cf. Elliott [1979] and *BGT*. One can use this fact to verify Theorem 2.3. Starting with relation (2.7), set

$$\lim_{x\to\infty} \{\log \Phi(e^{x+y}) - \log \Phi(e^x)\} = \log \psi(e^y) = k(y).$$

Then k will satisfy Cauchy's equation, hence $\log \psi(e^y) = Ay$, $\psi(v) = v^A$.

VERY SLOW VARIATION. The paper by Ash et al. contains interesting results for other kinds of slow variation. In the case of extremely slow variation, no condition such as measurability is required for uniform convergence. Let w be positive and nondecreasing on $[0, \infty)$. Taking as starting point relation (3.1) for the logarithm of a slowly varying function, a function f on $[0, \infty)$ is called *w-slowly varying* if

$$w(x)\{f(x + y) - f(x)\} \to 0 \quad \text{as } x \to \infty, \ \ \forall y. \tag{4.3}$$

(i) Let f be as in (4.3). If f is measurable *or* if w is such that

$$\sum_{n=0}^{\infty} \frac{w(x)}{w(x+n)} \leq M < \infty, \quad \forall x \geq 0$$

(example: $w(x) = e^x$), then (4.3) holds uniformly on every finite y-interval.

(ii) If

$$\sum_{n=0}^{\infty} \frac{1}{w(n)} = \infty,$$

then there is a function f which is w-slowly varying, but not even uniformly 1-slowly varying.

A related class of functions will be considered in Section 6.

5 Karamata's Characterization of Regularly Varying Functions

Karamata [1930b] has characterized regularly varying functions by their behavior under integration against powers; see Theorems 5.2 and 5.3. Cf. de Haan [1970], Bingham, Goldie and Teugels [1987] (BGT).

Proposition 5.1. *Let L be slowly varying on* $[a, \infty)$ *with* $a > 0$, *and let L be bounded on every finite interval starting at a. Then for* $u \to \infty$

$$\int_a^u t^{\beta-1} L(t)dt \sim u^\beta L(u)/\beta \quad \text{if } \beta > 0, \tag{5.1}$$

$$\int_u^\infty t^{\beta-1} L(t)dt \sim -u^\beta L(u)/\beta \quad \text{if } \beta < 0. \tag{5.2}$$

For $\beta = 0$ *one has* $\int_a^u t^{-1} L(t)dt/L(u) \to \infty$.

Proof. Let $\beta > 0$ and for the proof of (5.1), set $L(t) = 0$ for $0 \leq t < a$. Then dominated convergence shows that for $u \to \infty$,

$$\frac{1}{u^\beta L(u)} \int_0^u t^{\beta-1} L(t)dt = \int_0^1 v^{\beta-1} \frac{L(uv)}{L(u)} dv \to \int_0^1 v^{\beta-1} dv = \frac{1}{\beta}. \tag{5.3}$$

Indeed, it follows from (2.5) that in the present situation, $L(uv)/L(u) \leq Cv^{-\delta}$ for every number $\delta > 0$ when $0 < v < 1$ and u is large. The proof of (5.2) is similar. Here the boundedness condition is superfluous; cf. (2.5).

For $\beta = 0$ the left-hand side of (5.3) has $\liminf \geq \int_\varepsilon^1 v^{-1} dv$ for every $\varepsilon > 0$, so that it tends to ∞. One can show that in this case, the integral in (5.1) is slowly varying; cf. BGT.

Theorem 5.2. *Let* $\phi(t) = t^\alpha L(t)$ *be regularly varying for* $t \geq a > 0$ *with index* α, *and let* ϕ *be bounded on every finite interval starting at a. Then for* $u \to \infty$

$$\int_a^u t^{\gamma-1}\phi(t)dt \sim u^\gamma\phi(u)/(\alpha+\gamma) \quad \textit{if } \gamma > -\alpha, \tag{5.4}$$

$$\int_u^\infty t^{\gamma-1}\phi(t)dt \sim -u^\gamma\phi(u)/(\alpha+\gamma) \quad \textit{if } \gamma < -\alpha. \tag{5.5}$$

This is an immediate consequence of the Proposition. The following converse completes the characterization of regularly varying functions.

Theorem 5.3. *Let f be positive and integrable over every finite interval starting at the point $a > 0$. If there are numbers γ and $\alpha > -\gamma$ such that*

$$\int_a^u t^{\gamma-1}f(t)dt \sim u^\gamma f(u)/(\gamma+\alpha) \quad \textit{as } u \to \infty, \tag{5.6}$$

then f is regularly varying with index α. The same is true if there are numbers γ and $\alpha < -\gamma$ such that

$$\int_u^\infty t^{\gamma-1}f(t)dt \sim -u^\gamma f(u)/(\gamma+\alpha). \tag{5.7}$$

Proof. We consider the case of (5.6). Set

$$g(u) = \frac{u^\gamma f(u)}{\int_a^u t^{\gamma-1}f(t)dt}, \quad \text{so that} \quad \frac{g(u)}{u} = \frac{d}{du}\left\{\log\int_a^u t^{\gamma-1}f(t)dt\right\}$$

for almost all $u > a$. Then for $b > a$,

$$\int_b^u \frac{g(v)}{v}dv = \log\int_a^u t^{\gamma-1}f(t)dt - \log C, \quad C = \int_a^b t^{\gamma-1}f(t)dt.$$

Hence

$$\begin{aligned} f(u) &= u^{-\gamma}g(u)\int_a^u t^{\gamma-1}f(t)dt \\ &= Cu^{-\gamma}g(u)\exp\left\{\int_b^u \frac{g(v)}{v}dv\right\} \\ &= u^\alpha \cdot Cb^{-\gamma-\alpha}g(u)\exp\left\{\int_b^u \frac{g(v)-\gamma-\alpha}{v}dv\right\}. \end{aligned}$$

Now by the hypothesis, $g(u) \to \gamma+\alpha$, hence by the integral representation for slowly varying functions in Theorem 2.2, $f(u) = u^\alpha L(u)$ with slowly varying L. □

Slow and regular variation are preserved not only when one integrates against powers: one can use much more general functions. This has been an important theme of Karamata and his school since the 1930's. See for example Aljančić, Bojanić and Tomić [1954], Karamata [1962], and Bojanic and Karamata [1963a]; cf. also Vuilleumier [1963]. We refer to *BGT* for systematic treatment. Here we discuss an example that will be used later.

Proposition 5.4. *Let $\phi(v) = v^\alpha L(v)$ be regularly varying (at infinity) and let $k(v)v^\sigma$ be integrable over $(0, \infty)$ for every number σ in a neighborhood of α. Suppose also that for every number $B > 0$,*

$$\int_0^{B/u} k(v)\phi(uv)dv = \int_0^B k\left(\frac{v}{u}\right)\phi(v)\frac{dv}{u} \text{ exists and } = o\{\phi(u)\} \tag{5.8}$$

as $u \to \infty$. Then

$$I(u) = \int_0^\infty k(v)\phi(uv)dv \sim \phi(u)\int_0^\infty k(v)v^\alpha dv \quad \text{as } u \to \infty. \tag{5.9}$$

Observe that (5.8) will be satisfied if $\phi(v) = \mathcal{O}(v^\beta)$ on the intervals $(0, B)$ for some β less than but close to α, or if k is bounded near 0 while ϕ is locally integrable and $\alpha > -1$.

Proof of Proposition 5.4. Let χ_B denote the characteristic function of the interval $[B, \infty)$. By (5.8) it is sufficient to consider

$$\begin{aligned} I_B(u) &\stackrel{\text{def}}{=} \int_{B/u}^\infty k(v)\phi(uv)dv = \int_0^\infty k(v)u^\alpha v^\alpha L(uv)\chi_B(uv)dv \\ &= \phi(u)\int_0^\infty k(v)v^\alpha \frac{L(uv)}{L(u)}\chi_B(uv)dv. \end{aligned} \tag{5.10}$$

We know that $L(uv)\chi_B(uv)/L(u) \to 1$ as $u \to \infty$ for every $v > 0$. Also, for any number $\delta > 0$, we can determine $B = B(\delta)$ such that for $u \geq B$,

$$\frac{L(uv)}{L(u)}\chi_B(uv) \leq 2(v^\delta + v^{-\delta}) \qquad (0 < v < \infty); \tag{5.11}$$

cf. inequality (2.5). For small δ, the hypotheses, formulas (5.10), (5.11) and dominated convergence now show that

$$I_B(u)/\phi(u) \to \int_0^\infty k(v)v^\alpha dv.$$

6 Related Classes of Functions

Karamata and his school have introduced several extensions of regular variation. These were developed further by many authors, notably by a group in the Netherlands and by Bingham and Goldie. See the references below and cf. Bingham, Goldie and Teugels [1987] (BGT).

If f is the logarithm of a slowly varying function L, then $f(\lambda x) - f(x) \to 0$ as $x \to \infty$ for every number $\lambda > 0$. There is an extensive theory going back to Bojanic and Karamata [1963b], in which the difference $f(\lambda x) - f(x)$ is compared with an auxiliary function $g(x)$. Refinements are due to de Haan [1970], [1976], Seneta (see

his book [1976]), Geluk and de Haan [1981], [1987], and Bingham and Goldie [1982]. The book *BGT* devotes a whole chapter to the resulting *higher-order theory* or *de Haan theory.*

Suppose that there is a positive (measurable) function g such that for $x \to \infty$,

$$\frac{f(\lambda x) - f(x)}{g(x)} \text{ has a limit function } h(\lambda) \neq 0, \quad \forall\, \lambda > 0. \tag{6.1}$$

What sort of *comparison functions* g and *limit functions* h *can occur?* One has

$$\frac{f(\mu\lambda x) - f(x)}{g(x)} = \frac{f(\mu\lambda x) - f(\lambda x)}{g(\lambda x)} \frac{g(\lambda x)}{g(x)} + \frac{f(\lambda x) - f(x)}{g(x)}, \tag{6.2}$$

hence by (6.1)

$$\frac{g(\lambda x)}{g(x)} \to \frac{h(\mu\lambda) - h(\lambda)}{h(\mu)}, \quad \forall\, \lambda,\ \mu > 0. \tag{6.3}$$

If for some μ, the limit function is positive for all $\lambda > 0$, it follows from Theorem 2.3 that g must be *regularly varying*. Furthermore, if g has index α, then by (6.2)

$$h(\mu\lambda) = \lambda^\alpha h(\mu) + h(\lambda). \tag{6.4}$$

If $\alpha = 0$, so that g is *slowly varying*, (6.4) shows that $k(x) = h(e^x)$ satisfies the additive Cauchy equation (4.2). In this case $h(e^x) = Ax$, so that

$$h(\lambda) = A \log \lambda \quad \text{for some constant} \quad A \neq 0. \tag{6.5}$$

If $g \in RV_\alpha$ with $\alpha \neq 0$, one has

$$\lambda^\alpha h(\mu) + h(\lambda) = h(\mu\lambda) = \mu^\alpha h(\lambda) + h(\mu),$$

so that $h(\lambda)/(\lambda^\alpha - 1) = h(\mu)/(\mu^\alpha - 1)$. Setting the common value equal to A/α, one concludes that

$$h(\lambda) = A\,\frac{\lambda^\alpha - 1}{\alpha} \qquad (A \neq 0). \tag{6.6}$$

In our brief sketch of the theory we restrict ourselves to the case where g is a slowly varying function L.

Definition 6.1. We will say that a function f on $\mathbb{R}^+$ belongs to the *class* $\Pi(L, A)$ if it is measurable and

$$\lim_{x\to\infty} \frac{f(\lambda x) - f(x)}{L(x)} = A \log \lambda, \quad \forall \lambda > 0. \tag{6.7}$$

The *de Haan class* Π consists of the functions f which are in $\Pi(L, A)$ for some slowly varying function L and some constant $A \neq 0$. If $A > 0$ one speaks of Π^+, if $A < 0$ of Π^-.

By an adaptation of the argument used in Section 3, the convergence is uniform for $a \le \lambda \le b$ whenever $0 < a < b < \infty$. It follows that for any number $\delta > 0$,

$$\left| \frac{f(tx) - f(x)}{L(x)} \right| \le C(\delta) t^{\delta}, \quad \forall t \ge 1 \text{ when } x \ge x_0(\delta) > 0. \tag{6.8}$$

Indeed, by (6.7)

$$|f(\lambda u) - f(u)| \le (2|A| + 1) L(u) \quad \text{for } 1 \le \lambda \le e$$

when $u \ge u_0$. Apply this inequality with $u = e^{n-1}x$, $\lambda = e$ for $n = 1, \dots, N = [\log t]$, and with $u = e^N x$, $\lambda = t/e^N$. Add the results and apply inequality (2.5) to $L(e^{n-1}x)/L(x)$, etc.

Theorem 6.2. (de Haan) *The following statements involving asymptotics for $x \to \infty$ are equivalent:*

$$f \in \Pi(L, A), \tag{6.9}$$

$$f_1(x) = f(x) - \frac{1}{x} \int_{x_0}^{x} f(t)dt \sim AL(x) \quad \textit{for some } x_0 > 0, \tag{6.10}$$

$$f_2(x) = x \int_{x}^{\infty} f(t) \frac{dt}{t^2} - f(x) \sim AL(x), \tag{6.11}$$

$$f(x) = g(x) + \int_{x_0}^{x} g(t) \frac{dt}{t} \quad \textit{with } g(x) \sim AL(x). \tag{6.12}$$

Proof. We verify the equivalences that will be used later. After adjustment, L may be assumed positive and continuous for $x \ge 0$, cf. Section 2.

(6.9) $\Rightarrow$ (6.10). Take x_0 as in (6.8). Dividing the left-hand side of (6.10) by $L(x)$, one may write

$$\begin{aligned} \frac{f_1(x)}{L(x)} &= \int_{x_0}^{x} \frac{f(x) - f(t)}{xL(x)} dt + \frac{x_0 f(x)}{xL(x)} \\ &= \int_{x_0/x}^{1} \frac{f(x) - f(ux)}{L(x)} du + \frac{x_0 f(x)}{xL(x)}. \end{aligned} \tag{6.13}$$

Inserting a characteristic function χ, we consider the final integral as an integral over $\{0 \le u \le 1\}$. For any $\delta > 0$, the integrand is majorized by

$$\left| \frac{f(ux/u) - f(ux)}{L(ux)} \frac{L(ux)}{L(x)} \right| \chi_{[x_0/x, 1]}(u) \le C(\delta) \left(\frac{1}{u} \right)^{\delta} \cdot C'(\delta) u^{-\delta};$$

see (6.8) and (2.5). Hence by (6.7) and dominated convergence, the last integral tends to $-A \int_0^1 (\log u) du = A$. The final term in (6.13) tends to 0 (use (6.8) with $x = x_0$).

(6.10) $\Rightarrow$ (6.12). Integration and Fubini's theorem show that

$$\int_{x_0}^{x} \frac{f_1(u)}{u} du = \int_{x_0}^{x} \frac{f(u)}{u} du - \int_{x_0}^{x} \frac{du}{u^2} \int_{x_0}^{u} f(t)dt$$
$$= \int_{x_0}^{x} \frac{f(u)}{u} du - \int_{x_0}^{x} f(t)dt \int_{t}^{x} \frac{du}{u^2} = \frac{1}{x} \int_{x_0}^{x} f(t)dt$$
$$= f(x) - f_1(x).$$

This gives (6.12) with $g = f_1$; by (6.10), $f_1(x) \sim AL(x)$. If $A \neq 0$, the function f_1 is slowly varying.

(6.12) $\Rightarrow$ (6.9). The conclusion follows from the uniform convergence in (2.1). □

Corollary 6.3. *Let f in $\Pi(L, A)$ be locally bounded. If we adjust L in such a way that it becomes continuous (and positive) for $x \geq 0$, then for every number $\delta > 0$,*

$$\left| \frac{f(tx) - f(x)}{L(x)} \right| \leq C(\delta)(t^{\delta} + t^{-\delta}), \quad \forall x, t > 0. \tag{6.14}$$

Proof. We use the representation (6.12) with $g = f_1$ as in (6.10). Since f is locally bounded, so are f_1 and g. In view of the relation $g \sim AL$, it follows that $|g/L|$ is bounded on $\mathbb{R}^+$. Inequality (6.14) may now be derived from the formula

$$\frac{f(tx) - f(x)}{L(x)} = \frac{g(tx)}{L(tx)} \frac{L(tx)}{L(x)} - \frac{g(x)}{L(x)} + \int_{x}^{tx} \frac{g(v)}{L(v)} \frac{L(v)}{L(x)} \frac{dv}{v}$$

by standard arguments.

Every function f in Π which tends to infinity must be slowly varying. This can be derived from (6.12) and the case $\beta = 0$ of Proposition 5.1; cf. Geluk and de Haan [1987] (corollary 1.18).

In Section 10 we will need the class OR, of the *O-regularly varying functions*. It consists of the measurable, eventually positive functions Φ on $\mathbb{R}^+$ such that

$$0 < \Phi_*(\lambda) = \liminf_{u \to \infty} \frac{\Phi(\lambda u)}{\Phi(u)} \leq \limsup_{u \to \infty} \frac{\Phi(\lambda u)}{\Phi(u)} = \Phi^*(\lambda) < \infty, \quad \forall \lambda \geq 1; \tag{6.15}$$

cf. Avakumović [1935], Matuszewska, [1961/62], and Aljančić and Arandelović [1977]. Taking $\lambda = e$ one sees that functions of class OR are of at most polynomial growth. There is an integral representation which extends the one in Section 2; see *BGT*.

Another useful class is that of the Beurling slowly varying functions described in Section 11.

7 Integral Transforms and Regular Variation: Introduction

We will consider transforms of functions or measures which have their support on $[0, \infty)$. Let $S(v)$ be a nondecreasing function on $\mathbb{R}$ which vanishes for $v < 0$. It is

assumed that $S(\cdot)$ is *normalized* in the sense that it is continuous from the right. Then S defines a positive measure dS through the relation $dS(-\infty, v] = S(v)$. If we start with a positive measure dS with support on $[0, \infty)$ we call S its *mass function.*

The Laplace–Stieltjes transform of S, or Laplace transform of dS, is defined by

$$\mathcal{L}dS(t) = \int_{\mathbb{R}} e^{-tv} dS(v) = \int_{0-}^{\infty} e^{-tv} dS(v). \tag{7.1}$$

Suppose that the transform $\mathcal{L}dS(t)$ exists for $t > a \geq 0$. Then $S(v) = \mathcal{O}(e^{bv})$ for every $b > a$; cf. Proposition I.13.1. Integration by parts gives

$$\mathcal{L}dS(t) = t\mathcal{L}S(t) = t \int_0^{\infty} S(v)e^{-tv} dv \quad (t > a), \tag{7.2}$$

$$\begin{aligned} \mathcal{L}dS(1/x) &= (1/x) \int_0^{\infty} S(v)e^{-v/x} dv \\ &= \int_0^{\infty} e^{-v} S(xv) dv \quad (0 < x < 1/a). \end{aligned} \tag{7.3}$$

Our main concern is the relation between the *asymptotic behavior* of S at ∞ and that of $\mathcal{L}dS$ at a (usually 0). An important tool is provided by the so-called *Continuity Theorem*; cf. Feller [1966/71] (chapter 13).

Theorem 7.1. *If mass functions $S_k(v)$ are uniformly $\mathcal{O}(e^{av})$ and converge pointwise (almost everywhere) to a mass function $S(v)$, the Laplace transforms $\mathcal{L}dS_k(t)$ converge to $\mathcal{L}dS(t)$ for every $t > a$. Conversely, if for mass functions S_k the transforms $\mathcal{L}dS_k(t)$ converge to a function $F(t)$ for every $t > a$, then F is the transform of a mass function S and $S_k(v) \to S(v)$ at every point where S is continuous.*

For a proof of the first part one may use formula (7.2) and dominated convergence. For the second part one takes $t = b > a$ in formula (7.1) and derives that the functions $S_k(v)$ are uniformly $\mathcal{O}(e^{bv})$. One can then use Helly's selection principle to construct a subsequence $\{S_{n_k}\}$ of $\{S_k\}$ which converges on a dense set. The limit function can be extended to a nondecreasing function S on $\mathbb{R}$. Then $S_{n_k}(v) \to S(v)$ at every point v where S is continuous, and $\mathcal{L}dS = F$. Similarly $\mathcal{L}dS^* = F$ for any limit function S^*, obtained from a subsequence of $\{S_k\}$ that converges on a dense set. By the uniqueness theorem for Laplace transforms and monotonicity, $S^* = S$ wherever S is continuous. Thus the whole sequence $\{S_k\}$ converges to S where S is continuous. One can finally normalize S so that it becomes continuous from the right. □

In the Abelian and Tauberian theory of Laplace transforms, one may also consider more general functions S for which $\mathcal{L}S$ can be defined by formula (7.2).

8 Karamata's Theorem for Laplace Transforms

One of Karamata's best known theorems [1931] involves Laplace–Stieltjes transforms; cf. Feller [1966/71] (section 13.5), Bingham, Goldie and Teugels [1987] (section 1.7). It extends certain results of Hardy and Littlewood [1914a], [1929]; cf. Sections I.7, I.15.

Theorem 8.1. *Let $S(v)$ vanish for $v < 0$, be nondecreasing and such that the Laplace–Stieltjes transform $\mathcal{L}dS(t)$ exists for $t > 0$. Let $\phi(v) = v^\alpha L(t)$ with $\alpha \geq 0$ and slowly varying L. Then the following are equivalent:*

$$S(x) \sim A\phi(x) \quad as \ x \to \infty, \tag{8.1}$$

$$\mathcal{L}dS(1/x) \sim A\Gamma(\alpha+1)\phi(x) \quad as \ x \to \infty. \tag{8.2}$$

If $A \neq 0$, each of these relations implies

$$\mathcal{L}dS(1/x)/S(x) \to \Gamma(\alpha+1). \tag{8.3}$$

The theorem has an analog involving asymptotics for $x \searrow 0$; cf. Theorem I.15.3.

Proof of Theorem 8.1. (i) Let $S(v) \sim A\phi(v)$ with $A > 0$. Then S is regularly varying with index α. We now apply Proposition 5.4 with $k(v) = e^{-v}$ and ϕ replaced by S. Then condition (5.8) is satisfied; cf. the observation following the Proposition. Thus by (7.3) and (5.9)

$$\mathcal{L}dS(1/x) = \int_0^\infty e^{-v}S(xv)dv \sim S(x)\int_0^\infty e^{-v}v^\alpha dv \sim A\Gamma(\alpha+1)\phi(x).$$

The result is also true for $A = 0$. Indeed, since we deal with asymptotics at ∞, we may change ϕ to 0 on an initial interval $(0, B)$ beyond which ϕ is locally bounded. Applying the method above to $S_1 = S + \phi \sim \phi$, one finds that $\mathcal{L}dS_1(1/x)$ is asymptotic to $\Gamma(\alpha+1)\phi(x)$, hence $\mathcal{L}dS(1/x)/\phi(x) \to 0$.

(ii) Let $\mathcal{L}dS(1/x) \sim A\Gamma(\alpha+1)\phi(x)$. Then for any $t > 0$ and $x \to \infty$

$$\frac{\mathcal{L}dS(t/x)}{\phi(x)} \sim A\Gamma(\alpha+1)\frac{\phi(x/t)}{\phi(x)} \to A\frac{\Gamma(\alpha+1)}{t^\alpha}. \tag{8.4}$$

As a function of t, the left-hand side is the Laplace transform of $d_vS(xv)/\phi(x)$:

$$\int_{0-}^\infty e^{-tv}d_vS(xv) = \int_{0-}^\infty e^{-(t/x)w}dS(w) = \mathcal{L}dS(t/x).$$

The right-hand side of (8.4) is the Laplace transform of Adv_+^α:

$$(\mathcal{L}dv_+^\alpha)(t) = t\int_0^\infty v^\alpha e^{-tv}dv = \Gamma(\alpha+1)/t^\alpha.$$

Hence by Continuity Theorem 7.1 (now with a continuous parameter),

$$\frac{S(xv)}{\phi(x)} \to Av^\alpha \ \text{ as } \ x \to \infty, \quad \forall\, v > 0. \tag{8.5}$$

One may in particular take $v = 1$, which gives the desired asymptotic result.

□

We will see in Section 13 that (8.3) by itself already implies relations of the form (8.1), (8.2).

Theorem 8.1 has a counterpart for positive nonincreasing functions S on $\mathbb{R}^+$. Using formula (7.2) to define the Laplace transform, one has

Theorem 8.2. *Let $S(v)$ vanish for $v < 0$ and be locally integrable, positive and nonincreasing for $v \geq 0$. Let $\phi(v) = v^\alpha L(v)$ with $-1 < \alpha \leq 0$ and slowly varying L. Then for $x \to \infty$,*

$$\begin{aligned} &S(x) - S(\infty) \sim A\phi(x) \ \textit{if and only if} \\ &\mathcal{L}dS(1/x) - S(\infty) \sim A\Gamma(\alpha+1)\phi(x). \end{aligned} \tag{8.6}$$

The proof may be based on Theorem 8.1. One would start with the formula

$$\mathcal{L}dS(t) - S(\infty) = t\int_0^\infty \{S(v) - S(\infty)\}e^{-tv}dv = t\int_0^\infty e^{-tv}dU(v),$$

$$\text{where} \quad U(v) = \int_0^v \{S(w) - S(\infty)\}dw.$$

The first relation (8.6) implies

$$U(x) \sim Ax^{\alpha+1}L(x)/(\alpha+1); \tag{8.7}$$

cf. Theorem 5.2. By Theorem 8.1 with $\alpha + 1$ instead of α, relation (8.7) is equivalent to

$$\mathcal{L}dS(1/x) - S(\infty) = \frac{1}{x}\mathcal{L}dU(1/x) \sim \frac{1}{x}A\Gamma(\alpha+2)\frac{x^{\alpha+1}L(x)}{\alpha+1}. \tag{8.8}$$

This gives the second relation (8.6).

Conversely, the second relation (8.6) implies (8.8) and hence (8.7). Since we deal with asymptotics at ∞, it may be assumed that L has been made smooth and that $xL'(x)/L(x) = \varepsilon(x) \to 0$; cf. Section 2. By the monotonicity of S, the asymptotic formula for $U(x)$ may now be differentiated to give the first relation (8.6); cf. Section I.17. □

Geluk [1985] proved a Karamata-type result for O-regularly varying functions; cf. also de Haan and Stadtmüller [1985]. Weiermann [2003] has applied Karamata's theorem to a problem of enumeration.

9 Stieltjes and Other Transforms

Theorems of Abelian and Tauberian type for general Stieltjes transforms

$$F_\rho(x) = \int_{0-}^\infty \frac{dS(v)}{(x+v)^\rho} \tag{9.1}$$

go back to the work of Valiron [1914] on the growth and zeros of entire functions; cf. Applications III.19.4. Later developments are due to Titchmarsh [1927] (cf. [1958]) and Hardy and Littlewood [1929]. Extending their work, Karamata [1931] proved

Theorem 9.1. *Let $S(v)$ vanish for $v < 0$, be nondecreasing, continuous from the right and such that the Stieltjes transform $F_\rho(x)$ exists for every $x > 0$. Let $L(\cdot)$ be slowly varying and $0 \leq \alpha < \rho$. Then for $x \to \infty$*

$$
\begin{aligned}
&S(x) \sim Ax^{\alpha}L(x) \quad \textit{if and only if} \\
&F_\rho(x) \sim A\frac{\Gamma(\alpha+1)\Gamma(\rho-\alpha)}{\Gamma(\rho)}x^{\alpha-\rho}L(x).
\end{aligned} \tag{9.2}
$$

The result can be obtained by repeated application of theorems for the Laplace transform that involve asymptotics at 0 or ∞. One may start with the formula

$$\frac{1}{(x+v)^\rho} = \frac{1}{\Gamma(\rho)}\int_0^\infty e^{-(x+v)t}t^{\rho-1}dt;$$

cf. the special case $\rho = 1,\ L = 1$ treated in Section I.21.

Theorem 9.1 can be considered as a special case of the *general-kernel theorems* for Stieltjes integrals, due to Bingham and Teugels [1979]. Below we state their results in the elegant form presented in Bingham, Goldie and Teugels [1987] (BGT).

The *Mellin–Stieltjes convolution* $k * dS$ is defined by

$$k * dS(x) = \int_0^\infty k\left(\frac{x}{v}\right)dS(v) \tag{9.3}$$

for all $x > 0$ for which the integral exists. It is assumed that k is continuous almost everywhere and that S is monotonic; for definiteness we will take S continuous from the right. Since S has only countably many jumps, the values of x for which a discontinuity of $k(x/v)$ coincides with one of $S(v)$ form a set of measure zero. The additional conditions imposed on k and S below will then ensure that $k * dS(x)$ exists for (at least) almost all $x > 0$.

An important role is played by the *Mellin transform*

$$\hat{k}(s) = \int_0^\infty v^{-s}k(v)\frac{dv}{v} = \int_0^\infty v^s k\left(\frac{1}{v}\right)\frac{dv}{v}. \tag{9.4}$$

Observe that it corresponds to the two-sided Laplace transform under exponential change of variable. In asymptotics involving arbitrary regularly varying functions of index α, one requires that the Mellin transform exist throughout some vertical strip $\{\beta \le \operatorname{Re} s \le \gamma\}$ which contains the line $\{\operatorname{Re} s = \alpha\}$ in its interior. We mention one reason: if $h(v)L(v)$ is integrable over $[1,\infty)$ for every locally bounded, slowly varying function L, then $h(v)v^\delta$ is integrable over $[1,\infty)$ for some number $\delta > 0$; cf. Vuilleumier [1963] and BGT (section 2.3). For theorems involving the convolution (9.3), it is convenient to require that a certain *amalgam norm* be finite:

$$\|k\|_{\beta,\gamma} \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \max(e^{-\beta n}, e^{-\gamma n}) \sup_{e^n \le v \le e^{n+1}} |k(v)| < \infty. \tag{9.5}$$

We begin with an *Abelian theorem*:

Theorem 9.2. *Let k be continuous almost everywhere on $(0,\infty)$ and such that $\|k\|_{\beta,\gamma} < \infty$ for some constants β, γ with $\beta < \gamma$. Let $S(v)$ be monotonic on $(0,\infty)$ and $\mathcal{O}(v^\beta)$ for $v \searrow 0$. Finally suppose that*

$$S(v) \sim A\phi(v) = Av^\alpha L(v) \quad \textit{as} \quad v \to \infty, \tag{9.6}$$

where $\alpha \in (\beta, \gamma)$ and L is slowly varying. Then the convolution $k * dS$ exists almost everywhere on $(0, \infty)$, and

$$k * dS(x) = \int_0^\infty k(x/v)dS(v) \sim A\alpha\hat{k}(\alpha)\phi(x) \quad \textit{as} \ \ x \to \infty. \tag{9.7}$$

The proof by Bingham, Goldie and Teugels depends on approximation of k by piecewise constant functions of compact support. If χ_I is the characteristic function of the interval $I = [a, b)$ with $0 < a < b < \infty$, then by (9.6)

$$\int_0^\infty \chi_I(x/v)dS(v) = \int_{(x/b, x/a]} dS(v) = S(x/a) - S(x/b);$$
$$\chi_I * dS(x)/\{x^\alpha L(x)\} \to A(a^{-\alpha} - b^{-\alpha}) = A\alpha\hat{\chi}_I(\alpha) \quad \text{as} \ \ x \to \infty.$$

Tails in the integral for $k * dS$ are estimated with the aid of (9.5) and the estimates for $S(v)$ for small and large v which follow from the hypotheses.

We now state the corresponding *Tauberian theorem* of BGT. In view of Wiener's theory, one will expect (and can verify) that a converse of Theorem 9.2 requires a Wiener-type condition,

$$\hat{k}(s) \neq 0 \quad \text{on the line} \ \ \operatorname{Re} s = \alpha. \tag{9.8}$$

Theorem 9.3. *Let $\phi(v) = v^\alpha L(v)$ with $\alpha \neq 0$ and slowly varying L. Let k be nonnegative and continuous almost everywhere on $(0, \infty)$, satisfy the norm condition (9.5) for some $\beta < \alpha < \gamma$, as well as the Wiener condition (9.8). Let $S(v)$ be monotonic on $(0, \infty)$, $\mathcal{O}(v^\beta)$ for $v \searrow 0$, and such that the Mellin–Stieltjes convolution $k * dS$ exists almost everywhere and satisfies the asymptotic relation*

$$k * dS(x) \sim A\alpha\hat{k}(\alpha)\phi(x) = A\alpha\hat{k}(\alpha)x^\alpha L(x) \quad \textit{as} \ \ x \to \infty. \tag{9.9}$$

Then $S(x) \to 0$ if $\alpha < 0$ and whenever $\alpha \neq 0$,

$$S(x) \sim A\phi(x) \quad \textit{as} \ \ x \to \infty. \tag{9.10}$$

Bochner [1934], and Bochner and Chandrasekharan [1949] had dealt with the special case $\alpha > 0$ and $L(v) = \mathcal{O}\{(\log v)^C\}$.

In their proof of Theorem 9.3, Bingham, Goldie and Teugels used the following intermediate result:

Theorem 9.4. *Let the hypotheses of Theorem 9.3 be satisfied, except that one does not need the condition $\alpha \neq 0$ here. Let h be continuous almost everywhere with $\|h\|_{\beta',\gamma'} < \infty$ for some $\beta' < \alpha < \gamma'$. Then*

$$h * dS(x) \sim A\alpha\hat{h}(\alpha)\phi(x) \quad \textit{as} \ \ x \to \infty. \tag{9.11}$$

As an illustration of the general theorems we consider the case $\alpha > 0$ of Theorem 9.1 for the Stieltjes transform. The appropriate kernel is $k(1/v) = (1 + v)^{-\rho}$, with Mellin transform

$$\hat{k}(s) = \int_0^\infty \frac{v^{s-1}dv}{(1+v)^\rho} = \int_0^\infty v^{s-1}dv \int_0^\infty e^{-(1+v)t} \frac{t^{\rho-1}}{\Gamma(\rho)} dt = \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\rho)}; \tag{9.12}$$

cf. Whittaker and Watson [1927/96] (section 12.4). Since Γ is never zero, the transform is clearly zero-free. Taking $S(0) = 0$, formula (9.1) gives

$$k * dS(x) = \int_0^\infty \frac{dS(v)}{(1 + v/x)^\rho} = x^\rho F_\rho(x). \tag{9.13}$$

Thus Theorems 9.2 and 9.3 confirm the equivalence (9.2) if $\alpha > 0$ and $S(v) = \mathcal{O}(v^\beta)$ for some number $\beta \in (0, \alpha)$.

Remarks 9.5. The monotonicity condition on S in Theorems 9.1 and 9.4 can be relaxed; cf. Remarks I.21.3.

Bingham and Teugels [1979] also proved general Abelian and Tauberian theorems for 'ordinary' Mellin convolution,

$$k * S(x) = \int_0^\infty k\left(\frac{x}{v}\right) S(v) \frac{dv}{v}.$$

This kind of convolution on $\mathbb{R}^+$ corresponds to ordinary convolution on $\mathbb{R}$; it is more convenient for theorems involving regular variation than the definition used in Section II.2. In the Tauberian direction, the results require Tauberian conditions which match the index of regular variation; cf. Delange [1950] and *BGT*.

In Sections 13, 14 we will discuss a Mercerian theorem related to Theorem 9.1.

10 The Ratio Theorem

The following result is an extension of Karamata's Theorem 8.1. It involves nondecreasing functions on $\mathbb{R}$ with support on $\mathbb{R}^+$ as in Section 7, as well as their Laplace transforms. In the Theorem, the functions S and T themselves need not be regularly varying. However, it is required that S be in the related class OR of O-regularly varying functions described in Section 6. Thus S is measurable, (eventually) positive, and such that

$$0 < S_*(v) = \liminf_{u\to\infty} \frac{S(uv)}{S(u)} \le \limsup_{u\to\infty} \frac{S(uv)}{S(u)} = S^*(v) < \infty, \quad \forall\, v \ge 1. \tag{10.1}$$

Theorem 10.1. (Ratio theorem) *Let S and T be mass functions (hence nondecreasing), with S in OR. If for slowly varying L and constant $A \ge 0$,*

$$T(v) \sim AL(v)S(v) \quad \text{as } v \to \infty, \tag{10.2}$$

then for $x \to \infty$

$$\mathcal{L}dT(1/x) \sim AL(x)\mathcal{L}dS(1/x). \tag{10.3}$$

If $S^*(1+) = 1$ the converse is valid as well.

Proof. One may assume that $A > 0$ and $S(u) > 0$ for u greater than or equal to some number u_0. [If $A = 0$ or $S \equiv 0$ one may apply Theorem 8.1 with $A = 0$ to T.] For $u \geq u_0$, the nondecreasing functions

$$S_u(v) \stackrel{\text{def}}{=} \frac{S(uv)}{S(u)} \tag{10.4}$$

are bounded by a fixed polynomial in v by the definition of OR. Thus every sequence $x_k \to \infty$ has a subsequence $\{u_k\}$ such that

$$\Sigma_k(v) \stackrel{\text{def}}{=} S_{u_k}(v) = \frac{S(u_k v)}{S(u_k)} \quad \text{tends to a limit function} \quad \Sigma(v) \tag{10.5}$$

on a dense set, and in particular at each point where the limit function is continuous. By Continuity Theorem 7.1,

$$\mathcal{L}d\Sigma_k(t) = \int_{0-}^{\infty} e^{-tv} d\Sigma_k(v) = \frac{\mathcal{L}dS(t/u_k)}{S(u_k)} \to \mathcal{L}d\Sigma(t), \qquad \forall t > 0. \tag{10.6}$$

One now defines

$$T_k(v) = \frac{T(u_k v)}{L(u_k)S(u_k)}. \tag{10.7}$$

(i) Suppose that (10.2) holds. Then $T \in OR$ and by the slow variation of L,

$$T_k(v) \sim \frac{AL(u_k v)S(u_k v)}{L(u_k)S(u_k)} \sim A\Sigma_k(v) \quad \text{as } k \to \infty,$$

hence $T_k(v) \to A\Sigma(v)$ wherever Σ is continuous. Also, by the Continuity Theorem, $\mathcal{L}dT_k(t) \to A\mathcal{L}d\Sigma(t)$, that is,

$$\frac{\mathcal{L}dT(t/u_k)}{L(u_k)S(u_k)} \to A\mathcal{L}d\Sigma(t), \qquad \forall t > 0. \tag{10.8}$$

For $t = 1$ it follows from (10.8), (10.6) and the nonvanishing of $\hat{\Sigma}(1)$ that

$$\frac{\mathcal{L}dT(1/u_k)}{L(u_k)\mathcal{L}dS(1/u_k)} \to A.$$

Now every sequence $x_k \to \infty$ has a subsequence $\{u_k\}$ with this property, hence (10.3) follows.

(ii) Suppose that $S^*(1+) = 1$. Then 1 will be a point of continuity for Σ. Indeed, by (10.5) and the definition of S^*, $\Sigma(v) \leq S^*(v)$ for every $v > 1$, so that $\Sigma(1+) \leq 1$ and hence $\Sigma(1+) = 1$. Likewise $\Sigma(1-) = 1$, since

$$\Sigma(w) \geq \liminf_{u\to\infty} \frac{S(uw)}{S(u)} = \liminf \frac{S(u)}{S(u/w)} = \frac{1}{S^*(1/w)} \quad \text{for } w < 1.$$

Suppose now that we also have (10.3). Then

$$\mathcal{L}dT_k(t) = \frac{\mathcal{L}dT(t/u_k)}{L(u_k)S(u_k)} \sim \frac{A\mathcal{L}dS(t/u_k)}{S(u_k)} = A\mathcal{L}d\Sigma_k(t) \to A\mathcal{L}d\Sigma(t), \qquad \forall t > 0.$$

Hence by the Continuity Theorem, $T_k(v) \to A\Sigma(v)$ at each point of continuity for Σ, in particular at $v = 1$. One finally uses (10.7).

Remarks 10.2. In essence, the ratio theorem goes back to Feller [1963]. The present form is due to Stadtmüller and Trautner [1979], who also showed that the converse need not hold if $S^*(1+) > 1$. Predecessors are in Keldysh [1951] and Korenblum [1953], [1955]. There is a ratio theorem with remainder in Omey [1985]. Generalizations of Theorem 10.1 were obtained by Stadtmüller [1982], de Haan and Stadtmüller [1985], and Stadtmüller and Trautner [1985], [1999]; see also Lyttkens [1986] and Frennemo [1999], [2002]. For the arrangement of the proof above, cf. Bingham, Goldie and Teugels [1987].

11 Beurling Slow Variation

Unpublished Lecture Notes of Beurling [1957] contain an extension of Wiener's Tauberian theorem which is useful for various applications (Borel summability, probability theory); cf. Moh [1972], Peterson [1972], Bingham [1981], Bingham and Goldie [1983]. A positive measurable function ϕ on $(0, \infty)$ will be called *Beurling-slowly varying* if $\phi(x) = o(x)$ and

$$\frac{\phi\{x+\phi(x)y\}}{\phi(x)} \to 1 \quad \text{as } x \to \infty, \quad \forall\, y \in \mathbb{R}. \tag{11.1}$$

A sufficient condition for Beurling slow variation is that $\phi > 0$ have derivative $\phi'(x) \to 0$ as $x \to \infty$. Examples: x^α with $\alpha < 1$ and e^{-x}. Beurling's Tauberian theorem is as follows.

Theorem 11.1. *Let K be a Wiener kernel (Section II.8), let S be bounded and suppose that ϕ is Beurling-slowly varying. Then the relation*

$$\begin{aligned}\int_{\mathbb{R}} S\{x-\phi(x)y\}K(y)dy &= \int_{\mathbb{R}} S(t)K\left(\frac{x-t}{\phi(x)}\right)\frac{dt}{\phi(x)} \\ &\to A\int_{\mathbb{R}} K(y)dy \quad \textit{as } x \to \infty \end{aligned}\tag{11.2}$$

implies that

$$\int_{\mathbb{R}} S\{x-\phi(x)y\}H(y)dy \to A\int_{\mathbb{R}} H(y)dy \quad \textit{as } x \to \infty \tag{11.3}$$

for every function H in $L^1(\mathbb{R})$.

Proof. Replacing S by $S - A$ one may assume $A = 0$. Since S is bounded, the admissible L^1 functions H in (11.3) form a closed (linear) subspace. Now by Wiener's Approximation Theorem II.8.3, the linear span of the translates of K is dense in L^1. Hence it is sufficient to prove that each translate of K is an admissible function H. In other words, it is enough to prove that

$$\int_{\mathbb{R}} S\{x-\phi(x)y\}K(y-\lambda)dy \to 0 \quad \text{as } x \to \infty \tag{11.4}$$

for arbitrary real λ. Setting

$$x - \phi(x)\lambda = x', \quad \phi(x')/\phi(x) = m(x), \quad y - \lambda = m(x)z,$$

one may rewrite the desired relation (11.4) as

$$\int_{\mathbb{R}} S\{x' - \phi(x')z\}m(x)K\{m(x)z\}dz \to 0 \quad \text{as } x' \to \infty. \tag{11.5}$$

Now $m(x) \to 1$ by (11.1), and for our $K \in L^1$,

$$\int_{\mathbb{R}} |\rho K(\rho z) - K(z)|dz \to 0 \quad \text{as } \rho \to 1. \tag{11.6}$$

Indeed, one can approximate K arbitrarily well in L^1 by continuous functions with compact support. Next, observe that by (11.2), denoting the variable of integration by z instead of y,

$$\int_{\mathbb{R}} S\{x' - \phi(x')z\}K(z)dz \to 0 \quad \text{as } x' \to \infty.$$

This relation and (11.6) imply (11.5).

Remarks 11.2. Bloom [1976] has shown that the convergence in (11.1) for each y implies uniform convergence on every finite y-interval if ϕ is continuous. (Cf. Remark 3.4 for the case of smooth ϕ; it is an *open problem* whether measurability of ϕ is enough.) For 'uniformly Beurling-slowly varying functions' or 'self-neglecting functions', Bloom also obtained an integral representation. Cf. Bingham, Goldie and Teugels [1987]. Feichtinger and Schmeisser [1986] obtained weighted (that is, quantitative) versions of Beurling's theorem.

12 A Result in Higher-Order Theory

The following result is due to de Haan [1976]; cf. Geluk and de Haan [1981], [1987], Bingham, Goldie and Teugels [1987] (BGT).

Theorem 12.1. *Let $S(v)$ vanish for $v < 0$, be nondecreasing, continuous from the right, and such that the Laplace–Stieltjes transform $\mathcal{L}dS(t)$ (7.1) exists for $t > 0$. Let L be slowly varying and $A \geq 0$. Then the following two relations for $x \to \infty$ are equivalent:*

$$\frac{S(\lambda x) - S(x)}{L(x)} \to A \log \lambda, \quad \forall \lambda > 0, \tag{12.1}$$

$$\frac{\mathcal{L}dS\{1/(\lambda x)\} - \mathcal{L}dS(1/x)}{L(x)} \to A \log \lambda, \quad \forall \lambda > 0. \tag{12.2}$$

Each of these relations implies

$$\frac{S(x) - \mathcal{L}dS(1/x)}{L(x)} \to A\gamma, \tag{12.3}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant.

Proof. It is sufficient to consider $\lambda > 1$ and convenient to introduce the notation

$$G(x) \stackrel{\text{def}}{=} \mathcal{L}dS(1/x) = \int_{0-}^{\infty} e^{-v/x} dS(v). \tag{12.4}$$

(i) We begin by deriving some formulas. Define a nondecreasing function T by

$$T(x) = \int_{0-}^{x} y dS(y) = xS(x) - \int_{0}^{x} S(y)dy. \tag{12.5}$$

Then

$$H(x) \stackrel{\text{def}}{=} \mathcal{L}dT(1/x) = \int_{0-}^{\infty} e^{-v/x} v dS(v) = x^2 G'(x), \tag{12.6}$$

hence

$$\frac{G(\lambda x) - G(x)}{L(x)} = \int_{x}^{\lambda x} \frac{H(t)}{L(x)} \frac{dt}{t^2} = \int_{1}^{\lambda} \frac{H(ux)}{uxL(ux)} \frac{L(ux)}{L(x)} \frac{du}{u}. \tag{12.7}$$

Relation (12.1) is equivalent to the statement that S belongs to $\Pi(L, A)$ (Section 6). By the first part of Theorem 6.2, this is equivalent to the relation $T(x) \sim AxL(x)$. By Karamata's Theorem 8.1, the latter is equivalent to

$$H(x) \sim AxL(x). \tag{12.8}$$

We finally show that (12.8) implies (12.2), and conversely. For this we use formula (12.7). The direct part follows immediately from the fact that L is slowly varying.

For the converse, from (12.2) to (12.8), one estimates H from above and below, using its monotonicity. By (12.7)

$$\frac{H(x)}{\lambda x L(x)} \log \lambda \le \frac{G(\lambda x) - G(x)}{L(x)} \le \frac{H(\lambda x)}{\lambda x L(\lambda x)} \frac{L(\lambda x)}{L(x)} \lambda \log \lambda.$$

Hence by (12.4) and (12.2),

$$\limsup_{x \to \infty} \frac{H(x)}{xL(x)} \le A\lambda, \quad \liminf_{y \to \infty} \frac{H(y)}{yL(y)} \ge A/\lambda, \quad \forall \lambda > 1.$$

For $\lambda \searrow 1$ this gives (12.8).

(ii) Let S be as in the first part of the Theorem and satisfy (12.1). For the derivation of (12.3), write

$$G(x) = (1/x) \int_{0}^{\infty} S(v) e^{-v/x} dv = \int_{0}^{\infty} S(xu) e^{-u} du.$$

Thus

$$\frac{S(x) - G(x)}{L(x)} = \int_{0}^{\infty} \frac{S(x) - S(ux)}{L(x)} e^{-u} du.$$

Here we may assume that $L(x)$ has been adjusted to make it positive and continuous for $x \ge 0$; cf. Section 2. Since S is in $\Pi(L, A)$ and locally bounded, the quotient in the

integrand is majorized by an expression $C(u^\delta + u^{-\delta})$, where we may take $\delta = 1/2$; see Corollary 6.3. Hence by (12.1) and dominated convergence, the integral has the following limit as $x \to \infty$:

$$\int_0^\infty (-A \log u) e^{-u} du = -A\Gamma'(1) = A\gamma;$$

cf. Whittaker and Watson [1927/96] (section 12.3). □

There are numerous other results in higher-order theory; cf. Geluk [1981a] and *BGT*. For more recent refinements, we mention de Haan and Stadtmüller [1996].

13 Mercerian Theorems

If a sequence $\{s_n\}_1^\infty$ is Cesàro limitable to A, that is,

$$s_n^{(-1)}/n = (s_1 + s_2 + \cdots + s_n)/n \to A,$$

one needs a Tauberian condition in order to conclude that the sequence is convergent; cf. Chapter I. However, as Mercer [1907] observed, a relation

$$(1-\lambda)\{s_n^{(-1)}/n\} + \lambda s_n \to A, \quad \text{with } \lambda > 0$$

implies convergence $s_n \to A$ *without* any additional condition. Cf. Pólya and Szegő [1925/78] (chapter 3, problem 49), Hardy [1949] (who gives several proofs in sections 5.9, 5.10), Zeller and Beekmann [1958/70] (section 43), and Marić and Tomić [1984]. We sketch a proof for the simple case $\lambda = 1/2$. Since one may clearly assume $A = 0$, it is enough to prove that

$$s_n^{(-1)} + n s_n = o(n) \ \text{ implies } \ s_n^{(-1)} = o(n). \tag{13.1}$$

Notice how easily the corresponding relation $F(x) + xF'(x) = o(x)$ implies $xF(x) = o(x^2)$ by integration, so that $F(x) = o(x)$. In the case of (13.1), the relation

$$(n+1)s_n^{(-1)} - n s_{n-1}^{(-1)} = s_n^{(-1)} + n s_n = o(n)$$

gives $(n+1)s_n^{(-1)} = o(n^2)$ by summation!

An interesting generalization may be found in Paley and Wiener [1934] (section 18). Many other Tauberian theorems have 'Mercerian' analogs, see the special chapters on Mercerian theorems in Pitt [1958], and Bingham, Goldie and Teugels [1987] (*BGT*). Mercerian theorems in Banach spaces are considered by Sõrmus [1998].

Here we state Mercerian analogs to Karamata's Theorems 8.1 and 9.1 and to a 'general-kernel theorem' for ordinary Mellin convolution. A proof of Theorem 13.2 will be given in Section 14; for Theorem 13.1 see *BGT*.

Theorem 13.1. *Let $S(v)$ be nondecreasing on $\mathbb{R}$, equal to zero for $v < 0$ but eventually positive. Suppose that the Laplace–Stieltjes transform $\mathcal{L}dS(t)$ exists for all $t > 0$. If for some constant $c > 0$,*

$$\mathcal{L}dS(1/x) = \int_{0-}^{\infty} e^{-v/x} dS(v) \sim cS(x) \quad as \quad x \to \infty, \tag{13.2}$$

then $c = \Gamma(\alpha+1)$ for some number $\alpha \geq 0$, and $S(x) = x^{\alpha}L(x)$ with a slowly varying function L.

Theorem 13.2. *Let $S(v)$ be nondecreasing on $\mathbb{R}$, equal to zero for $v \leq 0$ but eventually positive. Suppose that the Stieltjes transform $F(x) = F_1(x)$ exists for all positive x; cf.* (9.1) *with $\rho = 1$. If for some constant $c > 0$,*

$$xF(x) = \int_{0}^{\infty} \frac{dS(v)}{1+v/x} \sim cS(x) \quad as \quad x \to \infty, \tag{13.3}$$

then $c = \alpha\pi/\sin\alpha\pi$ for some number $\alpha \in [0, 1)$ (so that $c \geq 1$), and $S(x) = x^{\alpha}L(x)$ with a slowly varying function L.

We now formulate a general Mercerian theorem involving Mellin convolution. The conditions given here can actually be relaxed in various ways; cf. Drasin and Shea [1976] (theorem 6.2), Jordan [1974] and *BGT* (section 5.2).

Theorem 13.3. *Let $k(\cdot)$ be a nonnegative kernel whose Mellin transform*

$$\hat{k}(s) = \int_{0}^{\infty} k(v)v^{-s-1}dv$$

exists for $\beta < s < \gamma$, where $\beta < 0 < \gamma$ and (β, γ) is the maximal (open) interval of existence. Let $S(\cdot)$ on $\mathbb{R}^+$ be nonnegative, bounded on every finite interval, of finite order

$$\alpha = \limsup_{v\to\infty} \frac{\log S(v)}{\log v} \quad with \quad \alpha \in (\beta, \gamma),$$

and 'of limited decrease':

$$\liminf_{1\leq\lambda\leq 2,\, v\to\infty} \frac{S(\lambda v)}{S(v)} = \delta > 0.$$

Suppose that

$$k * S(x) = \int_{0}^{\infty} k\left(\frac{x}{v}\right) S(v)\frac{dv}{v} \sim cS(x) \quad as \quad x \to \infty, \tag{13.4}$$

with $c \neq 0$. Then $c = \hat{k}(\alpha)$ and $S(x) = x^{\alpha}L(x)$ with a slowly varying function L.

Remarks 13.4. As we mentioned in Section 9, Abelian and Tauberian theorems for Stieltjes transforms go back to the work of Valiron [1914] on the growth and zeros of entire functions. The precise relation between growth and zeros has been investigated

by many authors. It was in that context that Edrei and Fuchs [1966] discovered Theorem 13.2. The subsequent extensive generalizations are known as 'Drasin–Shea theorems'; cf. Drasin [1968], Shea [1969], Ganelius [1970], Drasin and Shea [1976]. Kernels which change sign made their appearance in Jordan [1974], [1976]; cf. BGT. For recent advances, see Bingham and Inoue [1997], [1999], [2000a], [2000b], [2000c].

There are various Mercerian theorems related to higher-order theory; see Arandelović [1976], Embrechts [1978], Bingham and Teugels [1980], Omey and Willekens [1988], and cf. BGT.

14 Proof of Theorem 13.2

To show the flavor of the 'Drasin–Shea' work, we describe Shea's proof [1969] of Theorem 13.2.

STEP I. Let S satisfy the conditions of Theorem 13.2. For $0 < B < \infty$ and $x > 0$, integration by parts gives

$$\int_0^B \frac{dS(v)}{x+v} = \frac{S(B)}{x+B} + \int_0^B \frac{S(v)}{(x+v)^2} dv.$$

Since S is nonnegative, nondecreasing and the left-hand side has finite limit $F(x)$ as $B \to \infty$, both terms on the right also have a finite limit. The limit of $S(v)/v$ must then be zero, so that

$$F(x) = \int_0^\infty \frac{S(v)}{(x+v)^2} dv. \tag{14.1}$$

Clearly $F(tx) \le F(x)$ for $t \ge 1$, hence by (13.3)

$$1 \le \liminf_{v\to\infty} \frac{S(tv)}{S(v)} \le \limsup_{v\to\infty} \frac{S(tv)}{S(v)} = \limsup_{v\to\infty} \frac{tvF(tv)}{vF(v)} \le t, \quad \forall t \ge 1.$$

Fix any number $\lambda > 1$ and choose a sequence $v_n \to \infty$ such that

$$\frac{S(\lambda v_n)}{S(v_n)} \text{ converges to a limit } \mu. \tag{14.2}$$

The nondecreasing functions $g_n(t) = S(tv_n)/S(v_n)$ form a uniformly bounded family on every finite t-interval. One now applies Helly's selection principle to obtain a subsequence $\{p_n\}$ of the positive integers such that $g_{p_n}(t)$ converges to a nondecreasing function $g(t)$ wherever g is continuous; cf. Section 7. Taking a further subsequence if necessary, we may assume that $g_{p_n}(t) \to g(t)$ everywhere. For convenience we rename the numbers $v_{p_n} : v_n$ and the functions $g_{p_n} : g_n$. One may then write

$$g_n(t) = \frac{S(tv_n)}{S(v_n)} \to g(t), \quad \forall t. \tag{14.3}$$

Observe that $g(1) = 1$ and $g(\lambda) = \mu$.

Replacing x by xv_n and v by uv_n in (14.1), one obtains

$$x\int_0^\infty \frac{S(uv_n)}{S(v_n)}\frac{du}{(x+u)^2} = \frac{xv_nF(xv_n)}{S(xv_n)}\cdot\frac{S(xv_n)}{S(v_n)}. \tag{14.4}$$

Passing to the limit as $n\to\infty$, Fatou's lemma (cf. Rudin [1966/87]) and (13.3) imply that for any fixed $x>0$,

$$\begin{aligned} x\int_0^\infty g(u)\frac{du}{(x+u)^2} &= x\int_0^\infty \lim\left\{\frac{S(uv_n)}{S(v_n)}\frac{1}{(x+u)^2}\right\}du \\ &\le x\lim\int_0^\infty\{\cdots\cdots\}du = \lim\frac{xv_nF(xv_n)}{S(xv_n)}\cdot\frac{S(xv_n)}{S(v_n)} = cg(x). \end{aligned} \tag{14.5}$$

STEP 2. In (14.5) one actually has *equality*. To prove this we use the inequality

$$F(y) < \int_0^B \frac{S(v)}{(y+v)^2}dv + 4F(B) \quad (y>0), \tag{14.6}$$

which follows from the estimates

$$\int_B^\infty \frac{S(v)}{(y+v)^2}dv < \int_B^\infty \frac{S(v)}{v^2}dv < 4\int_B^\infty \frac{S(v)}{(B+v)^2}dv \le 4F(B).$$

In (14.6) one now sets $y=xv_n$, $B=bxv_n$, $v=uv_n$. This results in a good upper bound for the right-hand side of (14.4):

$$\frac{xv_nF(xv_n)}{S(xv_n)}\cdot\frac{S(xv_n)}{S(v_n)} < x\int_0^{bx}\frac{S(uv_n)}{S(v_n)}\frac{du}{(x+u)^2} + \frac{4}{b}\frac{bxv_nF(bxv_n)}{S(bxv_n)}\cdot\frac{S(bxv_n)}{S(v_n)}.$$

Letting n go to ∞ and using bounded convergence, one thus obtains the inequality

$$cg(x) \le x\int_0^{bx}\frac{g(u)}{(x+u)^2}du + \frac{4c}{b}g(bx).$$

Finally let b go to ∞ and observe that by the convergence of the first integral in (14.5), the final term above goes to zero. [Indeed,

$$\frac{g(R)}{x+R} = g(R)\int_R^\infty \frac{du}{(x+u)^2} \le \int_R^\infty g(u)\frac{du}{(x+u)^2} \to 0$$

as $R\to\infty$.] In combination with the inequality in (14.5), it follows that

$$x\int_0^\infty \frac{g(u)}{(x+u)^2}du = cg(x), \quad \forall x>0. \tag{14.7}$$

STEP 3. Inspection shows that a nonnegative nondecreasing solution of (14.7) is given by $g(t)=t^\alpha$, provided $\alpha\in[0,1)$ satisfies the equation

$$\int_0^\infty \frac{v^\alpha}{(1+v)^2}dv = \frac{\alpha\pi}{\sin\alpha\pi} = c. \tag{14.8}$$

It will be shown below that this is the *only solution* which qualifies. Then by (14.2) and (14.3) $\mu = g(\lambda) = \lambda^\alpha$. Note that the form of $g(\lambda)$ is independent of the choice of the original sequence $\{v_n\}$. It follows that

$$\lim_{v\to\infty} \frac{S(\lambda v)}{S(v)} = \lambda^\alpha. \tag{14.9}$$

Furthermore, since our number $\lambda > 1$ was arbitrary, relation (14.9) will hold for all such λ. It will also hold with λ replaced by $1/\lambda$. The final conclusion is that $S(v) = v^\alpha L(v)$, with slowly varying L; cf. Definition 2.1.

STEP 4. It remains to show that $g(t) = t^\alpha$, with α determined by (14.8), is the only admissible solution of equation (14.7). By an exponential change of variable, $u = e^s$, $x = e^z$, the equation becomes

$$e^z \int_{\mathbb{R}} \frac{g(e^s)e^s}{(e^z + e^s)^2} ds = \int_{\mathbb{R}} G(s)K(z-s)ds = cG(z), \tag{14.10}$$

where

$$G(s) = g(e^s), \quad K(z) = \frac{1}{(e^{z/2} + e^{-z/2})^2}.$$

Formal use of Fourier analysis suggests that equation (14.10) can have a nonzero solution G only if

$$\hat{K}(\omega) = \int_{\mathbb{R}} K(z)e^{-i\omega z} dz = \int_0^\infty \frac{v^{-i\omega}}{(1+v)^2} dv = \frac{-i\pi\omega}{\sin(-i\pi\omega)} \tag{14.11}$$

is equal to c. For a full treatment of equation (14.10) one may use complex Fourier analysis, see Titchmarsh [1937/86] (section 11.2). Such analysis shows the following. *Every solution which satifies a growth condition*

$$G(s) = \mathcal{O}(e^{a|s|}) \ \ on\ \mathbb{R}\ \ with\ \ a < 1 \tag{14.12}$$

can be written in the form

$$G(s) = \sum_j \sum_{k=1}^{m_j} c_{jk} s^{k-1} e^{i\omega_j s}. \tag{14.13}$$

Here ω_j runs through the roots of the equation $\hat{K}(\omega) = c$ with $|\mathrm{Im}\,\omega| \le a$, and m_j is the multiplicity of the root ω_j.

Every qualifying solution $G(s) = g(e^s)$ *does* satisfy a growth condition (14.12); see Step 5. It follows that the equation $\hat{K}(\omega) = c$ can have only a finite number of admissible roots, because $\hat{K}(\xi + i\eta) \to 0$ uniformly for $|\eta| \le a < 1$ as $\xi \to \pm\infty$. Now $G(s)$ in (14.13) must be real and positive, hence we need only consider the roots that are purely imaginary, that is, the numbers $\omega_j = i\eta_j$ such that

$$\frac{\pi\eta_j}{\sin \pi\eta_j} = c \qquad (-1 < \eta_j < 1).$$

There are no roots if $c < 1$. If $c > 1$ there are two roots, $\eta_j = \mp\alpha$ with $\alpha > 0$, say, so that

$$G(s) = c_1 e^{\alpha s} + c_2 e^{-\alpha s}, \quad g(u) = c_1 u^\alpha + c_2 u^{-\alpha}.$$

Condition (14.12) shows that we must have $\alpha < 1$; since g must be nondecreasing and $g(1) = 1$, one must take $c_2 = 0$ and $c_1 = 1$. If $c = 1$ there is a double root $\eta = 0$, so that $g(u) = c_1 + c_2 \log u$; again c_2 must be 0 and $c_1 = 1$.

STEP 5. To complete the proof, it has to be shown that

$$\beta = \limsup_{x\to\infty} \frac{\log g(x)}{\log x} < 1 \tag{14.14}$$

for every nonnegative nondecreasing solution of equation (14.7). This is a delicate matter, for which Shea used a version of so-called Pólya-peak lemmas. 'Pólya peaks' are used to describe the fine structure of infinite sequences, such as zero sequences of entire functions. The theory goes back to Pólya [1923] and was developed by, among others, Edrei [1964], [1965] and Shea himself [1966] (section 3); cf. Bingham, Goldie and Teugels [1987] (section 2.5).

Addressing the problem of (14.14), we know that $g(x) = o(x)$, so that $\beta \leq 1$. If $\beta = 1$, it would follow from the peak lemmas that there are sequences $x_n,\ y_n \to \infty$ such that $y_n/x_n \to \infty$ and

$$g(u)/u \geq \{1 + o(1)\}g(x_n)/x_n \quad \text{for } x_n \leq u \leq y_n; \tag{14.15}$$

cf. Shea [1969]. This inequality and (14.7) would imply that

$$cg(x_n) = x_n \int_0^\infty \frac{g(u)}{(x_n+u)^2} du \geq \{1 + o(1)\}g(x_n) \int_{x_n}^{y_n} \frac{u du}{(x_n+u)^2}.$$

Hence for $n \geq n_0$,

$$c \geq \frac{1}{2} \int_1^{y_n/x_n} \frac{v dv}{(1+v)^2}. \tag{14.16}$$

Since this is impossible one has $\beta < 1$. □

15 Asymptotics Involving Large Laplace Transforms

As in Section 7, dS denotes a positive measure on $\mathbb{R}$ with support on $[0, \infty)$. We again write S for the 'mass function', $S(v) = (dS)(-\infty, v]$, and suppose that the Laplace or Laplace–Stieltjes transform $\mathcal{L}dS(\cdot)$ exists for all positive values of the variable. Anticipating the eventual use of a complex variable $\zeta = \xi + i\eta$, we will write ξ instead of t in the real case. Thus

$$F(\xi) = \mathcal{L}dS(\xi) = \int_{0-}^\infty e^{-\xi v} dS(v) = \xi \int_0^\infty S(v) e^{-\xi v} dv \tag{15.1}$$

is supposed to exist for $\xi > 0$. Assuming that $F(\xi)$ is 'of rapid growth' as $\xi \searrow 0$, we will study the relation between the growth of F and that of S. Here it is often convenient to replace ξ by $1/x$, and to consider the growth of

$$G(x) = \mathcal{L}dS(1/x) = \int_{0-}^{\infty} e^{-v/x} dS(v) = \frac{1}{x} \int_{0}^{\infty} S(v) e^{-v/x} dv \tag{15.2}$$

as $x \to \infty$. Abusing the language, we sometimes refer to G as the Laplace transform of dS.

The motivation for studying asymptotics involving Laplace transforms of exponential growth came from number theory, in particular, the study of partition functions; cf. Section 23. In their first joint article, Hardy and Ramanujan [1917] studied the relation between the growth of $\log F$ and $\log S$. The logarithmic approach has been refined and extended by many authors, including Kohlbecker [1958], de Bruijn [1959] (who introduced a notion of 'conjugate' slowly varying functions), Wagner [1966], [1968], Schwarz [1968], Bingham and Teugels [1975], Kasahara [1978], Balkema, Geluk and de Haan [1979] (who worked with complementary convex functions; cf. Section 24), and Geluk, de Haan and Stadtmüller [1986]. Typical results will be discussed in Sections 16 and 20. We add the remark that Kosugi [1999a], [1999b], and Kasahara and Kosugi [2000], have studied the relation between the asymptotic behavior of sequences $\{\log F_n\}$ and $\{\log S_n\}$; there is a connection with the theory of 'large deviations' in probability theory.

A number of authors have related the growth of the functions F (or G) and S themselves, rather than their logarithms ('strong asymptotics'). In the real case the conclusion will then involve certain running averages of S. An initial result of this type was obtained by Avakumović and Karamata [1936]. The general theory is due to (Pitt and) Martin and Wiener [1938]; see Sections 17–19.

Much stronger results on S can be obtained if one has information on the growth of the Laplace transform in a suitable region of the complex plane. Here the fundamental paper was that of Ingham [1941]; see Sections 21, 22.

Various results involving large Laplace transforms have counterparts for the case of small transforms; cf. Avakumović [1950a], [1950b], de Bruijn [1959], Parameswaran [1961], Kasahara [1978]. For comprehensive treatment, see Bingham, Goldie and Teugels [1987]. In Section 24 we discuss some recent results for two-sided Laplace transforms of finite measures.

Finally we mention that there are asymptotic results for Laplace transforms of different character; see for example Mel'nik [1982].

16 Transforms of Exponential Growth: Logarithmic Theory

The following result involving the Laplace transform is contained in Kohlbecker's paper [1958]. We have adjusted the formulation.

Theorem 16.1. *Let $\phi(v) = v^{\alpha} L(v)$ be regularly varying of index $\alpha > 1$. Assuming ϕ monotonic (as we may), let $\phi^{\leftarrow}(v)$ denote its inverse. Let $S(v)$ be nondecreasing*

on $\mathbb{R}$, *with* $S(v) = 0$ *for* $v < 0$, *and such that the Laplace–Stieltjes transform* $G(x) = \mathcal{L}dS(1/x)$ *in* (15.2) *exists for* $x > 0$. *Then*

$$\log G(x) \sim (\alpha - 1)\phi(x)/x \quad \textit{as} \;\; x \to \infty \tag{16.1}$$

if and only if

$$\log S(v) \sim \alpha v/\phi^{\leftarrow}(v) \quad \textit{as} \;\; v \to \infty. \tag{16.2}$$

For proofs of this theorem and related results of de Bruijn [1959] and Kasahara [1978], see Bingham, Goldie and Teugels [1987] (section 4.12). As an illustration of Theorem 16.1 we will treat the following *model result*:

$$\log G(x) \sim x \quad \text{if and only if} \quad \log S(v) \sim 2\sqrt{v}. \tag{16.3}$$

There are related results involving more rapid growth, but in that case the characterizations are more complicated. Early work of Avakumović [1936] had shown that $\log S(v) \approx v/\log v$ corresponds roughly to $\log\log G(x) \approx x$. As an example of more precise logarithmic analysis we will show in Section 20 that for the case of nonincreasing $\{\log S(v)\}/v$,

$$\begin{aligned} &\log G(x) \sim e^x/x^2 \quad \text{if and only if} \\ &\log S(v) = \frac{v}{\log v} + \{1 + o(1)\}\frac{v}{\log^2 v}. \end{aligned} \tag{16.4}$$

Section 20 also describes a general logarithmic result by Geluk, de Haan and Stadtmüller [1986].

THE CASE $\log G(x) \sim x$. We begin with a *boundedness result* that will also be useful later. Suppose that for some constant $a > 0$,

$$e^{-ax}G(x) = \int_{0-}^{\infty} e^{-(v/x)-ax}dS(v) = \mathcal{O}(1) \;\text{ or }\; o(1) \quad \text{as } x \to \infty. \tag{16.5}$$

For given $v > 0$, the weight $e^{-(v/x)-ax}$ of $dS(v)$ is maximal when $ax^2 = v$, which gives the value $e^{-2\sqrt{av}}$. Thus it may be natural to introduce the auxiliary measure $dT_a(v) = e^{-2\sqrt{av}}dS(v)$, given by the mass function

$$T_a(v) \stackrel{\text{def}}{=} \int_{0-}^{v} e^{-2\sqrt{aw}}dS(w) = e^{-2\sqrt{av}}S(v) + \int_0^v S(w)e^{-2\sqrt{aw}}\sqrt{a/w}\,dw. \tag{16.6}$$

Proposition 16.2. *Let* G *satisfy one of the conditions* (16.5). *Then for all positive numbers* c *and* λ, *for* $3/2 \le \gamma \le 2$ *and for* $x,\ y,\ z \to \infty$,

$$\int_y^{y+cy^{3/4}} dT_a(v) = \mathcal{O}(1) \;\; \textit{or} \;\; o(1), \qquad \int_z^{z+\lambda} dT_a(w^4) = \mathcal{O}(1) \;\; \textit{or} \;\; o(1),$$

$$\int_{x^2}^{x^2+x^\gamma} dT_a(v) = \mathcal{O}(x^{\gamma-3/2}) \;\; \textit{or} \;\; o(x^{\gamma-3/2}), \quad \textit{respectively}. \tag{16.7}$$

Furthermore, the relation

$$\limsup_{z\to\infty} \int_z^{z+\lambda} dT_a(w^4) = \limsup_{x\to\infty} \int_x^{x+\lambda} e^{-2\sqrt{a}w^2} dS(w^4) = M_\lambda < \infty \tag{16.8}$$

implies that

$$\limsup_{x\to\infty} S(x^4)e^{-2\sqrt{a}x^2} \le M_\lambda. \tag{16.9}$$

Proof. For $b > 0$, $x > 0$ and $ax^2 \le v \le ax^2 + bx^{3/2}$ one has

$$(v/x) + ax = (\sqrt{v/x} - \sqrt{ax})^2 + 2\sqrt{av} < b^2/(4a) + 2\sqrt{av},$$

because

$$\sqrt{v/x} - \sqrt{ax} \le \sqrt{ax + b\sqrt{x}} - \sqrt{ax} < b/(2\sqrt{a}).$$

Hence

$$e^{-ax}G(x) = \int_{0-}^{\infty} e^{-(v/x)-ax} dS(v) \ge e^{-b^2/(4a)} \int_{ax^2}^{ax^2+bx^{3/2}} e^{-2\sqrt{av}} dS(v).$$

Setting $ax^2 = y$ one concludes that the conditions (16.5) and definition (16.6) imply the first part of (16.7). The other parts follow by setting $v = w^4$ and $y^{1/4} = z$, and by summation.

Suppose now that we have (16.8). Then for any $M > M_\lambda$,

$$\int_{x-\lambda}^{x} e^{-2\sqrt{a}w^2} dS(w^4) \le M \quad \text{provided } x \ge x_0 = x_0(M, \lambda), \quad \text{say.}$$

For such x

$$e^{-2\sqrt{a}x^2}[S(x^4) - S\{(x-\lambda)^4\}] \le M,$$

$$\begin{aligned} S(x^4) &\le Me^{2\sqrt{a}x^2} + S\{(x-\lambda)^4\} \\ &\le Me^{2\sqrt{a}x^2} + Me^{2\sqrt{a}(x-\lambda)^2} + S\{(x-2\lambda)^4\} \le \cdots \\ &\le Me^{2\sqrt{a}x^2} \sum_{0\le n<(x-x_0)/\lambda} e^{2\sqrt{a}(-2n\lambda x+n^2\lambda^2)} + S(x_0^4). \end{aligned}$$

Hence

$$\{S(x^4) - S(x_0^4)\}e^{-2\sqrt{a}x^2} \le M \sum_{0\le n<x/\lambda} e^{2\sqrt{a}(-n\lambda x)} \le \frac{M}{1 - e^{-2\sqrt{a}\lambda x}}.$$

For $x \to \infty$ this gives (16.9) with any right-hand side $M > M_\lambda$.

Proof of Model Result (16.3). (i) Let $\log G(x) \sim x$ as $x \to \infty$. Then $e^{-ax}G(x) = \mathcal{O}(1)$ as $x \to \infty$ for every number $a > 1$. Hence by Proposition 16.2, see (16.7)–(16.9),

$$S(x^4)e^{-2\sqrt{a}x^2} \le M < \infty, \quad \limsup_{x\to\infty} \frac{1}{x^2} \log S(x^4) \le 2\sqrt{a},$$

from which one concludes that

$$\limsup_{v \to \infty} \frac{1}{\sqrt{v}} \log S(v) \le 2. \tag{16.10}$$

For $b \in (0, 1)$ one knows that $e^{-bx} G(x) \ge 2$ when $x > x_b$. For appropriate choice of $b < 1 < a$ and $\varepsilon > 0$, we will be able to derive that for large x,

$$I(a, b, \varepsilon) = \int_{(a-\varepsilon)x^2}^{(a+\varepsilon)x^2} e^{-(v/x)-bx} dS(v) \ge 1. \tag{16.11}$$

Indeed,

$$2 \le e^{-bx} G(x) \le I(a, b, \varepsilon) + \int_{|v-ax^2|>\varepsilon x^2} e^{-(v/x)-bx+2\sqrt{av}} dT_a(v).$$

For small ε and $a - b \le \varepsilon^2/10$, the exponent in the final integral will be majorized by $-c(\varepsilon)\sqrt{v}$ with positive $c(\varepsilon)$. This may be verified by setting $x = y\sqrt{v}$. For $0 < v < x$, the exponent is also majorized by $-cx$ with $c > 0$. Estimates based on (16.7) will now show that the integral is $o(1)$ as $x \to \infty$.

Inequality (16.11) implies that for large x and $v = (a + \varepsilon)x^2$,

$$e^{-(a-\varepsilon)x-bx} S\{(a + \varepsilon)x^2\} \ge 1, \quad \log S(v) \ge \frac{a - \varepsilon + b}{\sqrt{a + \varepsilon}} \sqrt{v},$$

from which one can conclude that

$$\liminf_{v \to \infty} \frac{1}{\sqrt{v}} \log S(v) \ge 2. \tag{16.12}$$

(ii) We now turn to the other direction in (16.3). For given $\varepsilon \in (0, 1)$ one may start with inequalities

$$e^{(2-\varepsilon)\sqrt{v}} - \mathcal{O}(1) \le S(v) \le e^{(2+\varepsilon)\sqrt{v}} + \mathcal{O}(1).$$

Estimates for the transform $G(x) = \mathcal{L}dS(1/x)$ can then be obtained from the last expression in (15.2). We focus on an upper bound for $x \to \infty$. Setting $a = 1 + 2\varepsilon$, one will have

$$G(x) = \int_0^\infty S(v) e^{-v/x} d(v/x) \le \int_0^\infty e^{2\sqrt{av}-v/x} d(v/x) + \mathcal{O}(1). \tag{16.13}$$

The final integral, which we call $G_a(x)$, can be estimated by *Laplace's method*; cf. de Bruijn [1958/81] (chapter 4). The exponent attains its maximum value ax for $v = ax^2$. One may now set $v/x = ax + t$ and expand around the point $t = 0$:

$$\begin{aligned} 2\sqrt{av} - v/x &= 2ax \left(1 + \frac{t}{ax}\right)^{1/2} - ax - t \\ &= ax - \frac{t^2}{4ax} + \mathcal{O}\left(\frac{t^3}{x^2}\right) \le ax - \frac{t^2}{8ax} \end{aligned}$$

when $|t| \le \delta x$ with small δ. A little work will show that the values $|t| > \delta x$ contribute relatively little to the integral $G_a(x)$; cf. part (i). One thus finds that

$$G(x) \le \{1 + o(1)\}e^{ax} \int_{|t|\le\delta x} e^{-t^2/(8ax)}dt \sim e^{ax}\sqrt{8\pi ax} \quad \text{as } x \to \infty;$$

cf. formula (I.25.7). Hence $\limsup \{\log G(x)\}/x \le a = 1 + 2\varepsilon$. There is a related estimate from below.

In the present case there is a simple alternative to Laplace's method. Completing a square and setting $\sqrt{v/x} - \sqrt{ax} = w$, one may estimate as follows:

$$e^{-ax}G_a(x) = \int_0^\infty e^{-(\sqrt{v/x}-\sqrt{ax})^2} d(v/x) = \int_{-\sqrt{ax}}^\infty e^{-w^2} 2(w + \sqrt{ax})dw$$
$$\sim \int_{\mathbf{R}} e^{-w^2} 2(w + \sqrt{ax})dw = 2\sqrt{\pi ax}.$$

17 Strong Asymptotics: General Case

We continue under the hypotheses on dS and the Laplace transform F or G made in Section 15. What can one say then about S on the basis of the asymptotic behavior of the transform *itself*, rather than its logarithm?

For the general analysis, it is a little more convenient to consider F rather than G. Essentially following Martin and Wiener [1938], we proceed as follows. Suppose that

$$e^{-f(\xi)}F(\xi) = \int_{0-}^\infty e^{-v\xi - f(\xi)} dS(v) \to 1 \quad \text{as } \xi \searrow 0. \tag{17.1}$$

Here it is assumed that $f(\xi)$ is smooth and that $f'(\xi)$ goes monotonically to $-\infty$ as $\xi \searrow 0$. Thus for large v, the exponent $-v\xi - f(\xi)$ has a unique maximum for

$$-f'(\xi) = v \quad \text{or, by inversion,} \quad \xi = h(v), \quad \text{say.} \tag{17.2}$$

For asymptotic results, one tries to convert (17.1) to a 'normal form' of the type

$$\int_{\mathbb{R}} e^{-(y-z)^2/2} dU(z) \quad \text{as } y \to \infty. \tag{17.3}$$

Then Wiener's 'second Tauberian theorem' becomes applicable if one can show that $\int_y^{y+1} |dU(z)| = \mathcal{O}(1)$; cf. Section II.13.

For the conversion of (17.1) one sets

$$\xi = \rho\{u + \sigma(v)\} \tag{17.4}$$

with unknown functions ρ and σ. These functions are to be determined in such a way that the negative of the exponent in the integrand,

$$v\xi + f(\xi) = v\rho\{u + \sigma(v)\} + f[\rho\{u + \sigma(v)\}], \tag{17.5}$$

becomes minimal for $u = 0$, and that the normal form $(y - z)^2/2$ in (17.3) may be obtained by suitable definition of y and z. Necessary conditions on ρ and σ are obtained by expansion with respect to u around the point $u = 0$. Thus, to begin with, the first derivative

$$v\rho'\{u + \sigma(v)\} + f'[\rho\{u + \sigma(v)\}]\rho'\{u + \sigma(v)\} \text{ must equal } 0 \text{ for } u = 0.$$

This confirms the expected relation

$$v = -f'[\rho\{\sigma(v)\}] \quad \text{or} \quad \rho\{\sigma(v)\} = h(v). \tag{17.6}$$

The second derivative of (17.5),

$$v\rho''\{u + \sigma(v)\} + f''[\rho\{u + \sigma(v)\}]\rho'\{u + \sigma(v)\}^2 + f'[\rho\{u + \sigma(v)\}]\rho''\{u + \sigma(v)\},$$

must be equal to 1 for $u = 0$. In view of (17.6) this condition reduces to

$$f''[\rho\{\sigma(v)\}]\rho'\{\sigma(v)\}^2 = 1.$$

Now by differentiation with respect to v in (17.6),

$$1 = -f''[\rho\{\sigma(v)\}]\rho'\{\sigma(v)\}\sigma'(v) \quad \text{and} \quad \rho'\{\sigma(v)\}\sigma'(v) = h'(v).$$

Combining, one obtains the relations

$$\rho'\{\sigma(v)\} = -\sigma'(v), \quad \sigma'(v)^2 = -h'(v).$$

For σ one may take any antiderivative of $(-h')^{1/2}$:

$$\sigma(v) = \int^v \{-h'(w)\}^{1/2} dw. \tag{17.7}$$

Furthermore by (17.6) and (17.2), denoting inverse functions by $(\cdot)^{\leftarrow}$,

$$\rho(t) = h \circ \sigma^{\leftarrow}(t), \quad \rho^{\leftarrow}(\xi) = \sigma \circ h^{\leftarrow}(\xi) = \sigma \circ \{-f'(\xi)\}. \tag{17.8}$$

The development of the exponent in the integrand of (17.1) now takes the form

$$-v\rho\{u + \sigma(v)\} - f[\rho\{u + \sigma(v)\}] = -H(v) - (u^2/2) + r(u, v). \tag{17.9}$$

Here

$$H(v) = v\rho\{\sigma(v)\} + f[\rho\{\sigma(v)\}] = vh(v) + f\{h(v)\} \tag{17.10}$$

is a certain antiderivative of $h(v)$. By (17.4) one has $u = \rho^{\leftarrow}(\xi) - \sigma(v)$, and the remainder $r(u, v)$ is of order u^3 for $u \to 0$. Under appropriate conditions on f, one can ignore $r(u, v)$ and the part of the integral (17.1) where $|u|$ is greater than or equal to some function $\tau(\xi)$; cf. Remarks 17.1. Formally then, relation (17.1) is equivalent to

$$\int_{|\sigma(v) - \rho^{\leftarrow}(\xi)| < \tau(\xi)} e^{-\{\rho^{\leftarrow}(\xi) - \sigma(v)\}^2/2} e^{-H(v)} dS(v) \to 1 \quad \text{as } \xi \searrow 0. \tag{17.11}$$

This leads to a formula of type (17.3) if one sets

$$\rho^{\leftarrow}(\xi) = y, \quad \sigma(v) = z, \quad U(z) = \int_{v_0}^{\sigma^{\leftarrow}(z)} e^{-H(v)} dS(v). \tag{17.12}$$

If $U(z+1) - U(z)$ is bounded, Wiener's 'second theorem' now shows that for $z \to \infty$,

$$U(z+\lambda) - U(z) = \int_{\sigma^{\leftarrow}(z)}^{\sigma^{\leftarrow}(z+\lambda)} e^{-H(v)} dS(v) \to \frac{\lambda}{\sqrt{2\pi}}, \quad \forall \lambda > 0. \tag{17.13}$$

In subsequent sections we will treat cases where $F(\xi) \sim e^{1/\xi}$ or $G(x) \sim e^x$ and $G(x) \approx e^{e^x}$. The general analysis will then serve as a guide, not as proof.

Remarks 17.1. Martin and Wiener *derived conclusion* (17.13) *from the asymptotic relation* (17.1) under the following conditions on the function $f(\xi)$ of class C^4, when $\xi_0 \geq \xi \searrow 0$:

$$f''(\xi) \geq c > 0, \quad \int_{\xi}^{\xi_0} \{f''(t)\}^{1/2} dt \to \infty,$$
$$f^{(3)}(\xi) = o[\{f''(\xi)\}^{3/2}], \quad f^{(4)}(\xi) = o[\{f''(\xi)\}^2].$$

Notice that as a result, for $\xi \searrow 0$ and $v \to \infty$,

$$-f'(\xi) + f'(\xi_0) = \int_{\xi}^{\xi_0} f''(t)dt \geq c^{1/2} \int_{\xi}^{\xi_0} \{f''(t)\}^{1/2} dt \to \infty, \quad h(v) \searrow 0.$$

For the analysis of the remainder $r(u, v)$, Martin and Wiener referred to related analysis in Wiener and Martin [1937]. The results on p. 217 of that paper imply that for $\xi = \rho\{u + \sigma(v)\} \searrow 0$,

$$\max_{|u| \leq C} |r(u, v)| \to 0, \quad \forall C > 0,$$
$$-(u^2/2) + r(u, v) \leq -|u|/2 \ \text{ for } \ |u| \geq 2, \ u \leq \rho^{-1}(\xi) - \sigma(v_0).$$

18 Application to Exponential Growth

In [1940a] (part I), Avakumović obtained a 'model result' of the following type; cf. Martin and Wiener [1938].

Theorem 18.1. *As before, let $S(v)$ be nondecreasing with $S(v) = 0$ for $v < 0$ and let $G(x) = \mathcal{L}dS(1/x)$ exist for $x > 0$. Then for $x \to \infty$ one has*

$$G(x) = \int_{0-}^{\infty} e^{-v/x} dS(v) \sim e^x \tag{18.1}$$

if and only if

$$\int_x^{x+\lambda} e^{-2w^2} dS(w^4) \to \frac{2\lambda}{\sqrt{\pi}}, \quad \forall \lambda > 0, \tag{18.2}$$

or equivalently,

$$\int_x^{x+\lambda} S(w^4)e^{-2w^2} w dw \to \frac{\lambda}{2\sqrt{\pi}}, \quad \forall \lambda > 0. \tag{18.3}$$

In a certain average sense, $S(v)$ behaves like $\{1/(2\sqrt{\pi})\}e^{2\sqrt{v}}v^{-1/4}$.

Proof. (i) Suppose we have relation (18.1), so that in terms of $T(v) = T_1(v) = \int_{0-}^v e^{-2\sqrt{w}} dS(w)$,

$$e^{-x}G(x) = \int_{0-}^{\infty} e^{-(\sqrt{v/x}-\sqrt{x})^2} dT(v) \to 1 \quad \text{as } x \to \infty; \tag{18.4}$$

cf. (16.5), (16.6). In the integral (18.4), only the values of v relatively close to x^2 will matter. More precisely, it follows from Proposition 16.2, see (16.7), that we need only integrate over an interval $|v - x^2| < x^\gamma$, where $\gamma = (3/2) + \varepsilon$ with small $\varepsilon > 0$. Indeed, for $0 < v < 2x^2$ the exponent in the integrand is comparable to $-\{(v - x^2)/x^{3/2}\}^2$, which is $\le -v^\delta$ with $\delta > 0$ when $|v - x^2|$ is greater than or equal to x^γ. For $v \ge 2x^2$ the exponent is likewise majorized by $-v^\delta$; the interval $0 < v < x$ may be dealt with separately.

At this stage one may either use the analysis of Section 17, or one may 'guess' a suitable substitution to convert relation (18.4) to normal form. For $\log G(x)$ asymptotic to $g(x) = x$, so that $\log F(\xi) \sim f(\xi) = 1/\xi$ and $f'(\xi) = -1/\xi^2$, the general theory gives $v = 1/\xi^2$,

$$\xi = h(v) = v^{-1/2}, \quad \sigma'(v) = \{-h'(v)\}^{1/2} = 2^{-1/2}v^{-3/4}, \quad \sigma(v) = 2^{3/2}v^{1/4},$$
$$h^{\leftarrow}(\xi) = 1/\xi^2, \quad \rho^{\leftarrow}(\xi) = \sigma \circ h^{\leftarrow}(\xi) = 2^{3/2}\xi^{-1/2}, \quad \rho(t) = 8/t^2.$$

Thus the 'canonical' substitution in (17.1) and (18.1) would be

$$\xi = \rho\{u + \sigma(v)\} = 8/(u + 2^{3/2}v^{1/4})^2, \quad x = 1/\xi = (2^{-3/2}u + v^{1/4})^2.$$

Experimentation with (18.4) would lead to the equivalent substitution $v = (x^{1/2} + u)^4$ (with slightly different u). We proceed with the latter, which transforms the exponential in (18.4) as follows:

$$e^{-(\sqrt{v/x}-\sqrt{x})^2} = e^{-4u^2} e^{-4(u^3/x^{1/2})-u^4/x}. \tag{18.5}$$

For the integration we could restrict ourselves to $|v - x^2| < x^\gamma = x^{(3/2)+\varepsilon}$, or to

$$|u| = |x^{1/2} - v^{1/4}| < |x^2 - v|/x^{3/2} < x^\varepsilon.$$

Thus by the third part of (16.7), the final factor in (18.5) may be replaced by 1:

$$\int_{|v-x^2|<x^\gamma} \mathcal{O}\{(|u|^3/x^{1/2}) + u^4/x\} dT(v) = \mathcal{O}(x^{3\varepsilon-1/2})\, \mathcal{O}(x^\varepsilon) = o(1),$$

provided we take $\varepsilon < 1/8$.

The upshot is that the limit relation (18.4) may be replaced by

$$\int_{|v^{1/4}-x^{1/2}|<x^{\varepsilon}} e^{-4(x^{1/2}-v^{1/4})^2}\,dT(v) = 1 + o(1).$$

Setting $x^{1/2} = y$ and $v^{1/4} = w$, one may rewrite this as

$$\int_{|w-y|<y^{2\varepsilon}} e^{-4(y-w)^2}\,dT(w^4) = 1 + o(1).$$

Now by (16.7)

$$\int_z^{z+1} dT(w^4) = \mathcal{O}(1), \tag{18.6}$$

so that the limit relation can be given the definitive form

$$\int_{\mathbb{R}} e^{-4(y-w)^2}\,dT(w^4) \to 1 \quad \text{as } y \to \infty. \tag{18.7}$$

Observe that the kernel

$$K(z) = e^{-4z^2}, \quad \text{with } \int_{\mathbb{R}} K(z)dz = \frac{1}{2}\sqrt{\pi},$$

is in the Wiener class M (described in Section II.13): K is continuous and

$$\sum_{n=-\infty}^{\infty} \sup_{n\le z\le n+1} |K(z)| < \infty.$$

K is also a 'Wiener kernel': the Fourier transform

$$\hat{K}(t) = \int_{\mathbb{R}} e^{-4z^2} e^{-itz} dz = \frac{1}{2}\sqrt{\pi} e^{-t^2/16}$$

is free of zeros. Because of condition (18.6), we can now apply Wiener's 'second Tauberian theorem', Theorem II.13.2. It shows that

$$\lim_{y\to\infty} \int_{\mathbb{R}} H(y-w)\,dT(w^4) = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} H(z)dz \tag{18.8}$$

for *every* kernel H of class M. One may in particular use trapezoidal functions H; cf. the trapezoidal function $\hat{U}$ in Section II.9. Using approximation from above and below by trapezoidal functions, we may also take H equal to the characteristic function of an interval. For the interval $[-\lambda, 0]$, the result is

$$\int_y^{y+\lambda} dT(w^4) \to \frac{2\lambda}{\sqrt{\pi}} \quad \text{as } y \to \infty, \tag{18.9}$$

which is equivalent to (18.2).

(ii) For the proof in the other direction one may basically reverse the steps of part (i). Relation (18.2) or (18.9) implies (18.8) for piecewise constant functions H of compact support, from which one can go to continuous piecewise linear functions of compact support. Approximation in the class M then gives (18.8) for the kernel K. Alternatively one could derive (18.7) from the analog of Theorem II.13.2 for the Wiener family of all trapezoidal functions. Having (18.7), condition (18.6) allows one to derive (18.4), hence (18.1).

(iii) Assuming (18.2) we finally derive (18.3). This is a matter of integrating by parts:

$$\int_x^{x+\lambda} e^{-2w^2} dS(w^4) = \left[S(w^4)e^{-2w^2}\right]_x^{x+\lambda} + \int_x^{x+\lambda} S(w^4)e^{-2w^2} 4w dw. \tag{18.10}$$

The left-hand side has limit $2\lambda/\sqrt{\pi}$, but does the integrated term go to zero? An affirmative answer can be derived from Proposition 16.2. Indeed, it follows from (18.2) that we have (16.8) with $a = 1$ and $M_\lambda < 2\lambda$. Hence by (16.9)

$$\limsup_{x\to\infty} S(x^4)e^{-2x^2} \le 2\lambda, \quad \forall \lambda > 0.$$

Relation (18.3) now follows from (18.10). One can also go back from (18.3) to (18.2).

Remark 18.2. For applications one would like to know if there are conditions on S or G which imply that actually

$$S(v) \sim \{1/(2\sqrt{\pi})\}e^{2\sqrt{v}}v^{-1/4} \quad \text{as } v \to \infty. \tag{18.11}$$

This relation would follow from (18.3) if one knows that either the supremum or the infimum of $S^*(w) = S(w^4)e^{-2w^2}w$ over intervals $[x, x+\lambda]$ is asymptotic to $S^*(x)$ as $x \to \infty$. Alternatively, it would be enough if the asymptotic relation (18.1) for G holds uniformly in every angular region $|\arg x| \le \beta < \pi/2$ of the complex plane; see Section 21.

19 Very Large Laplace Transforms

Recent work on partition functions by Weiermann [2002] leads to the question of what one can say about S if $G(x) = \mathcal{L}dS(1/x)$ behaves like an iterated exponential such as e^{e^x}; cf. also Burris [2001]. Inequalities for this case by Avakumović [1936] can be made more precise.

A Model Result Where $\log G(x) \approx e^x$. We consider a special case where the analysis of Section 17 can be carried out explicitly. Suppose that

$$e^{-g(x)}G(x) = \int_{0-}^{\infty} e^{-(v/x)-g(x)}dS(v) \to 1 \quad \text{as } x \to \infty, \tag{19.1}$$

where

$$g(x) = \int_1^x \frac{e^t}{t^2}dt = \operatorname{li}(e^x) - \frac{e^x}{x} + e \sim \frac{e^x}{x^2}. \tag{19.2}$$

Here we have defined the logarithmic integral as

$$\mathrm{li}(v) \stackrel{\mathrm{def}}{=} \int_e^v \frac{dw}{\log w} = \frac{v}{\log v} + \frac{v}{\log^2 v} + \mathcal{O}\left(\frac{v}{\log^3 v}\right). \tag{19.3}$$

Normalization of relation (19.1) by the method of Section 17. Setting $x = 1/\xi$, $f(\xi) = g(x)$, one starts with the equation

$$v = -f'(\xi) = g'(1/\xi)\xi^{-2} = e^{1/\xi}.$$

It follows that

$$\begin{aligned} &\xi = h(v) = 1/\log v, \quad \sigma'(v) = \{-h'(v)\}^{1/2} = 1/(v^{1/2}\log v), \\ &\sigma(v) = \mathrm{li}(v^{1/2}), \quad h^{\leftarrow}(\xi) = e^{1/\xi}, \quad \rho^{\leftarrow}(\xi) = \sigma \circ h^{\leftarrow}(\xi) = \mathrm{li}(e^{1/(2\xi)}), \\ &vh(v) + f\{h(v)\} = (v/\log v) + g(\log v) = \mathrm{li}(v) + e. \end{aligned} \tag{19.4}$$

Setting

$$\mathrm{li}(e^{1/(2\xi)}) = \mathrm{li}(e^{x/2}) = y, \quad \mathrm{li}(v^{1/2}) = z, \quad T(v) = \int_{v_0}^{v} e^{-\mathrm{li}(w)-e} dS(w), \tag{19.5}$$

one obtains the following normal form for (19.1):

$$\int_{\mathbb{R}} e^{-(y-z)^2/2} dU(z) \to 1 \text{ as } y \to \infty, \text{ with } U(z) = T\{\sigma^{\leftarrow}(z)\}. \tag{19.6}$$

Here $\sigma^{\leftarrow}(z) \approx z^2 \log^2 z$. For the justification of (19.6) one may use the estimate

$$I(x, \lambda) = \int_{e^x}^{e^x + \lambda x e^{x/2}} dT(v) = \mathcal{O}(1), \quad \forall \lambda > 0, \tag{19.7}$$

which can be derived from the inequality

$$e^{-g(x)}G(x) = \int_{0-}^{\infty} e^{-(v/x)-\mathrm{li}(e^x)+(e^x/x)+\mathrm{li}(v)} dT(v) \ge \int_{e^x}^{e^x+\lambda x e^{x/2}} \cdots.$$

Indeed, in terms of $u = v - e^x$ the exponent may be written as

$$\int_{e^x}^{e^x+u} \left(\frac{1}{\log w} - \frac{1}{x}\right) dw,$$

which is greater than $-\lambda^2$ when $0 \le u \le \lambda x e^{x/2}$. Next, a closer look at the exponent shows that in limit relation (19.1), one may restrict the integration to intervals of the form $|v - e^x| < e^{\{(1/2)+\varepsilon\}x}$ where $\varepsilon > 0$. In (19.6) one may limit oneself to intervals $|z - y| < e^{\varepsilon x}$, so that the normal form is valid.

We continue with Wiener theory as in Section 18. The kernel $e^{-z^2/2}$ has integral $\sqrt{2\pi}$, and by (19.6), (19.7)

$$\int_z^{z+1} dU(\zeta) = \int_{\sigma^{\leftarrow}(z)}^{\sigma^{\leftarrow}(z+1)} dT(v) \approx \int_{z^2 \log^2 z}^{z^2 \log^2 z + 2z \log^2 z} dT(v) = \mathcal{O}(1).$$

Thus (19.6) implies the limit relations

$$\int_z^{z+\lambda} dU(\zeta) \to \frac{\lambda}{\sqrt{2\pi}}, \quad \int_{e^x}^{e^x + \lambda x e^{x/2}} e^{-\mathrm{li}(v) - e} dS(v) \to \frac{\lambda}{\sqrt{2\pi}}. \tag{19.8}$$

In the manner of Section 18 it follows that $S(e^x)e^{-\mathrm{li}(e^x)} \to 0$; cf. Proposition 16.2. Hence also

$$\int_{e^x}^{e^x + \lambda x e^{x/2}} S(v) e^{-\mathrm{li}(v) - e} dv / (\log v) \to \frac{\lambda}{\sqrt{2\pi}}. \tag{19.9}$$

One can of course go in the other direction as well.

For positive dS, we have thus established the *equivalence of the asymptotic relation* (19.1) *in the case* (19.2) *to each of the limit relations* (19.8) *and* (19.9).

20 Logarithmic Theory for Very Large Transforms

We begin with a model result. Continuing the discussion of Section 19, we derive the characterization presented in formula (16.4).

Example 20.1. If $\{\log S(v)\}/v$ is nonincreasing, the relation

$$\log G(x) \sim e^x / x^2 \quad \text{for } x \to \infty \tag{20.1}$$

implies

$$\log S(v) = \frac{v}{\log v} + \{1 + o(1)\} \frac{v}{\log^2 v} \quad \text{as } v \to \infty. \tag{20.2}$$

The converse is true even without monotonicity of $\{\log S(v)\}/v$.

The method of proof is similar to the method used in Section 16 for the equivalence (16.3).

(i) Let $\log G(x) \sim e^x / x^2$ and $a > 1$. Then with $g(x) = \int_1^x (e^t / t^2) dt$ as in (19.2),

$$e^{-ag(x)} G(x) = \int_{0-}^{\infty} e^{(-v/x) - ag(x)} dS(v) \le 1 \quad \text{for all large } x.$$

From this one derives by 'normalization' as in Section 19 that

$$\int_{ae^x}^{ae^x + xe^{x/2}} e^{-a\,\mathrm{li}(v/a) - ae} dS(v) = \mathcal{O}(1), \quad S(v) = \mathcal{O}\{e^{a\,\mathrm{li}(v/a)}\},$$

$$\log S(v) \le a\,\mathrm{li}(v/a) + \mathcal{O}(1) = \frac{v}{\log v} + \{1 + \log a + o(1)\} \frac{v}{\log^2 v}.$$

Since a can be taken arbitrarily close to 1, this inequality implies the asymptotic upper bound for $\log S$ desired for (20.2).

The lower bound requires more work. Choosing $\varepsilon > 0$ small and $b < 1 < a$, where $a - b < c\varepsilon^2$ with sufficiently small c, one can prove the following analog to (16.11) for large x:

$$J(a, b, \varepsilon) = \int_{(a-\varepsilon)e^x}^{(a+\varepsilon)e^x} e^{-(v/x)-bg(x)} dS(v) \geq 1. \tag{20.3}$$

Suppose now that (20.2) is false. Then there is a number $\delta > 0$ (which we take small) and a sequence $v_n \to \infty$ such that

$$\log S(v_n) \leq \frac{v_n}{\log v_n} + (1 - 3\delta)\frac{v_n}{\log^2 v_n}. \tag{20.4}$$

At this stage we use the hypothesis that $\{\log S(v)\}/v$ is nonincreasing. Thus for $v_n \leq v \leq \rho v_n$,

$$\begin{aligned} \log S(v) &\leq \frac{v}{v_n} \log S(v_n) \leq \frac{v}{\log(v/\rho)} + (1 - 3\delta)\frac{v}{\log^2(v/\rho)} \\ &\leq \frac{v}{\log v} + (1 - \delta)\frac{v}{\log^2 v}, \end{aligned} \tag{20.5}$$

provided we take $\log \rho = \delta$, say, and v_n large. We next integrate by parts in (20.3), taking $x = x_n$ such that for appropriate a and ε

$$v_n \leq (a - \varepsilon)e^{x_n} \leq (a + \varepsilon)e^{x_n} \leq \rho v_n.$$

Then for $b < 1$ sufficiently close to a and sufficiently small ε, the integrated terms will tend to 0 by (20.5), so that

$$\int_{(a-\varepsilon)e^{x_n}}^{(a+\varepsilon)e^{x_n}} e^{-(v/x_n)-bg(x_n)} S(v)dv \geq \{1 - o(1)\}x_n \to \infty. \tag{20.6}$$

However, it can be derived from (20.5) that the left-hand side of (20.6) tends to 0 when $b > e^{-\delta}$. Indeed, Laplace's method will show that for $A > e^{\delta}$,

$$\log \int_{e^x/A}^{Ae^x} e^{-(v/x)-bg(x)+(v/\log v)+(1-\delta)v/\log^2 v} dv \sim (e^{-\delta} - b)e^x/x^2$$

as $x \to \infty$. The resulting contradiction completes the proof of (20.2).

(ii) That (20.2) implies (20.1) also follows by Laplace's method.

A General Result. We now describe a general result of Geluk, de Haan and Stadtmüller [1986]; cf. Geluk and de Haan [1987]. It involves a notion of *strong asymptotic equivalence*.

Definition 20.2. We will say that measurable functions f_1 and f_2 are strongly asymptotically equivalent, notation $f_1 \overset{s}{\sim} f_2$, if for every number $c > 1$, there is a number $v_0 = v_0(c)$ such that

$$f_1(cv) \leq cf_2(v), \quad f_2(cv) \leq cf_1(v), \quad \forall\, v \geq v_0. \tag{20.7}$$

The result below also makes use of the class Π^- (Definition 6.1): f is in Π^- if there is a slowly varying function L such that

$$\lim_{x\to\infty} \frac{f(\lambda x) - f(x)}{L(x)} = -\log\lambda, \quad \forall\lambda > 0.$$

Theorem 20.3. *Let $S(v)$ be nondecreasing on $\mathbb{R}$, with $S(v) = 0$ for $v < 0$, and such that the Laplace transform $G(x) = \mathcal{L}dS(1/x)$ of* (15.2) *exists for $x > 0$. Let f in Π^- be positive, decreasing and continuous with $f(\infty) = 0$. Then the relation*

$$\log S(v) \overset{\mathrm{s}}{\sim} \int_{v_0}^{v} f(u)du \quad \text{as } v \to \infty \tag{20.8}$$

for some number v_0 implies

$$\log G(x) \sim \int_{x_0}^{x} f^{\leftarrow}(1/t)\frac{dt}{t^2} \quad \text{as } x \to \infty \tag{20.9}$$

for some number x_0. If $\{\log S(v)\}/v$ is nonincreasing, the converse is also true.

For the application of the Theorem one may use

Proposition 20.4. *Let g be positive. Then $g(v)/v$ is in Π^- if and only if there is a positive decreasing function f in Π^- such that for some number v_0,*

$$g(v) \overset{\mathrm{s}}{\sim} \int_{v_0}^{v} f(u)du. \tag{20.10}$$

If the positive decreasing function f is in $\Pi(L)^-$ and $f(\infty) = 0$, condition (20.10) *is equivalent to*

$$g(v) = vf(v/e) + o\{vL(v)\}. \tag{20.11}$$

Applications 20.5. Example 20.1 may be obtained from Theorem 20.3 by taking $f(u) = 1/(\log u)$ and $v_0 = e$. In this case $f^{\leftarrow}(1/t) = e^t$. Thus (20.9) gives $\log G(x) \sim \int_1^x (e^t/t^2)dt \sim e^x/x^2$ and (20.8) becomes $\log S(v) \overset{\mathrm{s}}{\sim} \mathrm{li}(v)$. By the Proposition, the latter relation is equivalent to (20.2). Indeed, the present function f is in $\Pi(L)^-$ for $L(x) = 1/\log^2 x$.

Geluk, de Haan and Stadtmüller gave several applications, which include the following as special cases:

$$\log G(x) \sim e^x \quad \text{corresponds to}$$
$$\log S(v) = \frac{v\{\log v + 2(\log\log v) + 1 + o(1)\}}{\log^2 v};$$

$$\log S(v) = \frac{v}{\log v} + \frac{o(v)}{\log^2 v} \quad \text{corresponds to} \quad \log G(x) \sim e^{-1}e^x/x^2.$$

Their analysis can also be used to treat the case where $G(x)$ behaves like an iterated exponential of higher order.

21 Large Transforms: Complex Approach

As before, we assume that $S(v)$ is nondecreasing, with $S(v) = 0$ for $v < 0$, and such that the Laplace transform $F(\zeta) = \mathcal{L}dS(\zeta)$ exists for $\zeta > 0$. This time we also consider complex values of ζ:

$$F(\zeta) = \int_{0-}^{\infty} e^{-\zeta v} dS(v) = \zeta \int_{0}^{\infty} S(v) e^{-\zeta v} dv \quad \text{for } \operatorname{Re}\zeta > 0. \tag{21.1}$$

In [1941] Ingham obtained precise estimates for $S(v)$ from the asymptotic behavior of $F(\zeta)$ for angular approach to the origin. His proof involved a substantial refinement of the Wiener–Ikehara method discussed in Section III.4. We will follow Ingham, but use a notation similar to that of Section 17.

Theorem 21.1. *Let S and F be as above and suppose that*

$$e^{-f(\zeta)} F(\zeta) = e^{-f(\zeta)} \int_{0-}^{\infty} e^{-\zeta v} dS(v) \to 1 \quad \textit{as } \zeta \to 0, \tag{21.2}$$

uniformly in every angle $|\arg\zeta| \le \beta < \pi/2$. *Here one supposes that* $f(\zeta)$ *is analytic in the intersection of the right half-plane with a neighborhood of* 0, *that*

$$F(\zeta) = \mathcal{O}(e^{f(|\zeta|)}) \quad \textit{as } \zeta \to 0, \quad \textit{uniformly in every angle } |\arg\zeta| \le \beta < \pi/2, \tag{21.3}$$

and that $f(\xi)$ *satisfies the following conditions for real* $\xi \searrow 0$, *in which* $0 < \delta(\xi) \le \xi/2$:

$$f(\xi) \ \textit{is real and positive},$$

$$-\xi f'(\xi) \nearrow \infty, \quad \textit{hence } \xi f''(\xi) \ge -f'(\xi) > 0, \tag{21.4}$$

$$\sqrt{f''(\xi)}/|f'(\xi)| = o\{\delta(\xi)/\xi\}, \tag{21.5}$$

$$f''(\xi + z) = \mathcal{O}\{f''(\xi)\} \ \textit{uniformly for } |z| \le \delta(\xi). \tag{21.6}$$

Then

$$S(v) \sim S_0(v) = \frac{e^{vh(v) + f\{h(v)\}}}{h(v)\sqrt{2\pi f''\{h(v)\}}} \quad \textit{as } v \to \infty, \tag{21.7}$$

where h is the inverse of $-f'$: *if* $v = -f'(\xi)$, *then* $\xi = h(v)$.

Example 21.2. Suppose that

$$F(\zeta) \sim e^{f(\zeta)} = e^{1/\zeta} \quad \text{as } \zeta \to 0,$$

uniformly in every angle $|\arg\zeta| \le \beta < \pi/2$. In this case one can take $\delta(\xi) = \xi/2$. One has $h(v) = v^{-1/2}$ and by (21.7),

$$S(v) \sim \{1/(2\sqrt{\pi})\} e^{2\sqrt{v}} v^{-1/4} \quad \text{as } v \to \infty; \tag{21.8}$$

cf. Section 18.

Example 21.3. Suppose that, uniformly in every angle $|\arg\zeta| \le \beta < \pi/2$,

$$F(\zeta) \sim e^{f(\zeta)} = \exp\left(\int_1^{1/\zeta} \frac{e^t}{t^2} dt\right) \quad \text{as } \zeta \to 0.$$

Here one can take $\delta(\xi) = \xi^2$. One has $h(v) = 1/\log v$ and by (21.7),

$$S(v) \sim \{e^e/\sqrt{2\pi}\} e^{\mathrm{li}(v)} v^{-1/2} \quad \text{as } v \to \infty. \tag{21.9}$$

The result is in accordance with relation (19.9) which involves certain averages of S; cf. (19.2).

Proof of Theorem 21.1. Let $k(z)$ be analytic in a domain D which contains the segment $[0, i]$ and set

$$K(w) = \int_0^i k(z) e^{-wz} dz, \tag{21.10}$$

where we integrate along a path $\gamma \subset D$ close to the segment $[0, i]$. We now consider small $\xi > 0$ and $Y = B\xi$ with large $B = \tan\beta$ such that the path $\xi + Y \cdot \gamma = \xi + \Gamma$ lies in the right half-plane (cf. Figure IV.22 below). Then by (21.1), integrating along Γ,

$$\begin{aligned} J(\xi) &\overset{\text{def}}{=} \int_0^{iY} \frac{F(\xi + z)}{\xi + z} k\left(\frac{z}{Y}\right) e^{uz} dz \\ &= \int_0^\infty S(v) e^{-\xi v} dv \int_0^{iY} k\left(\frac{z}{Y}\right) e^{-(v-u)z} dz \\ &= \int_0^\infty S(v) e^{-v\xi} K\{Y(v-u)\} Y dv. \end{aligned} \tag{21.11}$$

In order to proceed we need a good estimate for the integral $J(\xi)$. We formulate such an estimate as a proposition which will be established later.

Proposition 21.4. *For* $\xi \searrow 0$ *and*

$$Y = B\xi = (\tan\beta)\xi, \quad t = -f'(\xi), \quad u = u(\xi) = -f'(\xi) + \mathcal{O}(1/\xi), \tag{21.12}$$

one has

$$J(\xi) = \int_0^{iY} \frac{F(\xi + z)}{\xi + z} k\left(\frac{z}{Y}\right) e^{uz} dz \sim \frac{e^{f(\xi)} i k(0) \sqrt{\pi/2}}{\xi\sqrt{f''(\xi)}} = ik(0)\pi S_0(t) e^{-t\xi}, \tag{21.13}$$

where S_0 *is given by* (21.7).

We continue the proof of Theorem 21.1. Substituting relation (21.13) in (21.11), setting $v = u + w/Y$ and multiplying by $e^{u\xi}$, one obtains

$$\begin{aligned} e^{u\xi} J(\xi) &= e^{u\xi} \int_0^\infty S(v) e^{-v\xi} K\{Y(v-u)\} Y dv \\ &= \int_{-Yu}^\infty S(u + w/Y) e^{-w\xi/Y} K(w) dw \sim ik(0)\pi S_0(t) e^{(u-t)\xi}. \end{aligned} \tag{21.14}$$

Observe that by (21.12) and (21.4),

$$Yu = B\xi u = -B\xi f'(\xi) + \mathcal{O}(1) \to \infty \quad \text{as } \xi \searrow 0. \tag{21.15}$$

FIRST CHOICE FOR $k(z)$. In (21.10), let

$$k(z) = k_1(z) = 2(z - i), \quad \text{for which } k(0) = -2i.$$

We compute the transform, taking w real:

$$K(w) = K_1(w) = -\frac{2i}{w} + \frac{2(1 - e^{-iw})}{w^2}, \quad \text{Re}\, K_1(w) = \left(\frac{\sin w/2}{w/2}\right)^2.$$

Taking real parts in (21.14) and using (21.15), one finds that for any given number $\lambda > 0$ and $\xi \searrow 0$,

$$\begin{aligned} &S(u - \lambda/Y)e^{-\lambda\xi/Y} \int_{-\lambda}^{\lambda} \left(\frac{\sin w/2}{w/2}\right)^2 dw \\ &\le \int_{-\lambda}^{\lambda} S(u + w/Y)e^{-w\xi/Y}\, \text{Re}\, K_1(w)\, dw \\ &\le \int_{-Yu}^{\infty} \cdots \le \{1 + o(1)\} 2\pi S_0(t) e^{(u-t)\xi}. \end{aligned}$$

Now setting

$$u = t + \lambda/Y = t + \lambda/(B\xi)$$

[cf. (21.12)], one concludes that

$$\limsup_{t\to\infty} \frac{S(t)}{S_0(t)} \le \frac{2\pi e^{2\lambda/B}}{\int_{-\lambda}^{\lambda} \{(\sin w/2)/(w/2)\}^2 dw}. \tag{21.16}$$

For $\lambda = \sqrt{B}$ this implies that the lim sup in (21.16) does not exceed 1. Indeed, B can be taken arbitrarily large and $\int_{\mathbb{R}} \{(\sin x)/x\}^2 dx = \pi$.

SECOND CHOICE FOR $k(z)$. We next take

$$k(z) = k_2(z) = 2(z - i) - \frac{e^{\mu z} - e^{-\mu z}}{\mu}, \quad \mu > 0, \quad \mu \equiv 0 \,(\text{mod}\, 2\pi).$$

As before $k(0) = -2i$, but now the transform (21.10) is

$$K(w) = K_2(w) = -\frac{2i}{w} + (1 - e^{-iw})\left(\frac{2}{w^2} + \frac{1}{\mu(\mu - w)} + \frac{1}{\mu(\mu + w)}\right).$$

For real w,

$$\text{Re}\, K_2(w) = \left(\frac{\sin w/2}{w/2}\right)^2 \frac{\mu^2}{\mu^2 - w^2}.$$

Hence by (21.14)

$$\int_{-Yu}^{\infty} S(u + w/Y)e^{-w\xi/Y} \left(\frac{\sin w/2}{w/2}\right)^2 \frac{\mu^2}{\mu^2 - w^2} dw \sim 2\pi S_0(t)e^{(u-t)\xi}.$$

Taking into account the sign of Re $K_2(w)$, one concludes that for $\xi \searrow 0$,

$$\int_{-\mu}^{\mu} S(u + w/Y)e^{-w\xi/Y} \left(\frac{\sin w/2}{w/2}\right)^2 \frac{\mu^2}{\mu^2 - w^2} dw \geq \{1 + o(1)\}2\pi S_0(t)e^{(u-t)\xi}.$$

Thus

$$S(u + \mu/Y)e^{\mu\xi/Y} \int_{-\mu}^{\mu} \operatorname{Re} K_2(w)\, dw \geq \{1 + o(1)\}2\pi S_0(t)e^{(u-t)\xi}.$$

For

$$u = t - \mu/Y = t - \mu/(B\xi), \quad \xi \searrow 0,$$

it follows that

$$\liminf_{t\to\infty} \frac{S(t)}{S_0(t)} \geq \frac{2\pi e^{-2\mu/B}}{\int_{-\mu}^{\mu} \operatorname{Re} K_2(w)\, dw}. \tag{21.17}$$

The integral in the denominator is equal to

$$\int_{-\mu}^{\mu} 2(\sin w/2)^2 \left(\frac{2}{w^2} + \frac{1}{\mu(\mu - w)} + \frac{1}{\mu(\mu + w)}\right) dw$$
$$= \int_{-\mu}^{\mu} \left(\frac{\sin w/2}{w/2}\right)^2 dw + \frac{4}{\mu}\int_0^{2\mu} \frac{(\sin x/2)^2}{x} dx.$$

Hence for large μ the integral is close to 2π. Setting $\mu = 2\pi[\sqrt{B}]$, where B may be taken arbitrarily large, one concludes that the lim inf in (22.17) is at least equal to 1. Thus the proof is complete, modulo Proposition 21.4.

Remarks 21.5. Ingham's Tauberian theorem is actually somewhat more general than Theorem 21.1; cf. also the exposition in Postnikov [1988]. Where we have $F(\zeta)$ asymptotic to $e^{f(\zeta)}$ in (21.2), Ingham writes $F(\zeta) \sim g(\zeta)e^{f(\zeta)}$. Here g is positive real for real $\zeta = \xi$ and $\log g$ varies much more slowly than f near $\zeta = 0$. Thus the 'flat factor' g can usually be included in e^f.

We remark that Avakumović obtained related (but less general) complex Tauberian theorems in his papers [1940a] (parts II, III) and [1940c].

Ingham's method as it stands does not work if $F(\zeta) \sim \zeta^{-\alpha}$, $\alpha > 0$, because then one cannot satisfy condition (21.5). However, one can use a (simpler) adjusted version of the method. Thus one can obtain a(nother) complex-variable proof of the Hardy–Littlewood theorems for power series and Laplace transforms, such as Theorems I.7.3, I.7.4 and I.15.3. See Diamond [2001].

For the case of power series $F(\zeta) = \sum_0^\infty a_n e^{-n\zeta}$, Hayman [1956] obtained precise asymptotic estimates for the coefficients a_n from estimates for F in the complex domain. Results such as those of Ingham and Hayman have important applications to problems of 'asymptotic counting'; see for example Bender [1974] and Flajolet [2002], and cf. Section 23.

22 Proof of Proposition 21.4

We have to prove the asymptotic estimate (21.13) for the integral

$$J(\xi) = \int_0^{iY} \frac{F(\xi + z)}{\xi + z} k\left(\frac{z}{Y}\right) e^{uz} dz \quad \text{as } \xi \searrow 0. \tag{22.1}$$

By Cauchy's theorem, the integral is independent of the path Γ from 0 to iY in the half-plane $\{\operatorname{Re} z > -\xi\}$, provided Γ remains within the domain of analyticity of $k(z/Y)$ around the segment $[0, iY]$. Ingham's clever choice of Γ was essential for the proof of the desired estimate.

Step (i). The main contribution to the integral $J(\xi)$ will come from the part of Γ near the point 0, for which we take a straight segment $\Gamma_1 = [0, i\eta]$, with $\eta = \eta(\xi) = o\{\delta(\xi)\} = o(\xi)$ to be specified later. Using inequality (21.6) for $f''(\xi + z)$ and Cauchy's integral formula for the derivative of f'', one finds that

$$f^{(3)}(\xi + w) = \frac{1}{2\pi i} \int_{|z|=\delta(\xi)} \frac{f''(\xi + z)}{(w - z)^2} dz = \mathcal{O}\{f''(\xi)/\delta(\xi)\}$$

when $|w| \le \delta(\xi)/2$. Hence for $z \in \Gamma_1$, so that $z = o\{\delta(\xi)\}$,

$$\begin{aligned} W &\stackrel{\text{def}}{=} f(\xi + z) - f(\xi) - f'(\xi)z - f''(\xi)z^2/2 \\ &= (1/2) \int_0^z (z - w)^2 f^{(3)}(\xi + w)\, dw = o\{f''(\xi)|z|^2\}. \end{aligned}$$

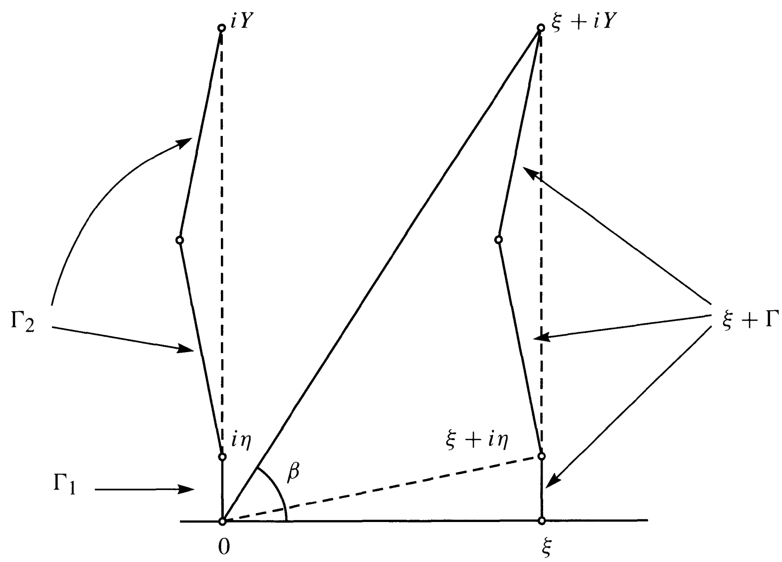

Fig. IV.22. The paths Γ and $\xi + \Gamma$

Also using the inequality $|e^W - 1| \le |W|e^{|W|}$, one concludes that for $\xi \searrow 0$,

$$e^{f(\xi+z)-f(\xi)} = e^{f'(\xi)z+f''(\xi)z^2/2}[1 + o\{f''(\xi)|z|^2\}e^{f''(\xi)|z|^2/4}]. \tag{22.2}$$

Taking $Y = B\xi$ and $u = u(\xi) = -f'(\xi) + \mathcal{O}(1/\xi)$ as in (21.12), we now use (21.2) and (22.2) to obtain the following uniform estimate for $z = iy \in \Gamma_1 = [0, i\eta]$:

$$\begin{aligned}
&\frac{F(\xi+z)}{e^{f(\xi)}}\frac{(\xi+z)^{-1}}{\xi^{-1}}\frac{k(z/Y)}{k(0)}e^{uz} \\
&= \frac{\{1+o(1)\}e^{f(\xi+z)}}{e^{f(\xi)}}\{1+o(1)\}\{1+o(1)\}e^{uz} \\
&= e^{f'(\xi)z+uz+f''(\xi)z^2/2}\cdot[1 + o\{f''(\xi)|z|^2 + 1\}e^{f''(\xi)|z|^2/4}] \\
&= e^{-f''(\xi)y^2/2}\cdot[1 + o\{f''(\xi)y^2 + 1\}e^{f''(\xi)y^2/4}].
\end{aligned} \tag{22.3}$$

Thus for $z = iy$, setting $y\sqrt{f''(\xi)} = s$,

$$\begin{aligned}
J_1(\xi) &\stackrel{\text{def}}{=} \int_{\Gamma_1} \frac{F(\xi+z)}{\xi+z}k\left(\frac{z}{Y}\right)e^{uz}dz = \int_0^{i\eta(\xi)} \cdots dz \\
&= \frac{e^{f(\xi)}}{\xi}k(0)\int_0^{\eta(\xi)\sqrt{f''(\xi)}} \{e^{-s^2/2} + o(s^2+1)e^{-s^2/4}\}\frac{i\,ds}{\sqrt{f''(\xi)}}.
\end{aligned} \tag{22.4}$$

Writing $\phi \gg \psi$ for positive functions ϕ and ψ with $\phi/\psi \to \infty$, we choose $\eta(\xi)$ such that

$$\delta(\xi) \gg \eta(\xi) \gg \frac{\xi\sqrt{f''(\xi)}}{|f'(\xi)|}; \tag{22.5}$$

cf. (21.5). Then by (21.4) $\eta(\xi)\sqrt{f''(\xi)} \to \infty$, hence by (22.4)

$$J_1(\xi) \sim \frac{e^{f(\xi)}ik(0)\sqrt{\pi/2}}{\xi\sqrt{f''(\xi)}} \quad \text{as } \xi \searrow 0. \tag{22.6}$$

Step (ii). For the second part of the path Γ, from $i\eta$ to $iY = iB\xi$, one takes a broken line Γ_2 which, apart from its end points, lies to the left of $[0, iY]$. It consists of two segments which each make a small angle $\arctan(\eta/\xi)$ with $[0, iY]$; see Figure IV.22. Let us consider the points $\xi + z$ with $z \in \Gamma_2$; for small ξ, they will lie in the angle $\{0 \le \arg\zeta \le \beta = \arctan B\}$. Also, $|\xi + z| > \xi$. For small ρ, $f(\rho)$ increases as ρ decreases; cf. (21.4), hence $f(|\xi + z|) < f(\xi)$. Thus by (21.3) and (21.12),

$$\begin{aligned}
\frac{F(\xi+z)}{\xi+z}k\left(\frac{z}{Y}\right)e^{uz} &= \mathcal{O}\left(\frac{e^{f(|\xi+z|)}}{|\xi+z|}\right)\mathcal{O}(1)e^{-f'(\xi)x+\mathcal{O}(Y/\xi)} \\
&= \mathcal{O}\left(\frac{e^{f(\xi)}}{\xi}e^{-f'(\xi)x}\right),
\end{aligned} \tag{22.7}$$

uniformly for $z = x + iy \in \Gamma_2$. Observe that for these points, $x \le 0$ and

$$\left|\frac{dz}{dx}\right| = |1 \mp i\xi/\eta| = \mathcal{O}(\xi/\eta).$$

Hence the contribution to $J(\xi)$ of the integral over Γ_2 can be estimated as follows:

$$J_2(\xi) \overset{\text{def}}{=} \int_{\Gamma_2} \frac{F(\xi+z)}{\xi+z} k\left(\frac{z}{Y}\right) e^{uz} dz = \mathcal{O}\left(\frac{e^{f(\xi)}}{\xi} \int_{-\infty}^{0} e^{|f'(\xi)|x} \frac{\xi}{\eta}\, dx\right)$$
$$= \mathcal{O}\left(\frac{e^{f(\xi)}}{\xi} \frac{\xi}{|f'(\xi)|\eta}\right) = o\left(\frac{e^{f(\xi)}}{\xi\sqrt{f''(\xi)}}\right). \tag{22.8}$$

Indeed, by (22.5) one has $|f'(\xi)|\eta(\xi) \gg \xi\sqrt{f''(\xi)}$.

By (22.6) and (22.8)

$$J(\xi) = J_1(\xi) + J_2(\xi) \sim J_1(\xi) \sim \frac{e^{f(\xi)} i k(0)\sqrt{\pi/2}}{\xi\sqrt{f''(\xi)}}, \tag{22.9}$$

which is the first part of (21.13). Setting $-f'(\xi) = t$ as in (21.12), so that $\xi = h(t)$, the definition of $S_0(\cdot)$ in (21.7) gives the second part of (21.13).

23 Asymptotics for Partitions

Let $p(n)$ denote the number of unrestricted partitions of the positive integer n, that is, the number of ways in which n can be written as a sum of positive integers; the order of the terms does not matter. In their first joint paper, Hardy and Ramanujan [1917] obtained an asymptotic estimate for $\log p(n)$. Later, several authors obtained logarithmic estimates for more general partition problems. In this context we mention Kohlbecker [1958], Schwarz [1968], Geluk [1981b], [1989]; cf. Bingham, Goldie and Teugels [1987] (*BGT*).

In a second article, Hardy and Ramanujan [1918] obtained the 'strong asymptotic result'

$$p(n) \sim \frac{1}{(4\sqrt{3})n} e^{\pi\sqrt{2n/3}}. \tag{23.1}$$

This relation was also discovered by Uspensky and later Rademacher even obtained an exact formula for $p(n)$; cf. Andrews [1976]. In [1941] Ingham obtained the strong estimate (23.1) and much more by his complex Tauberian method; cf. also Avakumović [1940b], de Bruijn [1948], and especially Schwarz [1969a]. One may also mention Newman's treatment [1998], an elementary approach by Erdős [1942], and the 'bare-handed approach' of Odlyzko [1992]. Báez-Duarte [1997] applied a probabilistic method.

Here we discuss Ingham's 'strong asymptotic result' for general unrestricted partitions. Let

$$0 < \lambda_1 < \lambda_2 < \cdots$$

be a given unbounded sequence of real numbers and let $N(v)$ be the number of $\lambda_k \le v$. For real $\mu \ge 0$, let $p(\mu)$ be the *number of representations*

$$\mu = m_1\lambda_1 + m_2\lambda_2 + \cdots \tag{23.2}$$

with integers $m_1, m_2, \cdots \geq 0$. Finally, for real v and for $\lambda > 0$, let

$$S(v) = \sum_{\mu \leq v} p(\mu), \qquad S_\lambda(v) = \frac{S(v) - S(v-\lambda)}{\lambda}, \tag{23.3}$$

where the summation is over the finite set of numbers $\mu \leq v$ for which $p(\mu)$ is different from zero. Assuming absolute convergence for $x = \operatorname{Re} z > 0$, one now has the following identities involving the *generating function* F:

$$F(z) = \int_{0-}^{\infty} e^{-zv} dS(v) = \sum_{\mu} p(\mu) e^{-\mu z} = \prod_{k=1}^{\infty} (1 - e^{-\lambda_k z})^{-1}, \tag{23.4}$$

$$\begin{aligned} \log F(z) &= -\sum_{k=1}^{\infty} \log(1 - e^{-\lambda_k z}) = -\int_0^{\infty} \log(1 - e^{-zv}) dN(v) \\ &= \int_0^{\infty} \frac{z}{e^{zv} - 1} N(v) dv. \end{aligned} \tag{23.5}$$

For the logarithmic theory based on these formulas and Theorem 16.1 we refer to Kohlbecker and BGT (section 6.1). We continue with Ingham's theory.

Theorem 23.1. *Suppose that for positive constants A, α and for real constants b, c,*

$$\begin{aligned} N(v) &= \sum_{\lambda_k \leq v} 1 = Av^\alpha + R(v), \\ Q(v) &\stackrel{\text{def}}{=} \int_0^v \frac{R(w)}{w} dw = b \log v + c + o(1) \quad \text{as } v \to \infty. \end{aligned} \tag{23.6}$$

Then for $z \to 0$ one has, uniformly in every angle $|\arg z| \leq \beta < \pi/2$,

$$\begin{aligned} F(z) &= \int_{0-}^{\infty} e^{-zv} dS(v) \sim e^{f(z)}, \quad \text{where} \\ f(z) &= A\Gamma(\alpha + 1)\zeta(\alpha + 1) z^{-\alpha} - b \log z + c. \end{aligned} \tag{23.7}$$

One can then apply Theorem 21.1 to obtain an asymptotic formula for $S(v)$. Furthermore, if λ belongs to the sequence $\{\lambda_k\}$, then $S_\lambda(v)$ is nondecreasing, and one can obtain an asymptotic formula for $S_\lambda(v)$ from the relation

$$F_\lambda(z) = \int_{0-}^{\infty} e^{-zv} dS_\lambda(v) \sim z e^{f(z)}, \tag{23.8}$$

which holds uniformly for $z \to 0$ in angles as before.

Example 23.2. Before we prove the Theorem we verify that it implies formula (23.1). In the case of ordinary partitions one has $\lambda_k = k$, $k = 1, 2, \ldots$, so that $N(v) = [v]$; taking $A = \alpha = 1$ we get $R(v) = [v] - v$. Thus, using Stirling's formula on the way,

$$\begin{aligned}
Q(v) &= \int_0^v \frac{[w]-w}{w}\,dw = \int_{1-}^v [w]d\log w - v \\
&= [v]\log v - \int_{1-}^v (\log w)d[w] - v = [v]\log v - \sum_{1\le k\le [v]} \log k - v \\
&= [v]\log v - [v]\log[v] + [v] - v - \frac{1}{2}\log(2\pi[v]) + o(1) \\
&= [v]\log\left(1 + \frac{v-[v]}{[v]}\right) + [v] - v - \frac{1}{2}\log(2\pi[v]) + o(1) \\
&= -\frac{1}{2}\log v - \frac{1}{2}\log(2\pi) + o(1) \quad \text{as } v \to \infty. \qquad (23.9)
\end{aligned}$$

That is, we have (23.6) with $b = -1/2$, $c = -(1/2)\log(2\pi)$. By (23.7) and (23.8) this gives

$$\begin{aligned}
&F(z) \sim e^{f(z)}, \quad F_1(z) \sim ze^{f(z)} = e^{f_1(z)}, \quad \text{where} \\
&f_1(z) = f(z) + \log z \\
&\quad = \Gamma(2)\zeta(2)\frac{1}{z} + \frac{3}{2}\log z - \frac{1}{2}\log(2\pi). \qquad (23.10)
\end{aligned}$$

We can now apply Theorem 21.1 with $F_1(z)$ instead of $F(z)$ or $F(\zeta)$: since $\lambda = 1$ belongs to $\{\lambda_k\}$, the function $S_1(v)$ is nondecreasing. The other conditions of the Theorem are satisfied with $\delta(\xi) = \xi/2$. Note also that when we take v equal to a positive integer, $v = n$, say, then $p(n) = S(n) - S(n-1) = S_1(n)$. For the application of Theorem 21.1 we have to compute the inverse function $h(v)$ of $-f_1'(\xi)$. The equation

$$v = -f_1'(\xi) = (\pi^2/6)\xi^{-2} - (3/2)\xi^{-1}$$

gives

$$\begin{aligned}
1/\xi &= \sqrt{6v}/\pi + 9/(2\pi^2) + o(1), \quad vh(v) = v\xi = \pi\sqrt{v/6} - (3/4) + o(1), \\
f_1(\xi) &= \pi\sqrt{v/6} - (3/4)\log v + (3/4) - (5/4)\log 2 \\
&\quad -(3/4)\log 3 + \log\pi + o(1).
\end{aligned}$$

From this one obtains the following expressions for the numerator and denominator of the fraction in (21.8) (using f_1 instead of f):

$$e^{2\pi\sqrt{v/6}}v^{-3/4}2^{-5/4}3^{-3/4}\pi\{1+o(1)\}, \qquad v^{1/4}2^{3/4}3^{-1/4}\pi\{1+o(1)\}.$$

Conclusion:

$$S_1(v) \sim e^{2\pi\sqrt{v/6}}v^{-1}2^{-2}3^{-1/2} \quad \text{as } v \to \infty,$$

which for $v = n$ gives the desired asymptotic formula (23.1) for $p(n) = S_1(n)$.

One can use the same method to estimate partitions in squares, etc.

Proof of Theorem 23.1. Observe first that by (23.6)

$$N(v) \le \int_v^{ev} \frac{N(w)}{w}dw = \int_v^{ev} Aw^{\alpha-1}dw + Q(ev) - Q(v) = \mathcal{O}(v^\alpha).$$

This inequality can be used to justify the basic formulas (23.4), (23.5) for $x = \operatorname{Re} z > 0$. By (23.5) and (23.6)

$$\begin{aligned}\log F(z) &= \int_0^\infty \frac{z}{e^{zv}-1}\{Av^\alpha dv + v\,dQ(v)\}\\ &= \frac{A}{z^\alpha}\int_0^\infty \frac{(zv)^\alpha}{e^{zv}-1} d_v(zv) - \int_0^\infty Q(v)d_v\frac{zv}{e^{zv}-1}\\ &= A\Gamma(\alpha+1)\zeta(\alpha+1)z^{-\alpha} - \int_0^\infty (b\log v + c)\omega'(zv)z\,dv\\ &\qquad - \int_0^\infty \rho(v)\omega'(zv)z\,dv, \end{aligned} \tag{23.11}$$

where $\omega(z) = z/(e^z-1)$, and $\rho(v)$ is continuous, $o(1)$ for $v \to \infty$ and $\mathcal{O}(|\log v|)$ for $v \searrow 0$.

The first integral in the final member of (23.11) can be evaluated explicitly:

$$\int_0^\infty (b\log v + c)\omega'(zv)z\,dv = \lim_{\varepsilon\searrow 0}\int_\varepsilon^\infty \cdots\cdots \tag{23.12}$$

$$= \lim_{\varepsilon\searrow 0}\left[(b\log v + c)\frac{zv}{e^{zv}-1} - b\log(1-e^{-zv})\right]_{v=\varepsilon}^\infty = b\log z - c.$$

To deal with the final integral in (23.11) one may observe that $e^{z/2}\omega'(z)$ and $|z|/x$ are bounded in any angle $|\arg z| \le \beta < \pi/2$. Thus

$$\left|\int_0^\infty \rho(v)\omega'(zv)z\,dv\right| \tag{23.13}$$

$$\le c_1\int_0^{1/\sqrt{x}} (|\log v|+1)x\,dv + o(1)\int_{1/\sqrt{x}}^\infty e^{-xv/2}x\,dv = o(1)$$

as $z = x + iy \to 0$ in such an angle.

Conclusion from (23.11)–(23.13):

$$\log F(z) = A\Gamma(\alpha+1)\zeta(\alpha+1)z^{-\alpha} - b\log z + c + o(1), \tag{23.14}$$

uniformly for $z \to 0$ in any angle $|\arg z| \le \beta < \pi/2$. This establishes (23.7) and will also prove (23.8). Indeed, since $S(v) = 0$ for $v < 0$,

$$\begin{aligned}F_\lambda(z) &= \frac{1}{\lambda}\left\{\int_{0-}^\infty e^{-zv}dS(v) - e^{-\lambda z}\int_{\lambda-}^\infty e^{-z(v-\lambda)}dS(v-\lambda)\right\}\\ &= \frac{1-e^{-\lambda z}}{\lambda}F(z) \sim zF(z). \end{aligned} \tag{23.15}$$

We finally take $\lambda = \lambda_j$. Then by (23.4)

$$F_\lambda(z) = \frac{1-e^{-\lambda z}}{\lambda}F(z) = \frac{1}{\lambda}\prod_{k\ne j}(1-e^{-\lambda_k z})^{-1}. \tag{23.16}$$

Expansion shows that the coefficients in the Dirichlet series for $F_\lambda(z)$ are positive, so that $S_\lambda(v)$ is nondecreasing. This completes the proof of the Theorem.

24 Two-Sided Laplace Transforms

We finally state some theorems of a different character, which involve transforms of finite measures on $\mathbb{R}$. The first result, due to Stef and Tenenbaum [2001], is a Laplace transform analog of the Berry–Esseen inequality. Of the other results, several are due to Balkema and coauthors; see the references below.

The Berry–Esseen Inequality of probability theory involves Fourier transforms; cf. Berry [1941], Esseen [1945]. Let S and U be arbitrary *distribution functions* on $\mathbb{R}$: nondecreasing functions with $S(-\infty+) = U(-\infty+) = 0$, $S(\infty-) = U(\infty-) = 1$. The Berry–Esseen inequality majorizes the supremum norm $\|S-U\|_\infty$ in terms of the difference of the so-called *characteristic functions* $\mathcal{F}dS$, $\mathcal{F}dU$ and a measure for the smoothness of U. The function U, which might correspond to the normal distribution, is usually supposed to have a bounded derivative. However, one can also use the 'concentration function',

$$Q_U(\delta) = \sup_v \{U(v+\delta) - U(v)\} \qquad (\delta > 0). \tag{24.1}$$

One then has a uniform estimate

$$\|S-U\|_\infty \le C\left\{Q_U\left(\frac{1}{\lambda}\right) + \int_{-\lambda}^{\lambda} |\mathcal{F}dS(t) - \mathcal{F}dU(t)|\,\frac{dt}{|t|}\right\}, \qquad \forall\lambda > 0.$$

See for example Faĭnleĭb [1968], Elliott [1979], Vaaler [1985], Postnikov [1988], and Tenenbaum [1995]. A proof will be given in Section VII.15.

Stef and Tenenbaum obtained a corresponding result in terms of the two-sided Laplace transforms

$$\mathcal{L}dS(\xi) = \int_{\mathbb{R}} e^{-\xi v}dS(v), \quad \mathcal{L}dU(\xi) = \int_{\mathbb{R}} e^{-\xi v}dU(v). \tag{24.2}$$

The estimate involves a continuous nondecreasing auxiliary function H on $\mathbb{R}^+$ which satisfies a condition of the form

$$H(\xi) \ge c_1\xi^4 \text{ with } c_1 > 0, \quad \forall\xi \ge 0. \tag{24.3}$$

In practice one may often take $H(\xi) = \mathcal{L}dU(\xi) + \mathcal{L}dU(-\xi)$.

Theorem 24.1. *Let S and U be distribution functions and let H be a continuous nondecreasing function on $\mathbb{R}^+$ which satisfies condition* (24.3). *Let ε, κ, λ be real numbers, with* $0 < \varepsilon < 1/\{3 + H(2)\}$, $0 < \kappa < \lambda$, *such that*

$$\begin{aligned} &|\mathcal{L}dS(\xi) - \mathcal{L}dU(\xi)| \le \varepsilon \quad \textit{for } 0 \le \xi \le \kappa, \\ &\mathcal{L}dS(\xi) + \mathcal{L}dU(\xi) \le c_2H(|\xi|) \quad \textit{for } -\lambda \le \xi \le \lambda. \end{aligned} \tag{24.4}$$

Then one has an inequality

$$\|S-U\|_\infty \le CQ_U\left(\frac{\log\lambda}{\lambda} + \frac{\log\mu}{\mu}\right), \tag{24.5}$$

where μ is any solution of the equation $H(\mu) = 1/\varepsilon$. The constant C depends only on κ, c_1 and c_2.

The proof is based on quantitative L^1 approximation by polynomials of the kind considered in Chapter VII. Stef and Tenenbaum applied their result to number-theoretic questions.

Finally we give a brief description of recent work motivated by probability theory. Let $dS(v)$ be a finite positive measure on $\mathbb{R}$ with *log-concave density* $s(v)$:

$$dS(v) = s(v)dv = e^{-\rho(v)}dv, \quad \text{with } \rho(v) \text{ convex.} \tag{24.6}$$

We also need the 'conjugate convex function'

$$\rho^*(\xi) \stackrel{\text{def}}{=} \sup_v \{-\rho(v) - \xi v\}; \tag{24.7}$$

cf. Balkema, Geluk and de Haan [1979]. Under appropriate conditions, there is a very close connection between the asymptotic behavior of $dS(v)$ or $s(v)$ and that of the two-sided transform $\mathcal{L}s(\xi)$ for just real ξ. It is assumed that the Laplace transform $\mathcal{L}s(\xi)$ exists on some nonempty open interval. The transform is log-convex:

$$F(\xi) = \mathcal{L}s(\xi) = e^{f(\xi)}, \quad \text{with } f(\xi) \text{ convex.} \tag{24.8}$$

Indeed, $f'' = \{FF'' - (F')^2\}/F^2 \geq 0$ by Cauchy–Schwarz, applied to the integral for F'. The results below involve the right-hand end point of the support of s or dS:

$$e_R = e_R(dS) = \sup\{v : dS[v, \infty) > 0\}, \tag{24.9}$$

(which may be ∞), and the left-hand end point of the interval of existence of $\mathcal{L}s$:

$$\hat{e}_L = \inf\{\xi : \mathcal{L}s(\xi) < \infty\}. \tag{24.10}$$

We begin with an Abelian theorem of Balkema, Klüppelberg and Resnick [1993], omitting an innocuous 'flat factor'.

Theorem 24.2. *Let $s(v) = e^{-\rho(v)}$ be log-concave and let its two-sided Laplace transform have a nondegenerate interval of existence. Suppose that $\sigma = 1/\sqrt{\rho''}$ is self-neglecting:*

$$\sigma\{v + x\sigma(v)\}/\sigma(v) \to 1 \quad \text{as } v \nearrow e_R, \tag{24.11}$$

locally uniformly in x. Then for $\xi \searrow \hat{e}_L$ or $v \nearrow e_R$,

$$\mathcal{L}s(\xi) \sim \sqrt{2\pi}\sigma(v)e^{-\xi v}s(v) \sim \sqrt{2\pi}\sigma(v)e^{\rho^*(\xi)}, \tag{24.12}$$

where $\xi = -\rho'(v)$, $v = -(\rho^*)'(\xi)$.

Example: $\rho(v) = v^2$. Balkema et al. (see also the references below) proved a number of related Tauberian results. To show the flavor we state an earlier result of Feigin and Yashchin [1983].

Theorem 24.3. *Let $s(v) = e^{-\rho(v)}$ be log-concave and let its two-sided Laplace transform $\mathcal{L}s(\xi) = e^{f(\xi)}$ (which is log-convex) have a nondegenerate interval of existence with $\hat{e}_L \leq 0$. Suppose that $\phi(\xi) = 1/\sqrt{f''(\xi)}$ is self-neglecting for $\xi \searrow \hat{e}_L$. Then for $v \nearrow e_R$,*

$$s(v) \sim \frac{e^{\xi v + f(\xi)}}{\sqrt{2\pi f''(\xi)}}, \qquad \xi = h(v),$$

where h is the inverse of $-f'$ as in Theorem 21.1.

It turns out that log-concavity of densities is a Tauberian condition for Laplace transforms. We quote a result of Balkema [2002]:

Theorem 24.4. *Let dS_j, $j = 1, 2$ be finite positive measures with log-concave densities s_j, $\mathcal{L}dS_j = \mathcal{L}s_j$. Then the following two statements are equivalent:*

(i) The measures dS_j have the same right-hand end point e_R and $s_1(v) \sim s_2(v)$ for $v \nearrow e_R$;

(ii) The Laplace transforms $\mathcal{L}dS_j$ have the same left-hand end point $\hat{e}_L$ and $\mathcal{L}dS_1(\xi) \sim \mathcal{L}dS_2(\xi)$ for $\xi \searrow \hat{e}_L$.

Additional references are Balkema, Geluk and de Haan [1995], Balkema, Klüppelberg and Resnick [1999], [2003], Balkema, Klüppelberg and Stadtmüller [1995]; cf. also Berg [1960], Stadtmüller [1993], and Báez-Duarte [1995].

V

Extensions of the Classical Theory

1 Introduction

This chapter deals with four different topics.

Our first subject is the algebraic approach to Wiener's theory (Sections 2–9). After some preliminaries on Banach algebras we present an algebraic form of Wiener's Approximation Theorem II.8.3. The subsequent treatment in the context of *Banach algebras* makes it natural to include Beurling's extension [1938] of the theorem to weighted L^1 spaces. Our discussion includes the necessary parts of Gelfand's theory [1939], [1941a] of maximal ideals, complemented by some of Shilov's results [1940], [1947] on so-called minimal ideals.

Sections 10–13 treat an extension of Wiener's theory to the case of *rapidly decreasing kernels*. It is due to Pitt [1938a], [1958] and plays an important role in Chapter VI. Let $K \in L^1(\mathbb{R})$ be a Wiener kernel, so that $\hat{K}(t) \neq 0$ for all real t, and let S be such that the convolution

$$\begin{aligned} K * S(x) = \int_{\mathbb{R}} K(x-y)S(y)dy \quad \text{exists} \\ \text{and tends to } A \int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty. \end{aligned} \tag{1.1}$$

In the Wiener–Pitt Theorem II.8.4 one obtains convergence of $S(x)$ to A from (1.1) under the condition that S be bounded and satisfy an appropriate Tauberian condition. In the case of rapidly decreasing kernels such as $K(x) = e^{-x^2}$, one need not postulate that S is bounded: the *boundedness* can be derived from a condition of at most exponential growth. At the same time, the function $K(x-y)$ may be replaced by a function $J(x, y)$ of more general type, but such a function must be well-approximated by a suitable 'difference kernel'. Important examples are provided by the kernels for Borel summability and other circle methods; see Section VI.16.

In Sections 14–21 we discuss a *functional-analytic method* to reduce the general case of a Tauberian theorem to the simpler case of bounded functions, or in this case, sequences. The method goes back to the Polish school of functional analysis;

for its implementation, an appropriate theory of so-called *FK-spaces* was developed by Wilansky, Zeller and others. Cf. the books by Zeller and Beekmann [1958/70], Wilansky [1984], and Boos [2000]. In the hands of Meyer-König and Zeller, the technique turned out to be effective in the treatment of Tauberian theorems for lacunary series.

The final Sections 22–26 are devoted to some *striking Tauberians* of different character. The first theorem, due to Erdős, Feller and Pollard [1949], was inspired by renewal theory; cf. Feller [1950/68] (chapter 13). The second result is an unusual Tauberian theorem due to Milin [1970], [1971]. It is important in the theory of univalent functions; cf. Duren [1983].

2 Preliminaries on Banach Algebras

For an algebraic formulation of Wiener's theorem we need some simple notions from the theory of (complex) Banach algebras. The basics can be found in many books, for example Rudin [1966/87] or [1973/91]. Additional references will be given in Section 5.

A BANACH ALGEBRA A is a complex Banach space in which any two elements can be multiplied. The multiplication must satisfy the usual associative and distributive laws, and it is customary to require that the norm of the product xy satisfy the inequality

$$\|xy\| \leq \|x\|\,\|y\|.$$

We only consider *commutative* Banach algebras: $xy = yx$ for all $x, y \in A$. A Banach algebra may or may not have a *unit element*, that is, an element e such that $ex = xe = x$ for all $x \in A$. If there is a unit e, we will require that $\|e\| = 1$.

Examples 2.1. The 'Wiener Algebra' A_W consists of the continuous functions f of period 2π with absolutely convergent Fourier series,

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{int}, \quad \text{with norm } \|f\| = \sum_{-\infty}^{\infty} |a_n| < \infty.$$

The product fg is defined in the ordinary way. Hence if g has Fourier coefficients b_n, then

$$f(t)g(t) = \sum_m a_m e^{imt} \sum_k b_k e^{ikt} = \sum_n \left(\sum_k a_{n-k} b_k \right) e^{int},$$

$$\|fg\| = \sum_n \left| \sum_k a_{n-k} b_k \right| \leq \sum_k \left(\sum_n |a_{n-k}| \right) |b_k| = \|f\|\,\|g\|.$$

The function $e \equiv 1$ serves as unit.

A more important example is the *convolution algebra* $L^1(\mathbb{R})$, that is, the normed space $L^1(\mathbb{R})$, furnished with the product given by convolution:

$$f * g(t) = \int_{\mathbb{R}} f(t-u)g(u)du; \tag{2.1}$$

cf. Section II.9. This algebra has no unit. Indeed, if $L^1(\mathbb{R})$ had a unit e, Fourier transformation of the relation $e * f = f$ would show that $\hat{e} \equiv 1$, and this would contradict the Riemann–Lebesgue lemma. (In the distributional theory of Fourier transformation, the Dirac measure δ plays the role of a unit, but it is not in L^1.)

An IDEAL I in A is a subalgebra with the property that xy is in I whenever x is in I and y is in A. The closure $\overline{I}$ of an ideal I is also an ideal.

A MAXIMAL IDEAL I is a proper ideal (that is, $I \neq A$) which is not contained in a larger proper ideal. As a linear subspace of A it must have codimension 1.

If A does not have a unit element one can adjoin one; cf. Section 6. In the remainder of this section we assume that our Banach algebra A has a unit e. An element $x \in A$ is called *invertible* if it has a multiplicative inverse. For example, if $\|x\| < 1$, then $e - x$ is invertible; the inverse is given by the sum of the geometric series $e + x + x^2 + \cdots$. The invertible elements of A form an open set: if x is invertible and $\|y\| < 1/\|x^{-1}\|$, then $z = x^{-1}y$ has norm less than 1 and hence $x + y = x(e+z)$ is invertible.

A proper ideal I cannot contain e or any other invertible element.

Proposition 2.2. *In a commutative Banach algebra A with unit, every proper ideal I is contained in a maximal ideal M, and every maximal ideal is closed.*

Proof. Proper ideals may be (partially) ordered by inclusion. By Hausdorff's maximality theorem or Zorn's lemma, there is a maximal ascending chain of proper ideals starting with any proper ideal I. The union M of the ideals in such a totally ordered family is itself an ideal. M is proper because it does not contain e, and M is maximal by the maximality of the chain.

Finally, a maximal ideal M does not contain an invertible element and since the set of the invertible elements is open, $\overline{M}$ does not contain an invertible element either. Thus $\overline{M}$ is a proper ideal, hence $M = \overline{M}$ because M was maximal.

Additional results on maximal ideals will be discussed in Section 5.

3 Algebraic Form of Wiener's Theorem

Henceforth we consider $L^1 = L^1(\mathbb{R})$ as an algebra under convolution.

For any number $\alpha \in \mathbb{R}$, the functions $f \in L^1$ whose Fourier transform $\hat{f}$ vanishes at α form a closed ideal M_α. This ideal is maximal. Indeed, let I be any ideal which contains M_α as a proper subset. Choose $g \in I$ with $\hat{g}(\alpha) \neq 0$ and let h be *any* function in L^1. Then the difference $f = h - \{\hat{h}(\alpha)/\hat{g}(\alpha)\}g$ is in M_α. It follows that h, as a linear combination of f and g, is in I. Thus $I = L^1$, hence M_α is a maximal ideal.

We will see below that *all* closed maximal ideals in L^1 are of the form M_α for some $\alpha \in \mathbb{R}$. An ideal which is not contained in one of the maximal ideals M_α is said to 'belong to ∞'. An example is given by the ideal J of the L^1 functions whose Fourier

transform has compact support. One can show that J is contained in every ideal which belongs to ∞; it is the so-called *minimal* ideal at ∞ (Section 8). This property can be used to give a Banach algebra proof for Wiener's theorem; see Sections 7, 8.

Definition 3.1. For a family $\mathcal{F}$ of L^1 functions, we let $Z(\hat{\mathcal{F}})$ denote the zero set of $\hat{\mathcal{F}}$, that is, the set of real numbers where the Fourier transform of every function in $\mathcal{F}$ is equal to zero.

For every nonempty compact set $K \subset \mathbb{R}$, there is a unique closed ideal $I = I(K)$ in L^1 for which $Z(\hat{I}) = K$. It consists of the functions whose Fourier transform vanishes on K.

Proposition 3.2. *The closed ideals I in L^1 coincide with the closed linear subspaces of L^1 that are invariant under translation.*

Proof. (i) Let Y be a closed translation invariant subspace of L^1 and let f be any function in Y. Then Y contains every finite linear combination $\sum c_k f(t-\lambda_k)$. For $g \in L^1$ and large B, the integral $\int_{|u|>B} |g(u)|du$ is small, so that also the norm

$$\begin{aligned} &\left\| f * g(t) - \int_{-B}^{B} f(t-u)g(u)du \right\| \\ &= \int_{\mathbb{R}} \left| \int_{|u|>B} f(t-u)g(u)du \right| dt \le \int_{|u|>B} |g(u)|du \cdot \|f\| \end{aligned} \tag{3.1}$$

is small. For small $|v|$, the number

$$\rho(v) = \rho(f, v) = \int_{\mathbb{R}} |f(t-v) - f(t)|dt \tag{3.2}$$

is small. Thus for a partitioning $-B = u_0 < u_1 < \cdots < u_n = B$ with small differences $u_k - u_{k-1}$, the norm

$$\begin{aligned} &\left\| \int_{-B}^{B} f(t-u)g(u)du - \sum_{k=1}^{n} f(t-u_k) \int_{u_{k-1}}^{u_k} g(u)du \right\| \\ &= \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \int_{u_{k-1}}^{u_k} \{f(t-u) - f(t-u_k)\} g(u)du \right| dt \\ &\le \sum_{k=1}^{n} \int_{u_{k-1}}^{u_k} |g(u)|\rho(u-u_k)du \end{aligned} \tag{3.3}$$

is small. Since the sum in the first member of this array is in Y, it follows from (3.1) and (3.3) that the convolution $f * g$ is in $\overline{Y} = Y$. Hence Y is an ideal.

(ii) Let $I \subset L^1$ be a closed ideal and let f be in I. For $a < b$, let $\chi_{[a,b]}$ denote the characteristic function of $[a, b]$. As $\varepsilon \searrow 0$,

$$f * \frac{1}{\varepsilon}\chi_{[a,a+\varepsilon]}(t) - f(t-a) = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} \{f(t-u) - f(t-a)\}du \to 0 \tag{3.4}$$

in L^1; cf. (3.2). The convolution on the left is in I. Thus the translate $f(t-a)$ is the L^1 limit of functions in I, hence itself in $\overline{I} = I$. It follows that I is translation invariant. □

In view of Definition 3.1, Wiener's approximation theorem for families $\mathcal{F}$ of L^1 functions may be stated as follows. The finite linear combinations of translates of functions in $\mathcal{F}$ are dense in L^1 if and only if the zero set $Z(\hat{\mathcal{F}})$ is empty; cf. Theorem II.8.3 and Section II.10. Combining this form of the result with Proposition 3.2, one obtains the following algebraic form of Wiener's theorem. It can be found in many books that deal with Banach algebras. Here we mention only the simple treatments in Goldberg [1961] and van de Lune [1986].

Theorem 3.3. *Let I be a closed ideal in L^1. Then $I = L^1$ if and only if $Z(\hat{I}) = \emptyset$.*

Corollary 3.4. *Every closed maximal ideal $I \subset L^1$ is of the form M_α with $\alpha \in \mathbb{R}$.*

Indeed, if $Z(\hat{I})$ would be empty, I would coincide with L^1, so it could not be a maximal ideal. Thus $Z(\hat{I})$ must contain a point α and then $I \subset M_\alpha$. By its maximality, I must coincide with M_α.

Remarks 3.5. Esterle [1980] has given an interesting proof for the 'if' part of Theorem 3.3. Using complex analysis, he found that any closed ideal I for which $Z(\hat{I})$ is empty must contain the functions of the approximate identity, given by the fundamental solution of the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$,

$$\delta_t(x) = \frac{1}{\sqrt{\pi t}} e^{-x^2/t} \quad \text{with } t \searrow 0.$$

Then for any $f \in L^1$, the convolutions $\delta_t * f$ are in I, so that also $f = \lim_{t\searrow 0} \delta_t * f$ is in I.

In the form of Theorem 3.3, Wiener's theorem has extensions to locally compact groups; cf. I.E. Segal [1947], Godement [1947], Rudin [1962/90], Eymard [1964], Hewitt and Ross [1970]. Several authors have investigated the closed span of the translates of the functions in $\mathcal{F}$ in cases where $Z(\hat{\mathcal{F}}) \neq \emptyset$; cf. Rudin.

We will see below that Theorem 3.3 extends to a large class of weighted L^1 spaces.

4 Weighted L^1 Spaces

In the following we discuss Beurling's generalization of Wiener's theorem to weighted L^1 spaces. Here the 'standard' *weight functions* $\omega(\cdot)$ are required to be positive measurable functions on $\mathbb{R}$, whose logarithms are subadditive:

$$\omega(t+u) \le \omega(t)\omega(u), \quad \forall t,\, u \in \mathbb{R}; \tag{4.1}$$

one may take $\omega(0) = 1$. The measurability implies that the weight functions are locally bounded; cf. Hille and Phillips [1957/74] (section 7.4). We consider the Banach space $L_\omega = L^1_\omega$ of the functions f on $\mathbb{R}$ such that

$$\|f\| = \|f\|_{1,\omega} = \int_{\mathbb{R}} |f(t)|\omega(t)dt < \infty. \tag{4.2}$$

The dual space L^∞_ω, of the continuous linear functionals on L_ω, is given by the (equivalence classes of the) functions ϕ for which

$$\|\phi\|_\infty = \|\phi\|_{\infty,\omega} = \text{ess sup}\,\frac{|\phi(t)|}{\omega(t)} < \infty.$$

With convolution as multiplication, L_ω becomes a Banach algebra, a so-called *Beurling algebra*. In fact, by (4.1)

$$\begin{aligned}\|f * g\| &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t-u)g(u)du \right| \omega(t)dt \\ &\leq \int_{\mathbb{R}^2} |f(t-u)g(u)|\omega(t-u)\omega(u)dtdu = \|f\|\|g\|.\end{aligned}$$

Condition (4.1) also implies the following relations for the growth indices a and b of ω; cf. Hille and Phillips:

$$a \stackrel{\text{def}}{=} \sup_{t<0} \frac{\log\omega(t)}{t} = \lim_{u\to-\infty} \frac{\log\omega(u)}{u}, \quad b \stackrel{\text{def}}{=} \inf_{t>0} \frac{\log\omega(t)}{t} = \lim_{u\to\infty} \frac{\log\omega(u)}{u}, \tag{4.3}$$

with $-\infty < a \leq b < \infty$. To prove the second part of (4.3), say, fix any $t > 0$ and set $u = nt + v$ with $n \in \mathbb{N}$ and $0 \leq v < t$. Then let u go to ∞ to show that

$$\limsup \frac{\log\omega(u)}{u} \leq \limsup \frac{n\log\omega(t) + \log\omega(v)}{nt+v} = \frac{\log\omega(t)}{t}.$$

It follows that $\limsup u^{-1}\log\omega(u) \leq b < \infty$. One may finally let t go to ∞ to conclude that

$$\limsup u^{-1}\log\omega(u) \leq \liminf t^{-1}\log\omega(t).$$

Thus $u^{-1}\log\omega(u)$ tends to a limit as $u \to \infty$. Since the limit is $\leq b$, it must be equal to b. The inequality $a \leq b$ follows from (4.1):

$$0 = \frac{\log\omega(0)}{u} \leq \frac{\log\omega(u) + \log\omega(-u)}{u} \to b - a \quad \text{as } u \to \infty.$$

We now introduce the *complex Fourier transform* of $f \in L_\omega$:

$$\hat{f}(z) = \int_{\mathbb{R}} f(t)e^{-izt}dt, \quad z = x + iy. \tag{4.4}$$

(In this section, x and y are real numbers, of course.) By (4.3), $e^{bt} \leq \omega(t)$ for $t > 0$ and $e^{at} \leq \omega(t)$ for $t < 0$. Hence if $y \leq b$ and $t > 0$,

$$|f(t)e^{-izt}| = |f(t)|e^{yt} \leq |f(t)|\omega(t).$$

Similarly for $y \geq a$ and $t < 0$. Thus it follows from (4.2) that the transform $\hat{f}(z)$ exists and is continuous throughout the closed strip

$$\Sigma = \{z = x + iy : x \in \mathbb{R},\ a \leq y \leq b\}. \tag{4.5}$$

The transform is analytic in the (possibly empty) interior Σ^0 of the strip.

Approximation Problem. Let $\mathcal{F}$ be a family of functions f_ν in L_ω. Under what conditions are the linear combinations of the translates of the functions f_ν dense in L_ω? Since Proposition 3.2 readily extends to the case of L_ω, one may also ask for conditions under which a closed ideal $I \subset L_\omega$ coincides with L_ω. Beurling's work [1938] implied the following answer.

Theorem 4.1. (Beurling's Approximation Theorem) *The analog of Wiener's Approximation Theorem 3.3 holds for L_ω whenever the weight function ω satisfies the condition of 'nonquasi-analyticity',*

$$\int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2} dt < \infty. \tag{4.6}$$

Under condition (4.6), the formulas (4.3) imply $a = b = 0$, so that the complex Fourier transform (4.4) becomes the ordinary Fourier transform.

The proof of Theorem 4.1 given below will use important results on Banach algebras due to Gelfand and Shilov. The reader might consider the following *open question*: Can Beurling's theorem be obtained also by appropriate extension of the distributional proof for Wiener's theorem in Section II.11?

5 Gelfand's Theory of Maximal Ideals

In the following A denotes a commutative (complex) Banach algebra with unit element e. We discuss the principal facts of Gelfand's theory for such algebras. Many books can serve as supplementary references. Besides Gelfand, Raikov and Shilov [1964/01], we mention Loomis [1953], Naimark [1956/72], Hille and Phillips [1957/74], Rickart [1960], Dunford and Schwartz [1963/88], Katznelson [1968/76], Rudin [1973/91], Palmer [1994], and Dales [2000].

Theorem 5.1. (Gelfand–Mazur) *For any maximal ideal M in A, the quotient ring $A' = A/M$ becomes a normed field under the coset norm*

$$\|X\| = \inf\{\|x\| : x \in X\}. \tag{5.1}$$

This (as well as any) complex normed field is isometrically isomorphic to the normed field $\mathbb{C}$ of the complex numbers.

Proof. Let I be a proper closed ideal in A. The quotient space A/I consists of the cosets X of I. Symbolically, if $x \in X$ then $X = x + I$. The quotient is a ring under the usual rules for addition and multiplication modulo I. With definition (5.1) for $\|X\|$, the ring A/I becomes a Banach algebra. Indeed, one will have completeness, and for the product coset XY, determined by the products xy with $x \in X$, $y \in Y$, one will have the inequality $\|XY\| \le \|X\|\|Y\|$. In the coset algebra, I is the zero element, O. The coset E containing e is unit element in A/I and has norm 1.

From here on, let $I = M$ be a *maximal ideal* in A. Then the quotient algebra $A' = A/M$ is a field. Indeed, take any $x \in A$ which is not in M. Let J_x be the set of all elements of the form

$$xy + m \quad \text{with } y \in A,\ m \in M.$$

Then J_x is an ideal which contains M as a proper subset (J_x contains x). Hence $J_x = A$, and thus e can be represented in the form $e = xy + m$. Passing to cosets of M, it follows that for any given $X \in A'$, the equation $XY = E$ has a solution $Y \in A'$.

We finally have to show that the elements X of the field $A' = A/M$ are in one-to-one correspondence with the complex numbers. To $\lambda \in \mathbb{C}$ there is a unique coset $X = \lambda E$. But can every element of A' be represented in the form λE with $\lambda \in \mathbb{C}$? Suppose there is an $X \in A'$ such that $X - \lambda E \neq M = O$ for all complex numbers λ. Then $(X - \lambda E)^{-1}$ exists for every λ. For λ close to λ_0 one can expand $(X - \lambda E)^{-1}$ as a convergent power series in $\lambda - \lambda_0$ with coefficients in A'. Also, $(X - \lambda E)^{-1} = -\lambda^{-1}(E - X/\lambda)^{-1} \to O$ as $\lambda \to \infty$. Now let l be any continuous linear functional on the Banach space A'. Then

$$f(\lambda) = l\{(X - \lambda E)^{-1}\}$$

is locally given by convergent power series, hence $f(\lambda)$ is an entire analytic function of λ. This function tends to 0 as $\lambda \to \infty$, so that $f \equiv 0$ by Liouville's theorem. Since this holds for every l one would have $(X - \lambda E)^{-1} = O$, which is impossible. One concludes that there is a one-to-one correspondence between the elements of the field $A' = A/M$ and the complex numbers λ. It is easy to verify that this correspondence is a norm preserving isomorphism.

Homomorphisms. For our Banach algebra A with identity, we consider the family of all *homomorphisms* h onto the complex numbers. Here 'homomorphism' means: a continuous linear functional which is also multiplicative. By the preceding, every maximal ideal M in A generates an isometric isomorphism between the field $A' = A/M$ and the complex field $\mathbb{C}$: the elements X of A' are the complex multiples $X = \lambda E$. Thus M determines a homomorphism h_M of A onto $\mathbb{C}$ by the rule that for $x \in X = \lambda E$, one sets $h_M(x) = \lambda$. Conversely, the kernel or 'null space' N of any nonzero homomorphism h of A onto $\mathbb{C}$ is a maximal ideal M_h in A. Indeed, the kernel N is an ideal because $h(x + y) = 0$ whenever $h(x) = h(y) = 0$, while $h(xy) = 0$ whenever $h(x) = 0$. Furthermore N is a maximal ideal, because N has codimension 1 as a closed linear subspace of A.

The correspondence $h \Leftrightarrow M_h$ between a homomorphism and its kernel identifies the set Δ of the homomorphisms with the set $\mathcal{M}$ of the maximal ideals M in A. We will show that the linear functional $h = h_M$ has norm ≤ 1:

$$|h_M(x)| \leq \|x\|, \quad \forall x \in A. \tag{5.2}$$

Suppose to the contrary that $h(x) = \lambda$ with $|\lambda| > \|x\|$ for some $x \in A$. Then $\|x/\lambda\| < 1$, so that $e - x/\lambda$ has an inverse given by the usual geometric series. However, $h(e - x/\lambda) = 1 - h(x/\lambda) = 0$, which gives a contradiction. Since $h_M(e) = 1$, the norm of h_M will actually be equal to 1.

Corollary 5.2. *An element $x \in A$ is invertible if and only if $h(x) \neq 0$ for every homomorphism $h \in \Delta$.*

Indeed, if x is invertible then $h(x)h(x^{-1}) = h(xx^{-1}) = h(e) = 1$, so that $h(x) \neq 0$. If x is not invertible, the set $\{xy,\ y \in A\}$ does not contain e, hence it is a

proper ideal $J \subset A$. This J belongs to a maximal ideal M. The latter is annihilated by the homomorphism $h_M \in \Delta$, so that $h_M(x) = 0$.

APPLICATION. As an application we give Gelfand's proof [1941c] of 'Wiener's Lemma' [1932] on absolutely convergent Fourier series:

Theorem 5.3. *Let f belong to the Wiener Algebra $A = A_W$:*

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{int}, \quad \textit{with} \quad \|f\| = \sum_{-\infty}^{\infty} |a_n| < \infty.$$

Suppose that $f(t) \neq 0$ for all (real) t. Then $1/f$ is also in A:

$$\frac{1}{f(t)} = \sum_{-\infty}^{\infty} b_n e^{int} \quad \textit{with} \quad \sum_{-\infty}^{\infty} |b_n| < \infty.$$

Proof. We saw in Examples 2.1 that A is a commutative Banach algebra, with unit $e \equiv 1$. For our given f and any real number α, the map $h_\alpha(f) = f(\alpha)$ is a complex homomorphism of A. We will show that every nonzero complex homomorphism h of A takes the function f into one of its values. Let us write v for the function e^{it} in A. Then v is invertible and $\|v\|_A = \|v^{-1}\|_A = 1$. Thus by (5.2), $|h(v)| \leq 1$ and $|h(v^{-1})| \leq 1$, so that $|h(v)| = 1$. It follows that $h(v) = e^{i\alpha}$ for a certain point $e^{i\alpha}$ on the unit circle $C(0, 1)$. Hence $h(v^n) = \{h(v)\}^n = e^{in\alpha}$ for all $n \in \mathbb{Z}$, and

$$h(f) = \sum_{-\infty}^{\infty} a_n h(v^n) = \sum_{-\infty}^{\infty} a_n e^{in\alpha} = f(\alpha).$$

Since f does not take the value 0, Corollary 5.2 shows that f is invertible, that is, $1/f$ belongs to A. □

Incidentally, there is a nice semi-classical proof of the Theorem in Newman [1975]. Wiener's Lemma will be put to good use in Section 23.

GELFAND REPRESENTATION. Let h_M be the homomorphism of A whose kernel is the maximal ideal M. Recall that the number $h_M(x)$ is obtained as follows. Let E_M be the unit element of the field A/M and let X be the coset in A/M which contains x. Then $h_M(x)$ is the complex number λ_x such that $X = \lambda_x E_M$.

If x is held fixed and M varies, $h_M(x)$ defines a function $\hat{x}$ on the set $\mathcal{M}$ of all maximal ideals by the formula

$$\hat{x}(M) = h_M(x), \quad M \in \mathcal{M}. \tag{5.3}$$

The function $\hat{x}$ is sometimes called the *Gelfand transform* of x.

It is customary to introduce a certain topology on $\mathcal{M}$ which makes the functions $\hat{x}$ continuous, and $\mathcal{M}$ a compact Hausdorff space. The space $\mathcal{M}$ then becomes the so-called *maximal ideal space* associated with A. In the case of the Wiener algebra, the maximal ideal space can be identified with the circle $C(0, 1)$.

6 Application to the Banach Algebra $A_\omega = (L_\omega, \mathbb{C})$

If a Banach algebra does not have a unit element one can *adjoin* one. We describe the procedure for the case of the algebra $L_\omega = L^1_\omega$ of Section 4, but the construction is quite general.

L_ω can be extended to the algebra $A = A_\omega = (L_\omega, \mathbb{C})$, whose elements have the form $F = (f, \lambda)$, with $f \in L_\omega$ and $\lambda \in \mathbb{C}$. Addition and multiplication by scalars are carried out componentwise in A. The algebra L_ω is represented in A by the elements $(f, 0)$, while the elements $(0, \lambda)$ represent the field $\mathbb{C}$. Identifying $(f, 0)$ with f and $(0, \lambda)$ with λ, the general element $(f, \lambda) \in A$ can be written as the sum

$$(f, \lambda) = (f, 0) + (0, \lambda) = f + \lambda. \tag{6.1}$$

Multiplication or convolution in $A = A_\omega$ is defined in accordance with this formula:

$$(f, \lambda)(g, \mu) = f * g + \lambda g + f\mu + \lambda\mu = (f * g + \lambda g + \mu f, \lambda\mu), \tag{6.2}$$

where $f * g$ is formed in L_ω. The element $(0, 1)$ acts as *unit element* e. Finally, A_ω is made into a Banach algebra by setting

$$\|F\| = \|(f, \lambda)\| = \|f\| + |\lambda|. \tag{6.3}$$

We will see that in the case of the Banach algebra $A = A_\omega = (L_\omega, \mathbb{C})$, the Gelfand transform of $F = (f, \lambda)$ is related to the complex Fourier transform of f. Let M be a maximal ideal in A_ω. By formulas (5.3) and (6.1),

$$\hat{F}(M) = h_M(F) = h_M\{(f, \lambda)\} = h_M(f + \lambda e) = h_M(f) + \lambda. \tag{6.4}$$

Observe first that L_ω is a maximal ideal in A_ω; it is the kernel of the homomorphism which maps the elements $F = (f, \lambda)$ onto λ. The following theorem characterizes the other maximal ideals.

Theorem 6.1. *Let M be any maximal ideal in A_ω different from L_ω, and let a and b be the indices associated with the weight function ω by formula* (4.3). *Then there is a complex number $z_M = x_M + iy_M$ with $a \le y_M \le b$ such that for all elements $F = (f, \lambda) \in A_\omega$,*

$$h_M(F) = \hat{f}(z_M) + \lambda = \int_{\mathbb{R}} f(t)e^{-iz_Mt}\,dt + \lambda. \tag{6.5}$$

M consists of those elements $F = (f, \lambda) \in A_\omega$ for which

$$F^*(z) \overset{\text{def}}{=} \hat{f}(z) + \lambda \quad \textit{is equal to } 0 \textit{ at } z = z_M. \tag{6.6}$$

Conversely, the elements $F = (f, \lambda) \in L_\omega$, whose transforms F^* satisfy an equation (6.6) *for some fixed number $z = x + iy$ with $a \le y \le b$, form a maximal ideal in A_ω.*

Since the maximal ideals different from L_ω are in one-to-one correspondence with the points $z = x + iy$ of the closed strip $a \le y \le b$, one may use the name 'Gelfand transform' also for F^*.

Proof of the Theorem. Because $M \ne L_\omega$, M must contain elements $F = (f, 0) = f$ with $h_M(f) \ne 0$. By (5.2), $|h_M(F)| \le \|F\| = \|f\| + |\lambda|$, so that by (6.4) with $\lambda = 0$, $|h_M(f)| \le \|f\|$. Thus h_M defines a continuous linear functional on the space L_ω of norm ≤ 1. Hence there is a function $\phi = \phi_M$ in the dual space L_ω^∞, with

$$\text{ess sup}\,\frac{|\phi(t)|}{\omega(t)} = \|\phi\|_\infty = \|h_M\| \le 1, \tag{6.7}$$

such that

$$h_M(f) = \int_{\mathbb{R}} f(t)\phi(t)dt, \quad \forall f \in L_\omega. \tag{6.8}$$

What else can we say about ϕ? One has

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(t)g(u)\phi(t+u)dtdu = \int_{\mathbb{R}^2} f(v-u)g(u)\phi(v)dvdu \\
&= h_M(f * g) = h_M(f)h_M(g) = \int_{\mathbb{R}} f(t)\phi(t)dt \int_{\mathbb{R}} g(u)\phi(u)du \\
&= \int_{\mathbb{R}^2} f(t)g(u)\phi(t)\phi(u)dtdu.
\end{aligned}$$

Since this holds for all f and g in L_ω, in particular for piecewise constant functions with compact support, one concludes that

$$\phi(t+u) = \phi(t)\phi(u) \quad \text{for almost all points } (t,u) \in \mathbb{R}^2. \tag{6.9}$$

It may actually be assumed that ϕ is continuous; see the end of the proof. Thus the equality in (6.9) holds for all t and u. It readily follows that

$$\phi(t) = \phi_M(t) = e^{ct} \quad \text{for some constant } c = c_M \in \mathbb{C}.$$

Writing $c_M = -iz_M = -i(x_M + iy_M)$, we finally use the inequality from (6.7),

$$\|\phi\|_\infty = \text{ess sup}\,\frac{|\phi(t)|}{\omega(t)} \le 1.$$

It shows that $e^{y_M t} \le \omega(t)$ almost everywhere or $y_M t \le \log \omega(t)$. Using the limit relations in (4.3) with t instead of u, one concludes that

$$\phi(t) = \phi_M(t) = e^{-iz_M t} = e^{-i(x_M + iy_M)t} \quad \text{with } a \le y_M \le b. \tag{6.10}$$

Formula (6.5) follows by combining (6.4), (6.8) and (6.10).

Since the elements $F = (f, \lambda) \in M$ form the kernel of h_M, they may be characterized by the relation

$$F^*(z_M) = \hat{f}(z_M) + \lambda = 0. \tag{6.11}$$

We still have to show that the elements F, whose transforms F^* satisfy an equation $F^*(z) = 0$ for a fixed number $z = x + iy$ with $a \le y \le b$, form a maximal ideal in A_ω. Let $F = (f, \lambda)$ and $G = (g, \mu)$ be in A_ω. Omitting the fixed argument z, one has the relations

$$\begin{aligned}(F+G)^* &= (f+g+\lambda+\mu)^* = (f+g)\hat{}+\lambda+\mu = F^*+G^*,\\ (F*G)^* &= \{(f+\lambda)*(g+\mu)\}^* = (f*g+\lambda g+\mu f+\lambda\mu)^*\\ &= (f*g+\lambda g+\mu f)\hat{}+\lambda\mu = (\hat{f}+\lambda)(\hat{g}+\mu) = F^*G^*;\end{aligned} \tag{6.12}$$

cf. (6.1), (6.2). That $(f+g)\hat{}$ and $(f*g)\hat{}$ are well defined at z follows from Section 4. Thus the elements $F \in A_\omega$ with $F^* = 0$ at z form an ideal. That this ideal is maximal follows by the same argument as we used at the beginning of Section 3.

It remains to verify that ϕ can be taken continuous. By (6.7) and (6.8) one may assume that ϕ is locally bounded. By (6.9) there is a subset $E \subset \mathbb{R}$ of measure 0 such that for $t \in \mathbb{R} \setminus E$, there is equality in (6.9) for almost all u. Since ϕ is not equivalent to the zero function, there is an interval (c, d) such that $\int_c^d \phi(u)du \neq 0$. For $t \in \mathbb{R} \setminus E$, we may integrate relation (6.9) over $c < u < d$ to show that

$$\int_{c+t}^{d+t} \phi(v)dv = \int_c^d \phi(t+u)du = \phi(t)\int_c^d \phi(u)du.$$

It follows that ϕ can be made continuous by changing its values on a set of measure 0, and this does not affect (6.8).

7 Regularity Condition for L_ω

In this and the next section we will use Shilov's method of regular Banach algebras [1940], [1947]. Such algebras are characterized by a separation property involving the Gelfand transforms; cf. also Gorin's exposition [1978]. The algebra L_ω is called *regular* if for every point $x_0 \in \mathbb{R}$ and every number $\delta > 0$, there is a function $f \in L_\omega$ such that

$$\hat{f}(x_0) \neq 0, \quad \text{while} \quad \hat{f}(x) = 0 \;\text{ for }\; |x - x_0| \ge \delta. \tag{7.1}$$

Theorem 7.1. *Let ω be a standard weight function as in* (4.1). *Then condition* (4.6), *that is, the condition of nonquasi-analyticity*

$$\int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2} dt < \infty, \tag{7.2}$$

is sufficient for the regularity of L_ω. (The condition is also necessary.)

The sufficiency of condition (7.2) in Theorem 7.1 may be derived from a theorem of Paley and Wiener [1934] (theorem 12) which we formulate as follows.

Theorem 7.2. *Let $g \in L^2$ be nonnegative but not equivalent to 0. Then there are an L^2 function h and a constant B such that $|h| = g$ and $\hat{h}(x) = 0$ for $x \geq B$ if and only if*

$$\int_{\mathbb{R}} \frac{|\log g(t)|}{1+t^2} dt < \infty. \tag{7.3}$$

Proof of Theorem 7.1. For a weight function ω as in (7.2) we apply Theorem 7.2 to the function

$$g(t) = \frac{1}{1+\omega(t)+\omega(-t)} \frac{1}{1+t^2}. \tag{7.4}$$

This g is in $L_\omega \cap L^1 \cap L^2$ and it satisfies condition (7.3). If h and B are as in Theorem 7.2 for our function g, then $\hat{h}$ is continuous and one may assume that $\hat{h}(x) \neq 0$ for values of x arbitrarily close to B. Now let $\delta > 0$ be given. Multiplying $h(t)$ by a suitable exponential e^{ict} if necessary, one may assume that $\hat{h}(0) \neq 0$ and $\hat{h}(x) = 0$ for $x \geq \delta$.

For the proof of Theorem 7.1 it is enough to construct a function $f \in L_\omega$ such that (7.1) holds with $x_0 = 0$: if f is in L_ω then so is $f(t)e^{ict}$. Now take f_1 such that $|f_1| = g$, $\hat{f}_1(0) \neq 0$ and $\hat{f}_1(x) = 0$ for $x \geq \delta$. Define $f_2(t) = f_1(-t)$. Then $|f_2| = g$, $\hat{f}_2(0) \neq 0$ and $\hat{f}_2(x) = 0$ for $x \leq -\delta$. Finally set $f = f_1 * f_2$, so that $\hat{f} = \hat{f}_1 \hat{f}_2$. Since the functions f_j are in L_ω so is f; furthermore $\hat{f}(0) \neq 0$ and $\hat{f}(x) = 0$ for $|x| \geq \delta$. It follows that L_ω is regular.

We indicate why condition (7.2) is necessary for the regularity of L_ω. The key is provided by the following 'quasi-analyticity' result of Levinson [1936], [1940]; cf. also Koosis [1988–92]. Let $f \in L^1$ be such that

$$\log |f(t)| \leq -\theta(t) \quad \text{as } t \to \infty,$$

where θ is a positive increasing function such that

$$\int_1^\infty \frac{\theta(t)}{t^2} = \infty.$$

Then the Fourier transform $\hat{f}$ cannot vanish on an interval unless it vanishes identically. One may deduce that under some mild conditions on ω, the algebra L_ω fails to be regular if the integral in (7.2) is divergent; cf. Vretblad [1973]. For the general case see Domar [1981], Dales and Hayman [1981].

THE MAXIMAL IDEALS IN A_ω. Let the regularity condition (7.2) be satisfied. In this case the indices a and b in (4.3) must be equal to 0. It now follows from Theorem 6.1 that all maximal ideals in $A = A_\omega$ different from L_ω have the form M_α for some $\alpha \in \mathbb{R}$. Here M_α consists of the elements $F = (f, \lambda)$ such that $h_\alpha(F) = \hat{f}(\alpha) + \lambda = 0$. It is convenient to denote the maximal ideal L_ω by M_∞; the homomorphism with kernel M_∞ is given by $h_\infty(F) = h_\infty\{(f, \lambda)\} = \lambda$. Under the usual topology, the maximal ideal space $\mathcal{M}$ of A_ω is homeomorphic to the one-point compactification $\mathbb{R}_e = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$. In the following we identify $\mathcal{M}$ with $\mathbb{R}_e$, and $\hat{F}$ of (6.4) with F^* of (6.6).

Every proper ideal $I \subset A$ belongs to one or more maximal ideals M_α.

Definition 7.3. The set of all maximal ideals M_α which contain I will be called the *hull* $H(I)$ of I. Equivalently one can consider $H(I)$ as the zero set $Z(\hat{I})$, the set where the Gelfand transform $\hat{F} = F^*$ of every element $F \in I$ is equal to zero (the 'skeleton' of I in the Russian literature).

By the continuity of the functions F^*, the hull $H(I) = Z(\hat{I})$ is a compact set $S \subset \mathbb{R}_e$. If I is closed, I consists of all elements $F \in A$ whose Gelfand transform vanishes on the hull S; we write $I = I(S)$. If I has hull S one also says that I *belongs* to the set S. Note that if I belongs to the point $\{\infty\}$, I is contained in L_ω.

8 The Closed Maximal Ideals in L_ω

Let ω be a weight function as in (4.1) which satisfies the regularity condition (4.6) = (7.2), so that the maximal ideals M_α in $A = A_\omega = (L_\omega, \mathbb{C})$ are in one-to-one correspondence with the points α of $\mathbb{R}_e = \mathbb{R} \cup \{\infty\}$. For the determination of the closed maximal ideals in L_ω and the proof of Beurling's Theorem 4.1, it is convenient to consider *minimal* ideals in A in the sense of Shilov [1947]. Other references for this section are Gelfand, Raikov and Shilov [1964/01], and Dunford and Schwartz [1963/88].

Theorem 8.1. *Among the ideals $I \subset A$ that belong to the point $\{\infty\}$, or to another compact set $S \subset \mathbb{R}_e$, there is a minimal ideal $J = J(\infty)$, or $J(S)$, respectively. It consists of all elements $F \in A$ whose Gelfand transform $\hat{F}$ vanishes on some neighborhood of $\{\infty\}$, or of S.*

For the proof we restrict ourselves to the case $S = \{\infty\}$. We need

Proposition 8.2. *Let I be any ideal in A belonging to $\{\infty\}$ and let K be any compact set in $\mathbb{R}$. Then there is a function $u \in I$ whose Fourier transform $\hat{u}$ is equal to* 1 *at all points of K.*

Proof. Since I belongs to the point $\{\infty\}$, it is contained in no maximal ideal other than $M_\infty = L_\omega$. Let J be the ideal $I(K)$ belonging to K. Then the hull $\{\infty\}$ of I is disjoint from the hull K of J. Now form the ideal $I + J$ consisting of all sums $u + v$ with $u \in I$ and $v \in J$. This ideal cannot belong to any maximal ideal M_α of A. Indeed, suppose that $h_\alpha(I + J) = h_\alpha(I) + h_\alpha(J) = 0$, or

$$h_\alpha(u + v) = h_\alpha(u) + h_\alpha(v) = 0 \quad \text{for all } u \in I \text{ and all } v \in J.$$

Then $h_\alpha(I)$ and $h_\alpha(J)$ must both be 0 since one can take $v = 0$ here, and similarly u. However, $h_\alpha(I) = 0$ only if $\alpha = \infty$ and $h_\alpha(J) = 0$ only for $\alpha \in K$.

Since the ideal $I + J$ does not belong to a maximal ideal, it follows from Proposition 2.2 that it coincides with A. In particular $I + J$ must contain the unit element e of A:

$$e = u + v \quad \text{with } u \in I \text{ and } v \in J.$$

Hence for any point $\alpha \in K$,

$$1 = h_\alpha(e) = h_\alpha(u) + h_\alpha(v) = \hat{u}(\alpha) + 0.$$

Proof of Theorem 8.1. Let $I \subset A$ be any ideal belonging to $\{\infty\}$, so that $I \subset L_\omega$. Let J consist of all functions $f \in L_\omega$ whose Fourier transform $\hat{f}$ has compact support (in $\mathbb{R}$). Fixing such an f, we have to show that it belongs to I. Let $K = \operatorname{supp} \hat{f}$. By Proposition 8.2, there is a function $u \in I$ such that $\hat{u}(\alpha) = 1$ for all $\alpha \in K$. Thus

$$(u * f)\hat{}(\alpha) = \hat{u}(\alpha)\hat{f}(\alpha) = \hat{f}(\alpha), \quad \forall \alpha \in K.$$

The same equality holds for α outside $K = \operatorname{supp}\hat{f}$, because then $\hat{f}(\alpha) = 0$. Hence

$$f = u * f, \quad \text{so that} \quad f \in I,$$

as had to be proved.

For the proof of Beurling's Approximation Theorem 4.1 we restate it as follows.

Theorem 8.3. *Let ω be a standard weight function as in* (4.1) *which satisfies the condition* (4.6) = (7.2) *of nonquasi-analyticity, so that L_ω is regular. Let I be any ideal in L_ω. Then the closure of I coincides with L_ω if and only if the zero set $Z(\hat{I})$ in $\mathbb{R}$ is empty.*

Proof. The ideal I can also be considered as an ideal in $A = A_\omega = (L_\omega, \mathbb{C})$; cf. formula (6.2). In particular I is contained in a maximal ideal M_α of A.

First suppose that $Z(\hat{I})$ contains a point $\alpha \in \mathbb{R}$. Then I belongs to the corresponding maximal ideal M_α, which consists of all elements $F = (f, \lambda)$ such that $\hat{F}(\alpha) = \hat{f}(\alpha) + \lambda = 0$. In particular M_α contains the closed ideal M'_α of all elements $(f, 0) = f \in L_\omega$ whose Fourier transform $\hat{f}$ vanishes at α. The ideal M'_α is a maximal ideal in L_ω; cf. the argument at the beginning of Section 3. Since $M'_\alpha = M_\alpha \cap L_\omega$ it contains I. In particular $\overline{I} \neq L_\omega$.

Next suppose that $Z(\hat{I}) \cap \mathbb{R}$ is empty. Then I is contained only in the maximal ideal $M_\infty = L_\omega$ of A_ω. In other words, I belongs to the point $\{\infty\}$, or equivalently, $H(I) = Z(\hat{I}) = \{\infty\}$. Since L_ω is regular, Theorem 8.1 now shows that I must contain the minimal ideal $J = J(\infty)$, which consists of all functions $f \in L_\omega$ whose Fourier transform has compact support (in $\mathbb{R}$).

To complete the proof that $\overline{I} = L_\omega$ we use the continuous linear functionals test (Hahn–Banach theorem) to show that $\overline{J} = L_\omega$. The continuous linear functionals l on L_ω have the form

$$l(f) = l_\phi(f) = \int_{\mathbb{R}} f(t)\phi(t)dt, \quad \text{with } \phi \in L^\infty_\omega; \tag{8.1}$$

cf. Section 4. Suppose then that $l_\phi(f) = 0$ for every function $f \in J$. Since $\hat{J}$ is translation invariant, $f \in J$ implies that $f(t)e^{-ixt} \in J$ for all $x \in \mathbb{R}$. We choose an $f \not\equiv 0$. Since $\hat{f}$ has compact support, f is the restriction of an entire function (cf. the easy part of the Paley–Wiener theorem), hence $f(t) \neq 0$ almost everywhere. By our hypothesis,

$$\int_{\mathbb{R}} f(t)e^{-ixt}\phi(t) = 0, \quad \forall x \in \mathbb{R}. \tag{8.2}$$

Thus $f(t)\phi(t) = 0$ almost everywhere and hence $\phi \equiv 0$. Conclusion: the only continuous linear functional l on L_ω which vanishes on J is the zero functional. Thus J is dense in L_ω, so that $\overline{I} = \overline{J} = L_\omega$.

Corollary 8.4. *Assuming that L_ω is regular, the closed maximal ideals in L_ω are the ideals M'_α described above, with $\alpha \in \mathbb{R}$.*

Remark 8.5. Dales and Hayman [1981] have given a proof of Theorem 8.3 by complex analysis. Their work extends a method used by Esterle [1980] for the case of the ordinary Wiener theorem. It also makes use of a result for a class of rapidly growing analytic functions on the unit disc by Hayman and Korenblum [1976].

9 Related Questions Involving Weighted Spaces

We consider 'Beurling algebras' L_ω, weighted L^1 spaces under convolution as in Section 4. In their study one distinguishes three cases, depending on ω.

(i) The nonquasi-analytic case, characterized by the condition

$$j(\omega) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2} dt < \infty. \tag{9.1}$$

In this case the indices a and b of (4.3) are both equal to 0. The quasi-analytic case, given by $j(\omega) = \infty$, gives rise to two separate cases.

(ii) The analytic case, given by the condition $a < b$.

(iii) The nonanalytic, quasi-analytic case, characterized by $j(\omega) = \infty$ and $a = b$.

In all three cases the maximal ideal space is homeomorphic to the closed strip $a \le \operatorname{Re} z \le b$. From the Tauberian point of view one would like to know when there are closed ideals associated with the point ∞. Many authors have studied the *primary ideals*, closed ideals which are contained in at most one maximal ideal.

Beurling's theorem implies that there are no primary ideals at ∞ in the nonquasi-analytic case. Nyman [1950], Korenblyum [1956], [1957a], [1957b], [1958], [1960], Geĭsberg and Konjuhovskiĭ [1971], Vretblad [1973] and Hedenmalm [1985] found and described primary ideals at infinity for large classes of weights. Domar [1983] proved that in the quasi-analytic case, there are always nontrivial primary ideals at infinity, hence there can be no analog to Theorem 3.3 in this case. Nyman was the first to give a necessary and sufficient condition on a function f in a certain nonregular algebra L_ω, under which its translates span the algebra. If $\omega(t) = e^{c|t|}$ with $c > 0$, the Fourier transform $\hat{f}(x+iy)$ must not only be free of zeros in the closed strip $\{|y| \le c\}$, but $\log|\hat{f}(x+iy)|$ must be $o(e^{\pi|x|/(2c)})$ as $x \to \pm\infty$; cf. also Korenblyum. There are recent applications to Tauberian theory; see Bingham and Inoue [1997], [1999], [2000c] .

Additional references are Wermer [1954], Gurariĭ [1976] and his survey [1998], Borichev and Hedenmalm [1995], Nikolski [1995], Carleson [1997], Frennemo [1999], [2002], Reiter and Stegeman [2000], Rolewicz [2000].

There are extensions of the results to other locally compact Abelian groups.

10 A Boundedness Theorem of Pitt

The theorem in question applies to rapidly decreasing kernels that are sufficiently close to 'difference kernels'. We begin by making this precise.

Definition 10.1. We will say that the kernel $J(x, y)$ is *well-approximated* by the difference kernel $K(x-y)$, relative to the weights $e^{-u(x-y)}$ with $0 \le u \le d$, if the products $J(x, x-z)e^{-uz}$ and $K(z)e^{-uz}$ are majorized by a fixed L^1 function $K^*(z)$ for $x \in \mathbb{R}$ and $0 \le u \le d$, and if the function

$$\rho(x) = \sup_{0 \le u \le d} \int_{\mathbb{R}} |J(x, x-z) - K(z)| e^{-uz} dz \tag{10.1}$$

is bounded on $\mathbb{R}$ and tends to 0 as $x \to \infty$.

For such kernels one has the following 'boundedness theorem'; see Pitt [1938a], [1958] (section 4.3) for the case $\phi \equiv 0$.

Theorem 10.2. *Let d be positive, let $J(x, y)$ be well-approximated by $K(x-y)$ relative to the weights $e^{-u(x-y)}$ with $0 \le u \le d$, and let $K(z)e^{-uz}$ be a Wiener kernel for $0 \le u \le d$. Let $S(y) = 0$ for $y < 0$ and*

$$S(y) = \mathcal{O}(e^{uy}) \quad \text{for } y \ge 0 \text{ and some number } u < d. \tag{10.2}$$

In addition, let S be slowly decreasing or satisfy the step function condition on $\mathbb{R}$ (Definition II.2.3). Finally suppose that

$$G(x) = \int_{\mathbb{R}} J(x, y)S(y)dy = \mathcal{O}\{e^{\phi(x)}\} \quad \text{for } x \ge 0, \tag{10.3}$$

where ϕ is nonnegative, nondecreasing and differentiable, with $\phi'(x) \searrow 0$ as $x \to \infty$. Then $S(y) = \mathcal{O}\{e^{\phi(y)}\}$.

If G is bounded for $x \ge 0$ (so that S is bounded) and

$$\int_{\mathbb{R}} J(x, y)S(y)dy - A \int_{\mathbb{R}} J(x, y)dy \to 0 \quad \text{as } x \to \infty, \tag{10.4}$$

then $S(x) \to A$ as $x \to \infty$.

The proof uses Pitt's method and consists of several steps which can be found in Sections 11–13. An important ingredient is the following refined form of Wiener's division theorem relative to convolution, which is due to Pitt [1958] (section 4.1).

Theorem 10.3. *Let H and K^u be in $L^1(\mathbb{R})$ with $|K^u(x)| \le K^*(x) \in L^1(\mathbb{R})$ for u in an index set I. Suppose that for the Fourier transforms, $\hat{H}(t) = 0$ for $|t| > \lambda$ and $|\hat{K}^u(t)| \ge m > 0$ for $|t| \le \lambda$ and $u \in I$. Then there are L^1 functions Q^u and an L^1 function Q^* independent of u such that for all $u \in I$,*

$$H = Q^u * K^u, \quad \text{with } |Q^u(y)| \le Q^*(y), \ \forall y. \tag{10.5}$$

The proof of Theorem 10.3 depends on careful analysis of the standard proof for Wiener's division theorem given in Section II.9; cf. Hardy [1949] or Pitt [1958]. The quotient Q^u is obtained as a sum of local quotients. For fixed numerator H and fixed u, the number of local quotients and their norms depend only on the modulus of continuity of $\hat{K}^u$ and on $\min |\hat{K}^u(t)|$ for $|t| \le \lambda$. The modulus of continuity can be bounded in terms of K^*.

11 Proof of Theorem 10.2, Part 1

Let $J,\ K,\ S,\ G$ and ϕ be as in Theorem 10.2. Then $\phi(x) = o(x)$ as $x \to \infty$, hence

$$G(x) = \int_{\mathbb{R}} J(x, y)S(y)dy = \mathcal{O}(e^{\varepsilon x}) \quad \text{as } x \to \infty, \quad \forall \varepsilon > 0. \tag{11.1}$$

Having (10.2), we let b be the *infimum* of the positive numbers u for which $S(y)e^{-uy}$ is bounded. We wish to prove that $b = 0$ and *suppose* for the time being that $b > 0$. Using Pitt's method [1958] (section 4.3), we will show that this supposition leads to a contradiction. For the proof we focus on numbers $a \in (0, b)$ and $c \in (b, d)$ which are so close together that $2c - a \leq d$.

STEP 1. Consider the following bounded functions (which vanish for $y < 0$ and at $+\infty$):

$$S_k(y) = S_{k,c}(y) = y^k e^{-cy} S(y), \tag{11.2}$$

and set

$$\sup_y |S_k(y)| = M_k = M_k(c). \tag{11.3}$$

Suppose for a moment that it has been shown already that for some numbers a and c as above, the sequence

$$\{M_k(c-a)^k/k!\} \text{ is } \textit{bounded}, \quad \text{with supremum } M, \text{ say.} \tag{11.4}$$

Then for $y \geq 0$

$$|S(y)|e^{-cy}\{(c-a)y\}^k/k! \leq M, \quad \forall k,$$

so that for $0 < \theta < 1$

$$|S(y)|e^{-cy} \sum_{k=0}^{\infty} \frac{\{(c-a)y\theta\}^k}{k!} \leq \frac{M}{1-\theta}, \quad |S(y)| \leq \frac{M}{1-\theta} e^{\{\theta a + (1-\theta)c\}y}.$$

Here one can choose θ such that $\theta a + (1-\theta)c < b$, but in view of the definition of b, this contradicts the assumption $b > 0$. In other words, *if we prove* (11.4), we can conclude that $b = 0$. The proof of (11.4) will run through Section 12.

STEP 2. In order to establish (11.4) for small $c - a$ we *suppose* that it is *false*. Then one can choose arbitrarily large n for which

$$(c-a)^n M_n \geq n!, \quad \frac{M_k(c-a)^k}{k!} \leq \frac{M_n(c-a)^n}{n!}, \quad k = 0, 1, \ldots, n-1. \tag{11.5}$$

In the sequel we consider *only such special* n. For those,

$$\sum_{k=0}^{n-1} \binom{n}{k} M_k |z|^{n-k} \leq M_n \sum_{k=0}^{n-1} \frac{|(c-a)z|^{n-k}}{(n-k)!} = M_n \sum_{k=1}^{n} \frac{|(c-a)z|^k}{k!}$$
$$\leq M_n\{e^{(c-a)|z|} - 1\}. \tag{11.6}$$

By the definition of G and (11.2),

$$G(x)e^{-cx}x^n = \int_{\mathbb{R}} J(x,y)e^{-c(x-y)}\{y+(x-y)\}^n e^{-cy}S(y)dy$$
$$= \int_{\mathbb{R}} J(x,y)e^{-c(x-y)} \sum_{k=0}^{n} \binom{n}{k} (x-y)^{n-k} S_k(y)dy. \tag{11.7}$$

Let us set $\sup_{x\geq 0} |G(x)|e^{-ax} = C_a$; cf. (11.1). Then by (11.7) and (11.6), for $x \geq 0$,

$$\left| \int_{\mathbb{R}} J(x,y)e^{-c(x-y)} S_n(y)dy \right|$$
$$\leq |G(x)|e^{-cx}x^n + \int_{\mathbb{R}} |J(x,y)|e^{-c(x-y)} \sum_{k=0}^{n-1} \binom{n}{k} |x-y|^{n-k} M_k dy$$
$$\leq C_a e^{-(c-a)x}x^n + M_n \int_{\mathbb{R}} |J(x,y)|e^{-c(x-y)} \sum_{k=0}^{n-1} \frac{\{(c-a)|x-y|\}^{n-k}}{(n-k)!}$$
$$\leq C_a \left\{ \frac{n}{(c-a)e} \right\}^n + M_n \int_{\mathbb{R}} |J(x, x-z)|e^{-cz}(e^{(c-a)|z|}-1)dz. \tag{11.8}$$

Step 3. We now introduce the convolution

$$T_n(x) = T_{n,c}(x) = Ke^{-c\cdot} * S_{n,c}(x) = \int_{\mathbb{R}} K(x-y)e^{-c(x-y)} S_{n,c}(y)dy. \tag{11.9}$$

By the hypotheses, the functions $K(z)e^{-uz}$ and $J(x, x-z)e^{-uz}$ are majorized by a fixed L^1 function $K^*(z)$ for $x \in \mathbb{R}$ and $0 \leq u \leq d$; cf. Definition 10.1. Setting $\|K^*\| = C$ we thus have in particular

$$|T_n(x)| \leq CM_n = CM_n(c), \quad \forall x,\ \forall n \geq 0,\ \forall c \in (b,d]. \tag{11.10}$$

We use (11.8) and relation (10.1) to estimate $|T_n(x)|$ for large x and (our special) n:

$$|T_n(x)| = \left| \int_{\mathbb{R}} \{K(x-y) - J(x,y)\}e^{-c(x-y)} S_{n,c}(y)dy \right.$$
$$\left. + \int_{\mathbb{R}} J(x,y)e^{-c(x-y)} S_{n,c}(y)dy \right|$$
$$\leq M_n \rho(x) + C_a (n/\{(c-a)e\})^n$$
$$+ M_n \int_{\mathbb{R}} |J(x,x-z)|e^{-cz}(e^{(c-a)|z|}-1)dz. \tag{11.11}$$

By the hypotheses the first term on the right is $M_n \cdot o(1)$ as $x \to \infty$. The second term is $o\{n!/(c-a)^n\}$ as $n \to \infty$, which by (11.5) is $o(M_n)$. For large x, the integral $I = I(x,a,c)$ at the end can be made arbitrarily small by making $c-a$ small. Indeed,

$$e^{-cz}(e^{(c-a)|z|}-1) < \begin{cases} e^{-az} & \text{if } z \geq 0, \\ e^{(2c-a)|z|} \leq e^{-dz} & \text{if } z < 0. \end{cases}$$

Hence

$$I \le \int_{\mathbb{R}} |J(x, x-z) - K(z)|(e^{-az} + e^{-dz})dz + \int_{\mathbb{R}} |K(z)|e^{-cz}(e^{(c-a)|z|} - 1)dz.$$

By (10.1) the first integral here is bounded by $2\rho(x)$. The final integral tends to 0 as $c - a \to 0$ by dominated convergence: the integrand is majorized by $2K^*(z)$. Combining these results we conclude that for given $\eta > 0$, one can fix $c - a$ so small that

$$|T_n(x)| \le \eta M_n \quad \text{for all large } x \text{ and (special) } n. \tag{11.12}$$

12 Theorem 10.2: Proof that $S(y) = \mathcal{O}(e^{\varepsilon y})$

STEP 4. On our way to a proof of (11.4) we next use Pitt's Division Theorem 10.3. Let H be the Fejér kernel for $\mathbb{R}$,

$$H(x) = D_\lambda(x) = \lambda D(\lambda x) = \frac{1 - \cos \lambda x}{\pi \lambda x^2},$$

where $\lambda > 0$ will be taken large. The Fourier transform $\hat{D}_\lambda(t)$ is equal to $1 - |t|/\lambda$ for $|t| \le \lambda$ and equal to 0 for $|t| > \lambda$; cf. Example II.7.1. For $0 \le u \le d$, the products $K^u(x) = K(x)e^{-ux}$ are majorized by the L^1 function $K^*(x)$ which is independent of u. Also by the hypotheses, the Fourier transforms $|\hat{K}^u(t)| = |\hat{K}(t - iu)|$ have a positive minimum $m = m_\lambda$ for $0 \le u \le d$, $|t| \le \lambda$. Hence by Theorem 10.3 there are L^1 functions Q^u_λ, as well as an L^1 function Q^*_λ independent of u, such that

$$D_\lambda(x) = \int_{\mathbb{R}} K(x-y)e^{-u(x-y)}Q^u_\lambda(y)dy, \quad \text{with } |Q^u_\lambda(y)| \le Q^*_\lambda(y). \tag{12.1}$$

Thus for $u = c$ and by (11.9), where we write S_n for $S_{n,c}$,

$$|D_\lambda * S_n| = |Ke^{-c\cdot} * Q^c_\lambda * S_n| = |Q^c_\lambda * T_n| \le Q^*_\lambda * |T_n|. \tag{12.2}$$

Let $\varepsilon \in (0, 1)$ be given and take $\eta = \varepsilon/\|Q^*_\lambda\|$. By (11.10) and (11.12), the last convolution may then be estimated as follows:

$$\int_{\mathbb{R}} Q^*_\lambda(y)|T_n(x-y)|dy \le \int_{|y|>Y} Q^*_\lambda \cdot CM_n dy + \int_{|y|\le Y} Q^*_\lambda \cdot \eta M_n dy \le 2\varepsilon M_n, \tag{12.3}$$

provided Y has been suitably chosen and x and n are large.

STEP 5. We finally use the Tauberian condition. By the hypotheses $S(\cdot)$ is slowly decreasing or satisfies the step function condition on $\mathbb{R}$. In the former case there is a number $\delta > 0$ such that for all large x, $S(y) - S(x - \delta) \ge -\varepsilon$ for $x - \delta \le y \le x + \delta$. In the latter case S is piecewise constant, and the intervals of constancy all contain an interval of the form $[x - \delta, x + \delta]$ with fixed $\delta > 0$. Taking our special n large, we now consider numbers x' for which $|S_n(x')| \ge (1 - \varepsilon)M_n$. Notice that by (11.2), (11.3) and (11.5)

$$|S_n(x')| = |x'S_{n-1}(x')| \le x'M_{n-1} \le x'(c-a)M_n/n,$$

so that

$$x' \ge (1-\varepsilon)n/(c-a) \ge n/d \quad \text{when } \varepsilon \text{ is small.}$$

Let us assume for definiteness that $S_n(x') > 0$. We then let x' be the initial point of an interval $x - \delta \le y \le x + \delta$ as above. (Otherwise we take $x' = x + \delta$, and then use the inequality $x' \ge n/d$ to estimate $(y/x')^n$ from below in the next lines.) For y in such an interval,

$$\begin{aligned} S_n(y) = y^n e^{-cy} S(y) &\ge y^n e^{-cy}\{S(x') - \varepsilon\} \\ &\ge e^{-2c\delta} S_n(x') - \varepsilon\{n/(ec)\}^n \\ &\ge e^{-2c\delta}(1-\varepsilon)M_n - \varepsilon M_n \ge e^{-4d\delta} M_n \end{aligned}$$

when ε is small and n is large. Recalling that $\int_{\mathbb{R}} D_\lambda(z)dz = 1$, we thus find

$$\begin{aligned} \int_{\mathbb{R}} D_\lambda(x-y)S_n(y)dy &\ge \int_{x-\delta}^{x+\delta} D_\lambda(x-y)e^{-4d\delta}M_n dy \\ &\quad -\left(\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right) D_\lambda(x-y)M_n dy \\ &= e^{-4d\delta}M_n \int_{-\delta}^{\delta} D_\lambda(z)dz - M_n\left(1 - \int_{-\delta}^{\delta} D_\lambda(z)dz\right) \\ &= \left((1 + e^{-4d\delta})\int_{-\lambda\delta}^{\lambda\delta} D(z)dz - 1\right)M_n. \end{aligned} \tag{12.4}$$

Combination with (12.2) and (12.3) gives

$$\left((1 + e^{-4d\delta})\int_{-\lambda\delta}^{\lambda\delta} D(z)dz - 1\right)M_n \le 2\varepsilon M_n$$

for large n. But this is impossible if ε (and possibly δ) is small and λ is taken large, because $\int_{\mathbb{R}} D(z)dz = 1$.

This contradiction establishes (11.4), so that $b = 0$ by Step 1. It follows that $S(y) = \mathcal{O}(e^{cy})$ for every number $c > 0$.

13 Theorem 10.2: Proof that $S(y) = \mathcal{O}\{e^{\phi(y)}\}$

STEP 6. Recall that the function $G(x)$ in (10.3) is $\mathcal{O}\{e^{\phi(x)}\}$ for $x \ge 0$, where ϕ is nonnegative, nondecreasing, differentiable and such that $\phi'(x) \searrow 0$ as $x \to \infty$. Thus we have an inequality

$$\sup_{x \ge 0} e^{-cx}|G(x)| \le B \max_{x \ge 0} e^{\phi(x) - cx} = B\psi(c), \quad \text{say}, \quad \forall c > 0. \tag{13.1}$$

Here $\psi(c)$ is nonincreasing, and for every large x there is a value of c [namely, $c = \phi'(x)$], for which $e^{\phi(x)-cx} = \psi(c)$. By (11.2), (11.3) and Section 12,

$$M_0(c) = \sup_y |S_{0,c}(y)| = \sup_y e^{-cy}|S(y)| < \infty, \quad \forall c > 0. \tag{13.2}$$

We wish to prove that $e^{-\phi(y)}S(y)$ is bounded. Suppose then that $e^{-\phi(y)}S(y)$, which is bounded on finite intervals, is unbounded for $y \to \infty$. This will imply that $M_0(c)/\psi(c)$ is unbounded for $c \searrow 0$. Indeed, let $e^{-\phi(y_k)}|S(y_k)| \to \infty$ where $y_k \to \infty$. Then by (13.1) and (13.2), for $c_k = \phi'(y_k) \to 0$,

$$M_0(c_k)/\psi(c_k) \geq e^{-c_k y_k}|S(y_k)|e^{-\phi(y_k)+c_k y_k} \to \infty. \tag{13.3}$$

In particular the function $M_0(c)$ will be unbounded, and being nonincreasing, must tend to ∞ as $c \searrow 0$. Hence the values x' of x for which $e^{-cx}|S(x)|$ is close to $M_0(c)$ must tend to ∞ as $c \searrow 0$.

By (11.9) and the first part of (11.11) with $n = 0$, and in view of (13.1), (13.2),

$$\begin{aligned} |T_0(x)| &= |Ke^{-c\cdot} * S_{0,c}(x)| \leq M_0(c)\rho(x) + e^{-cx}|G(x)| \\ &\leq M_0(c)\rho(x) + B\psi(c), \quad \forall x \geq 0. \end{aligned} \tag{13.4}$$

Combining this inequality with (12.2) for $n = 0$ one obtains

$$\begin{aligned} \left|\int_{\mathbb{R}} D_\lambda(x-y)S_{0,c}(y)dy\right| &= \left|\int_{\mathbb{R}} Q_\lambda^c(y)T_0(x-y)dy\right| \\ &\leq M_0(c)\int_{\mathbb{R}} Q_\lambda^*(y)\rho(x-y)dy + B\psi(c)\|Q_\lambda^*\|. \end{aligned} \tag{13.5}$$

Now let $\varepsilon \in (0, 1)$ be given. Recall that $\rho(x)$ in (10.1) tends boundedly to 0 as $x \to \infty$. Also using (13.3), we thus find that the right-hand side of (13.5) is less than $\varepsilon M_0(c)$ when x is large and c of the form c_k is small.

Knowing that the values of x' for which $|S_{0,c}(x')|$ is close to $M_0(c)$ are relatively large when c is small, we use the Tauberian condition as in Section 12. It follows that there are intervals $x - \delta \leq y \leq x + \delta$ with large x throughout which $|S_{0,c}(y)| \geq (1 - \varepsilon)M_0(c)$. As a result

$$\begin{aligned} &\left|\int_{\mathbb{R}} D_\lambda(x-y)S_{0,c}(y)dy\right| \\ &\geq \int_{x-\delta}^{x+\delta} D_\lambda(x-y)(1-\varepsilon)M_0(c)dy - M_0(c)\left(1 - \int_{-\delta}^{\delta} D_\lambda(z)dz\right) \\ &= \{(2-\varepsilon)M_0(c)\}\int_{-\delta}^{\delta} D_\lambda(z)dz - M_0(c). \end{aligned} \tag{13.6}$$

Combination of (13.5) and (13.6) shows that for small $c = c_k$,

$$\left((2-\varepsilon)\int_{-\lambda\delta}^{\lambda\delta} D(z)dz - 1\right)M_0(c) \leq 2\varepsilon M_0(c).$$

However, this is impossible if ε and possibly δ is small and λ is taken large.

The contradiction proves that $S(y) = \mathcal{O}\{e^{\phi(y)}\}$.

Completion of the Proof of Theorem 10.2. We now turn to the final statement in Theorem 10.2. By the hypotheses $\int_{\mathbb{R}} |J(x, y) - K(x-y)|dy \to 0$ as $x \to \infty$.

Furthermore G is bounded so that we may take $\phi = 0$, hence by the preceding, S too is bounded. Thus

$$\int_{\mathbb{R}} \{J(x, y) - K(x - y)\}\{S(y) - A\}dy \to 0 \quad \text{as } x \to \infty.$$

By hypothesis (10.4) one has $\int_{\mathbb{R}} J(x, y)\{S(y) - A\}dy \to 0$, hence

$$\int_{\mathbb{R}} K(x - y)\{S(y) - A\}dy \to 0.$$

Under a Tauberian condition as in the Theorem, the final relation implies that $S(x)$ tends to A as $x \to \infty$; see the Wiener–Pitt Theorem II.8.4.

14 Boundedness Through Functional Analysis

Systematic use of functional analysis in summability theory was initiated by the Polish functional analysts, notably Mazur [1930], Banach [1932], Mazur and Orlicz [1933], [1955]. The work was extended by Zeller [1951a], [1951b], [1953a], [1953b], [1953c], [1956] in Germany and by Wilansky in the United States; cf. the latter's book [1984] for references. For matrix methods, the limitable sequences form what Zeller called a 'Fréchet-coordinate space' or FK-space; cf. the brief reference in Köthe [1969] (end of section 30). Following Zeller, we give a short introduction to the theory of such spaces.

The basic *boundedness principle* for Tauberian theory says that the following phenomenon occurs in a large class of FK-spaces: 'If there is a divergent sequence among the elements of the space, then there is also a bounded divergent sequence among its elements'. Thus certain general Tauberian problems, in particular problems involving the partial sums of gap series, can be reduced to the case of bounded sequences (of partial sums). The final result was the outcome of work by many authors: see the articles by Mazur and Orlicz [1955], Wilansky and Zeller [1955], and especially Meyer-König and Zeller [1956], [1962]. For their proofs, Meyer-König and Zeller refined the powerful method of the *sliding hump* ('gleitender Buckel' in German); see Sections 18, 19. In a simpler form the method goes back to Lebesgue and Toeplitz, 1910–1913. Relatively recent publications include Bennett and Kalton [1972], Noll and Stadler [1989], and Swartz [1996]; see also Boos [2000].

Fairly detailed accounts of the theory of FK-spaces can be found in the books by Zeller and Beekmann [1958/70], Wilansky [1984], and Boos [2000]. However, the more refined applications to Tauberian theorems for gap series have appeared only in journal articles. References are given in Sections 20, 21 and Chapter VI.

15 Limitable Sequences as Elements of an FK-space

In the following X will denote a *sequence space*: a linear space whose elements, 'points' or 'vectors' are semi-infinite sequences

$$\mathbf{x} = \{x_n\}, \quad n = 0, 1, 2, \ldots$$

of complex numbers. The element or vector $\mathbf{x}$ is called bounded, convergent, divergent, ... if the sequence $\{x_n\}$ is bounded, convergent, divergent, ... in the ordinary sense.

In the vector space X, we introduce a topology by a countable family of seminorms q_j: nonnegative, subadditive functionals such that $q_j(\lambda\mathbf{x}) = |\lambda| q_j(\mathbf{x})$. The seminorms must form a 'separating family': for every vector $\mathbf{x} \neq \mathbf{0}$, at least one seminorm q_j must be $\neq 0$; cf. Rudin [1973/91]. Thus the formula

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{q_j(\mathbf{x}-\mathbf{y})}{1 + q_j(\mathbf{x}-\mathbf{y})}$$

will define a metric. It is required that the resulting space be complete. Equipped with this metric, the space X will be an F-space in the sense of Banach (Fréchet space). It is called an *FK-space* if the coordinate or component maps $\mathbf{x} \mapsto x_n$ are continuous ('F-space with componentwise convergence').

We now consider summability methods, or more precisely, limitation methods Γ given by semi-infinite matrices $\Gamma = [c_{kn}]$, $k, n = 0, 1, \ldots$:

$$\mathbf{y} = \Gamma\mathbf{x} \quad \text{means} \quad y_k = \sum_{n=0}^{\infty} c_{kn} x_n, \qquad k = 0, 1, \ldots. \tag{15.1}$$

The *domain* of definition D_Γ is the set of vectors $\mathbf{x}$ for which the numbers y_k in (15.1) are well-defined. The *effective domain* or limitation domain W_Γ ('Wirkfeld' in German) is the subset of D_Γ which consists of the Γ-limitable sequences. In other words, the vectors $\mathbf{x}$ for which the vectors $\mathbf{y} = \Gamma\mathbf{x}$ are well-defined and *convergent*.

We always assume that the method Γ is *regular*: if $x_n \to A$ then also $y_k \to A$. By a classical result which goes back to Toeplitz, Γ is regular if and only if the following conditions are satisfied:

$$\sup_k \sum_{n=0}^{\infty} |c_{kn}| < \infty, \qquad \sum_{n=0}^{\infty} c_{kn} \to 1 \quad \text{as } k \to \infty,$$
$$c_{kn} \to 0 \quad \text{as } k \to \infty, \ \forall n;$$

cf. Banach [1932] (p. 90). More generally one may consider *conservative* methods Γ, methods which assign *some* finite limit to every convergent sequence. See Boos [2000] for recent work on the subject.

A matrix method Γ is called *reversible* if for every convergent sequence $\mathbf{y}$ there is exactly one sequence $\mathbf{x}$ such that $\mathbf{y} = \Gamma\mathbf{x}$. For example, a method given by a (lower) triangular matrix is reversible if the diagonal elements are different from 0.

The following basic results go back to Mazur and Orlicz [1933], [1955] and Zeller [1951a]; see Zeller and Beekmann [1958/70] for details and additional references.

Proposition 15.1. *The effective domain W_Γ of a matrix method Γ is a separable FK-space X under the family of seminorms*

$$p(\mathbf{x}) = \sup_k |y_k| \qquad (\text{where } \mathbf{y} = \Gamma\mathbf{x}), \tag{15.2}$$

$$p_k(\mathbf{x}) = \sup_m \left| \sum_{n=0}^{m} c_{kn} x_n \right| \qquad (k = 0, 1, \ldots), \tag{15.3}$$

$$q_n(\mathbf{x}) = |x_n| \qquad (n = 0, 1, \ldots). \tag{15.4}$$

Remark 15.2. In special cases fewer seminorms may suffice to define the (same) topology. If the matrix Γ is reversible, one can show that the effective domain becomes a Banach space ('BK-space') under the norm

$$\|\mathbf{x}\| = \sup_k |y_k|. \tag{15.5}$$

Proof of the Proposition (Outline). The set of vectors $\mathbf{x}$ for which $\sum_n c_{0n} x_n$ converges is an FK-space under the seminorms p_0 and q_j, $j = 0, 1, \ldots$. There is a corresponding result for the other rows of Γ. From this one derives that the domain of Γ is an FK-space under the seminorms (15.3) and (15.4). The seminorm p serves to make the effective domain complete. The separability may be derived from the fact that the product space $X \times X \times X \times \cdots$, in which each factor is furnished with only one of the seminorms from (15.2), (15.3) or (15.4), is separable.

Proposition 15.3. *Let X be an FK-space as in Proposition 15.1. Then the continuous linear functionals on X have the form*

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} a_n x_n + \sum_{k=0}^{\infty} b_k y_k + b \cdot \lim_{k \to \infty} y_k, \tag{15.6}$$

where $\sum a_n x_n$ converges, $\mathbf{y} = \Gamma \mathbf{x}$ and $\sum |b_k| < \infty$.

If Γ is reversible one may take the coefficients a_n equal to 0.

16 Perfect Matrix Methods

A subset E of an F-space X is called *fundamental* if the finite linear combinations of the vectors of E lie dense in X, so that the closed (linear) span of E coincides with X.

Definition 16.1. A regular limitation method Γ is called *perfect* if the vectors

$$\mathbf{e} = \{1, 1, 1, \cdots\}, \tag{16.1}$$

$$\mathbf{e}_0 = \{1, 0, 0, \cdots\}, \ \mathbf{e}_1 = \{0, 1, 0, \cdots\}, \ \cdots \tag{16.2}$$

form a fundamental set in the effective domain W_Γ. More generally, the closed linear span of these vectors will be called the perfect part of W_Γ.

For every regular matrix method Γ there is a regular matrix method Δ, whose effective domain W_Δ is the perfect part of W_Γ; cf. Zeller [1956]. The notion of a perfect method and related concepts play an important role in consistency theorems; see Mazur [1930], Banach [1932], Hill [1937] and the books mentioned in Section 14. For us the notion is important because of Theorem 16.4 below.

One may use Proposition 15.3 to derive Mazur's criterion:

Corollary 16.2. *A regular triangular matrix method* Γ *whose diagonal elements are* $\neq 0$ *is perfect if and only if the equations*

$$\sum_{k=0}^{\infty} b_k c_{kn} = 0, \quad n = 0, 1, 2, \ldots, \tag{16.3}$$

with $\sum |b_k| < \infty$, *imply* $\mathbf{b} = \mathbf{0}$.

Examples 16.3. The Cesàro method $(C, 1)$ is perfect. The matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Here the system of equations (16.3) takes the form

$$\begin{array}{l} b_0 + b_1/2 + b_2/3 + b_3/4 + \cdots = 0 \\ 0 + b_1/2 + b_2/3 + b_3/4 + \cdots = 0 \\ 0 + \ \ 0 \ \ + b_2/3 + b_3/4 + \cdots = 0 \\ 0 + \ \ 0 \ \ + \ \ 0 \ \ + b_3/4 + \cdots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \end{array}$$

hence $\mathbf{b} = \mathbf{0}$.

As a sequence to sequence transformation, the Euler method has matrix elements $c_{kn} = \binom{k}{n}/2^k$; cf. Section VI.20. In this case the system (16.3) becomes

$$\sum_{k=n}^{\infty} \binom{k}{n} b_k \frac{1}{2^k} = 0, \quad \text{or} \quad \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) b_k \left(\frac{1}{2}\right)^{k-n} = 0,$$

$n = 0, 1, 2, \ldots$. The final relations say that the function $\sum_0^\infty b_k w^k$ (which is analytic for $|w| < 1$) and its derivatives must vanish at the point $w = 1/2$, so that $\mathbf{b} = \mathbf{0}$. The Euler method is perfect.

The following *boundedness principle* is basic for the application of FK-spaces to Tauberian theory. In the applications X would be the effective domain of a limitation method.

Theorem 16.4. *Let* X *be an* FK*-space in which the vectors* $\mathbf{e}$ *and* $\mathbf{e}_j$ *of* (16.1), (16.2) *together form a fundamental set. Then* either *every vector* $\mathbf{x}$ *in* X *is convergent,* or X *contains a bounded divergent vector* $\mathbf{x}$.

The theorem is due to Meyer-König and Zeller [1956], [1962]. Predecessors may be found in the work of Mazur and Orlicz [1955], and Wilansky and Zeller [1955]. Theorem 16.4 will be derived from Theorem 18.1 below.

17 Methods with Sectional Convergence

References for this section are Zeller [1951b], Meyer-König and Zeller [1956], and Wilansky and Zeller [1956]; cf. also Zeller and Beekmann [1958/70].

Definition 17.1. One says that an FK-space X_0 possesses *sectional convergence* if the vectors $\mathbf{e}_0,\ \mathbf{e}_1, \ldots$ of (16.2) form a *basis* of X_0. Equivalently,

$$\mathbf{x}^{(m)} = \{x_0, \cdots, x_m, 0, 0, \cdots\} \to \mathbf{x} = \{x_0, \cdots, x_m, x_{m+1}, \cdots\} \qquad (17.1)$$

as $m \to \infty$ for every vector $\mathbf{x} \in X_0$. A matrix method Γ is said to possess sectional convergence if the subspace $(W_\Gamma)_0$, of the vectors $\mathbf{x}$ that are limitable to $\mathbf{0}$, possesses sectional convergence.

It may be derived from Proposition 15.1 that a regular matrix method Γ has sectional convergence if and only if

$$\sup_k \left| \sum_{n=m+1}^{\infty} c_{kn} x_n \right| \to 0 \quad \text{as } m \to \infty, \quad \forall \mathbf{x} \in (W_\Gamma)_0. \qquad (17.2)$$

This is equivalent to 'sectional boundedness' in $(W_\Gamma)_0$:

$$\sup_{k,m} \left| \sum_{n=0}^{m} c_{kn} x_n \right| < \infty, \qquad \forall \mathbf{x} \in (W_\Gamma)_0. \qquad (17.3)$$

For a regular method Γ with sectional convergence, condition (17.1) is equivalent to

$$\{x_0, \cdots, x_m, A, A, \cdots\} \to \mathbf{x} \quad \text{as } m \to \infty, \quad \forall \mathbf{x} \in W_\Gamma, \qquad (17.4)$$

where $A = \lim y_k = \lim(\Gamma \mathbf{x})_k$. Indeed, (17.4) means that the method Γ limits the vector $\mathbf{x}' = \mathbf{x} - A\mathbf{e}$ to $\mathbf{0}$.

Corollary 17.2. *A regular method with sectional convergence is perfect.*

Example 17.3. We verify that the Cesàro method $(C, 1)$ possesses sectional convergence. Let Γ be the corresponding matrix, which is reversible (Example 16.3). By Remark 15.2, W_Γ can be considered as a Banach space under the norm

$$\|\mathbf{x}\| = \sup_k |y_k| = \sup_k \frac{|x_0 + \cdots + x_k|}{k+1}.$$

Now take $\mathbf{x}$ in $(W_\Gamma)_0$, so that $y_k \to 0$. Let $\mathbf{x}^{(m)}$ be as in (17.1) and set $\mathbf{y}^{(m)} = \Gamma \mathbf{x}^{(m)}$. Then

$$y_k^{(m)} = \begin{cases} (x_0 + \cdots + x_k)/(k+1) & \text{if } k \le m, \\ (x_0 + \cdots + x_m)/(k+1) & \text{if } k > m. \end{cases}$$

We have to show that

$$\|\mathbf{x} - \mathbf{x}^{(m)}\| = \sup_k |y_k - y_k^{(m)}| \to 0 \quad \text{as } m \to \infty.$$

For given $\varepsilon > 0$, let k_0 be so large that $|y_k| < \varepsilon$ when $k \geq k_0$. Clearly $y_k - y_k^{(m)} = 0$ for $k \leq m$, while for $k > m \geq k_0$,

$$|y_k - y_k^{(m)}| \leq |y_k| + \frac{m+1}{k+1}|y_m| < 2\varepsilon.$$

18 Existence of (Limitable) Bounded Divergent Sequences

In the formulation of Meyer-König and Zeller [1956], [1962], the *boundedness principle* reads as follows.

Theorem 18.1. *Let X be an FK-space in which the convergent vectors do not form a closed set. Then X contains both bounded divergent vectors and unbounded vectors.*

In the applications X will consist of the limitable sequences for a matrix method. For the proof of Theorem 18.1 we need some notation and terminology. It is convenient to call a vector $\mathbf{y}$ *eventually constant* if the components y_k become constant from a certain index on. This is equivalent to saying that $\mathbf{y}$ is a *finite* linear combination of $\mathbf{e}$ and vectors $\mathbf{e_j}$; cf. Definition 16.1.

If $\mathbf{x}$ is a bounded vector, we set

$$q(\mathbf{x}) = \sup_{n \geq 0} |x_n|. \tag{18.1}$$

We let S_B, S_C and S_N denote the BK-spaces (Banach spaces with componentwise convergence) of, respectively, the bounded sequences, the convergent sequences and the null sequences, always under the norm q.

Several authors have investigated which conservative matrix methods limit no bounded sequences besides convergent ones; see for example Garling and Wilansky [1972]. This question led to abstract notions of *Tauberian operators*; cf. Kalton and Wilansky [1976]. A bounded linear operator T from a Banach space X to a Banach space Y may be called Tauberian if every bounded sequence $\{x_n\} \subset X$, for which $\{Tx_n\} \subset Y$ is weakly convergent, has a weakly convergent subsequence. More recent papers on Tauberian operators include Alvarez and González [1991], Cross [1992], Holub [1993], and Hernando [1998].

Derivation of Theorem 16.4 from Theorem 18.1. Let X be as in Theorem 16.4, that is, every vector

$$\mathbf{x} \in X \quad \text{has the form } \mathbf{x} = c\mathbf{e} + \sum_{n=0}^{\infty} x_n \mathbf{e_n}.$$

Thus $\mathbf{x}$ is a limit of eventually constant, hence convergent, vectors $\mathbf{y}$. Suppose now that X contains a divergent vector $\mathbf{x}$. Since $\mathbf{x}$ can be approximated by convergent vectors, the set of the convergent vectors fails to be closed in X. Thus by Theorem 18.1, X contains a bounded divergent vector.

Proof of Theorem 18.1 (Meyer-König and Zeller [1962]). We assume that X satisfies the conditions of the Theorem.

STEP 1. Let the topology in X be given by the sequence of seminorms q_j, $j = 0, 1, \ldots$. We will show that the seminorms q_j may be replaced by an equivalent system of seminorms p_k which satisfy the convenient condition (18.4) below.

Observe that in the FK-space X, the functional $\mathbf{x} \mapsto x_0$ is continuous. Then by a general result on continuity of seminorms, there are an integer $j_0 \geq 0$ and a constant $\lambda_0 > 0$ such that

$$|x_0| \leq \lambda_0\{q_0(\mathbf{x}) + \cdots + q_{j_0}(\mathbf{x})\}, \quad \forall \mathbf{x} \in X; \tag{18.2}$$

cf. Zeller [1951a] (p. 467). We give an independent proof. If (18.2) would be false, there would be vectors $\{\mathbf{x}^{(n)}\}$ and constants $0 < \lambda_0^{(n)} \to \infty$ such that

$$|x_0^{(n)}| > \lambda_0^{(n)}\{q_0(\mathbf{x}^{(n)}) + \cdots + q_n(\mathbf{x}^{(n)})\}, \quad n = 0, 1, \ldots.$$

Replacing each $\mathbf{x}^{(n)}$ by a suitable scalar multiple, one may assume that $|x_0^{(n)}| = 1$. But then $q_j(\mathbf{x}^{(n)}) < 1/\lambda_0^{(n)}$, $j = 0, \ldots, n$. Thus $\mathbf{x}^{(n)} \to \mathbf{0}$ in X, but $x_0^{(n)} \not\to 0$, a contradiction.

By the same argument, there are $j_1 \geq 0$ and $\lambda_1 > 0$ such that

$$|x_1| \leq \lambda_1\{q_0(\mathbf{x}) + \cdots + q_{j_1}(\mathbf{x})\}, \quad \forall \mathbf{x} \in X;$$

we may take $j_1 > j_0$ and $\lambda_1 > \lambda_0$. Continuing in this way, one obtains indices j_k, $0 \leq j_0 < j_1 < \cdots$ and constants λ_k, $0 < \lambda_0 < \lambda_1 < \cdots$ such that

$$|x_k| \leq \lambda_k\{q_0(\mathbf{x}) + \cdots + q_{j_k}(\mathbf{x})\}, \quad \forall \mathbf{x} \in X; \quad k = 0, 1, \ldots.$$

Setting

$$p_k(\mathbf{x}) = \lambda_k\{q_0(\mathbf{x}) + \cdots + q_{j_k}(\mathbf{x})\}, \quad k = 0, 1, \ldots, \tag{18.3}$$

one obtains a system of seminorms p_k which defines the same topology as the family $\{q_j\}$. Thus we may replace the system $\{q_j\}$ by the equivalent system $\{p_k\}$; cf. Zeller [1953b] (criterion 2.5 and section 7). It is clear that

$$p_0(\mathbf{x}) \leq p_1(\mathbf{x}) \leq \cdots; \quad |x_k| \leq p_k(\mathbf{x}), \quad \forall \mathbf{x} \in X; \quad k = 0, 1, \ldots. \tag{18.4}$$

STEP 2. We next show that *to every pair of numbers* (B, k), where $B > 0$ and $k \in \mathbb{N}_0$, *there is a vector* $\mathbf{x}$ *in* $X \cap S_C$ (hence a convergent vector) *such that* $q(\mathbf{x}) = \sup|x_n|$ is larger than $Bp_k(\mathbf{x})$.

Suppose to the contrary that there is a pair (B, k) such that

$$q(\mathbf{x}) \leq Bp_k(\mathbf{x}), \quad \forall \mathbf{x} \in X \cap S_C. \tag{18.5}$$

Since the set of the convergent vectors in X fails to be closed, there are vectors $\mathbf{x}_n \in X \cap S_C$ which converge in X to a vector $\mathbf{x}' \notin S_C$. The vectors $\mathbf{x}_n$ form a Cauchy sequence in the FK-space X, hence by (18.5), also in the BK-space S_C. Thus $\mathbf{x}_n$ converges to a vector $\mathbf{x}^*$ in S_C. The componentwise convergence implies that $\mathbf{x}^* = \mathbf{x}'$, which contradicts the condition $\mathbf{x}' \notin S_C$.

STEP 3. One may conclude that *to every pair* (ε, k), $\varepsilon > 0$, $k \in \mathbb{N}_0$, *there is a vector* $\mathbf{x}$ *in* $X \cap S_C$ *such that*

$$p_k(\mathbf{x}) < \varepsilon, \quad q(\mathbf{x}) = 20. \tag{18.6}$$

Indeed, by Step 2 with $B = 20/\varepsilon$, there is a vector $\mathbf{y} \in X \cap S_C$ such that $q(\mathbf{y}) > (20/\varepsilon)p_k(\mathbf{y})$. Since $q(\mathbf{y}) > 0$ one can determine $\lambda > 0$ such that $q(\lambda\mathbf{y}) = 20$. The vector $\mathbf{x} = \lambda\mathbf{y}$ will satisfy (18.6).

19 Bounded Divergent Sequences, Continued

In the continuation of the proof of Theorem 18.1 we will apply the method of the 'sliding hump'.

STEP 4. When ε is small, the vector $\mathbf{x}$ in Step 3 has small initial component(s) and one or more large 'middle' components; cf. (18.4). Hence the sequence $\mathbf{x}$ exhibits at least a 'one-sided hump'. We will show that one can find such an $\mathbf{x}$ in S_N (a null sequence $\mathbf{x}$). Then the components eventually become small again; the sequence of components exhibits a 'two-sided hump'. The desired vector $\mathbf{x}$ will be obtained by combining two convergent vectors $\mathbf{y}$ and $\mathbf{z}$ as in Step 3 in such a way that the result is a null sequence. If $\mathbf{y}$ already has large middle component(s) where $\mathbf{z}$ still has small components, the combination will have the required hump behavior. More specifically, we will show that *to every pair* (ε, k), *there is a vector* $\mathbf{x}$ *in* $X \cap S_N$ *such that*

$$p_k(\mathbf{x}) < \varepsilon, \quad q(\mathbf{x}) = 5. \tag{19.1}$$

For the proof, one may take $0 < \varepsilon \le 1$ and then choose $\mathbf{y}$ in $X \cap S_C$ such that $p_k(\mathbf{y}) < \varepsilon$ (hence $|y_0| < \varepsilon, \cdots, |y_k| < \varepsilon$) and $q(\mathbf{y}) = 20$. We also choose $n > k$ such that at least one of the numbers $|y_{k+1}|, \cdots, |y_n|$ is larger than 18. Then we construct, again as in Step 3, a vector $\mathbf{z} \in X \cap S_C$ for which $p_n(\mathbf{z}) < \varepsilon$ (so that $p_k(\mathbf{z}) < \varepsilon, |z_{k+1}| < \varepsilon, \cdots, |z_n| < \varepsilon$) and $q(\mathbf{z}) = 20$. We now take $\mathbf{x} = \lambda\mathbf{y} + \mu\mathbf{z}$, where λ and μ are chosen such that $\mathbf{x}$ is a *null sequence* (the vectors $\mathbf{y}$ and $\mathbf{z}$ are convergent!) and $|\lambda| + |\mu| = 1$. Then $p_k(\mathbf{x}) < \varepsilon$ and

$$\sup_{k<m\le n} |x_m| > \frac{1}{3}18 - \frac{2}{3}\varepsilon > 5 \quad \text{if } \frac{1}{3} \le \lambda \le 1,$$

$$\sup_{n<m<\infty} |x_m| > -\frac{1}{3}20 + \frac{2}{3}19 > 5 \quad \text{if } 0 \le \lambda < \frac{1}{3}.$$

It follows that $q(\mathbf{x}) > 5$. We adjust this inequality to $q(\mathbf{x}) = 5$ through multiplication of $\mathbf{x}$ by a suitable scalar.

STEP 5. After these preparations and following Meyer-König and Zeller [1962], we construct a bounded divergent vector $\mathbf{w} \in X$ by the method of the *sliding hump*; cf. Figure V.19. To $\varepsilon = \varepsilon_0 = 1/2^0$ and $k = k_0 = 0$ one forms a null sequence $\mathbf{x} = \mathbf{x}_0$ as in Step 4, so that $p_{k_0}(\mathbf{x}_0) < 1/2^0$ and $q(\mathbf{x}_0) = 5$. One also chooses a number $k_1 > k_0$ such that $|x_{0k}| < 1/2^0$ for $k \ge k_1$. Since we know that $|x_{0k}| \le p_k(\mathbf{x}_0) < 1/2^0$ for

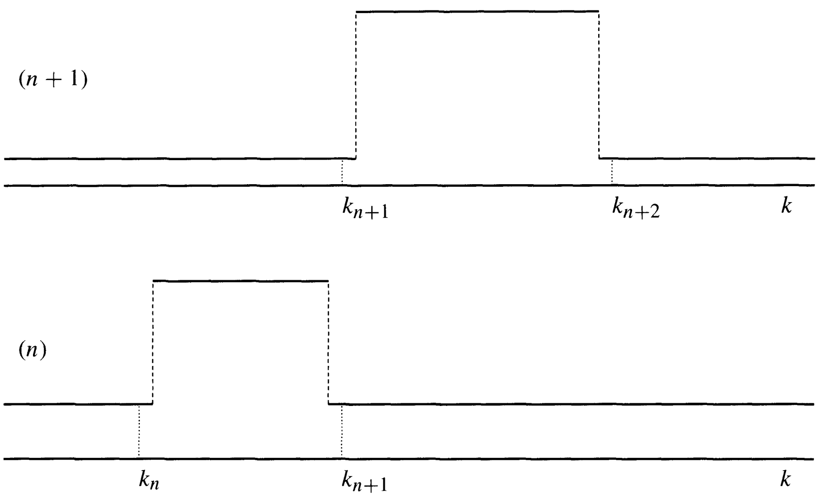

Fig. V.19. Majorants for $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$

$k \le k_0$ while $q(\mathbf{x}_0) = \sup |x_{0,k}| = 5$, one must have $k_1 \ge k_0 + 2$. To $\varepsilon_1 = 1/2^1$ and k_1 one next forms a null sequence $\mathbf{x} = \mathbf{x}_1$ as in Step 4, so that $p_{k_1}(\mathbf{x}_1) < 1/2^1$ and $q(\mathbf{x}_1) = 5$. One now chooses $k_2 > k_1$ such that $|x_{1k}| < 1/2^1$ for $k \ge k_2$; as a consequence, $k_2 \ge k_1 + 2$. Continuing in this manner, one obtains a sequence of null vectors $\mathbf{x}_n \in X$ and indices k_n such that

$$p_{k_n}(\mathbf{x}_n) < \frac{1}{2^n}, \quad q(\mathbf{x}_n) = 5, \quad |x_{nk}| < \frac{1}{2^n} \text{ for } k \ge k_{n+1} \ge k_n + 2. \tag{19.2}$$

As a result, cf. (18.4),

$$|x_{nk}| \le p_k(\mathbf{x}_n) < \frac{1}{2^n} \text{ for } k \le k_n, \qquad \max_{k_n < k < k_{n+1}} |x_{nk}| = 5. \tag{19.3}$$

The desired vector $\mathbf{w}$ is obtained as the sum $\mathbf{w} = \sum_{n=0}^{\infty} \mathbf{x}_n$. It follows from (19.3) that the series is convergent in the FK-space X. Indeed, for any given k and for n so large that $k_n \ge k$, one has $p_k(\mathbf{x}_n) < 1/2^n$.

The sequence $\mathbf{w} = \{w_k\}$ is *bounded*. To verify this, we look at a fixed value of k and the corresponding element $w_k = \sum_{n=0}^{\infty} x_{nk}$. We first take $k \in \{k_n\}$, so that $k = k_s$, say. Letting n run from 0 to ∞, the term x_{nk} will 'never meet a hump', so that $|x_{nk}| < 1/2^n$ for all n. Indeed, for $n \le s - 1$, so that $s \ge n + 1$, one has $k = k_s \ge k_{n+1}$, while for $n \ge s$ one has $k = k_s \le k_n$; the result now follows from (19.2) and (19.3). Hence

$$|w_k| = \left| \sum_{n=0}^{\infty} x_{nk} \right| < 2 \quad \text{for } k = k_s. \tag{19.4}$$

We next take $k \notin \{k_n\}$, so that $k_s < k < k_{s+1}$, say. For n running from 0 to ∞, the term x_{nk} will now meet exactly one hump, namely, for $n = s$, and then $|x_{nk}| \leq 5$. For all other values of n one has $|x_{nk}| < 1/2^n$. Indeed, for $n+1 \leq s$ one has $k > k_s \geq k_{n+1}$ and for $n \geq s+1$ one has $k < k_{s+1} \leq k_n$. Thus

$$|w_k| \leq |x_{s,k}| + \sum_{n \neq s} |x_{nk}| < 5 + 2 = 7 \quad \text{for } k_s < k < k_{s+1}. \tag{19.5}$$

We now show that the bounded sequence $\{w_k\}$ *fails to converge*. Indeed, for every integer $s \geq 0$, the interval (k_s, k_{s+1}) contains at least one integer k for which $|x_{sk}| = 5$; cf. (19.3). Selecting such a special number k one will have $|x_{nk}| < 1/2^n$ for $n \neq s$ as before. Hence for the special k

$$|w_k| \geq |x_{sk}| - \sum_{n \neq s} |x_{nk}| > 5 - 2 = 3. \tag{19.6}$$

This holds for infinitely many integers k; combination with (19.4) shows that the sequence $\{w_k\}$ is divergent.

STEP 6. Although we will make no use of the fact, we finally observe that X must contain unbounded vectors. Suppose to the contrary that $X \subset S_B$. Now let $\mathbf{x}^*$ be any vector in X which is a limit (in X) of vectors $\mathbf{x}_n \in X \cap S_C$. Then one also has $\mathbf{x}_n \to \mathbf{x}^*$ in the BK-space S_B; cf. Zeller [1951a] (theorem 4.5). Observe that S_C is closed in S_B, hence $\mathbf{x}^* \in S_C$, so that $\mathbf{x}^* \in X \cap S_C$. It would follow that $X \cap S_C$ is closed in X, but this contradicts the hypothesis of the Theorem.

Related more general questions on the existence of unbounded limitable sequences were treated by Kuttner and Parameswaran [1994], and Boos and Parameswaran [1997].

20 Gap Tauberian Theorems

Following Meyer-König and Zeller [1956], [1962], we consider sequence pairs

$$(\mathbf{x}, \mathbf{a}), \quad \text{where } x_n = a_0 + \cdots + a_n. \tag{20.1}$$

Thus the vector $\mathbf{x}$ will be eventually constant precisely if the series $\sum a_k$ breaks off. For a given matrix $\Gamma = [c_{kn}]$, the limitation method Γ always denotes the sequence to sequence method defined by Γ. The Γ-limitable sequences $\mathbf{x}$ constitute the effective domain W_Γ, here considered as an FK-space.

Definition 20.1. Let $\mathbf{n} = \{n_k\} = \{n_0, n_1, \cdots\}$ be an increasing sequence of non-negative integers. We say that $\mathbf{x}$ or $\mathbf{a}$ satisfies the *gap condition* $G(\mathbf{n})$ if $a_n = 0$ for $n \neq n_0, n_1, \cdots$.

A *gap Tauberian theorem* for Γ and $\mathbf{n}$ is a theorem which asserts the following: Vectors $\mathbf{x} \in W_\Gamma$ which satisfy the gap condition $G(\mathbf{n})$ are convergent. In this case the gap condition becomes a Tauberian condition for the limitation method Γ.

Definition 20.2. Let Γ be regular and let $\mathbf{n}$ be given. The method Γ is called $\mathbf{n}$ *gap-perfect* if every vector in W_Γ which satisfies condition $G(\mathbf{n})$ can be approximated in W_Γ by *eventually constant* vectors $\mathbf{x}$, which likewise satisfy condition $G(\mathbf{n})$.

A regular method Γ will be called *gap-perfect* if it is $\mathbf{n}$ gap-perfect for every sequence $\mathbf{n}$.

Theorem 20.3. *Let the method Γ be regular and $\mathbf{n}$ gap-perfect for a certain sequence $\mathbf{n}$ as in Definition 20.1. Suppose that the gap condition $G(\mathbf{n})$ is a Tauberian condition for bounded vectors in W_Γ. Then it is also a Tauberian condition for arbitrary vectors in W_Γ.*

Proof. The vectors in W_Γ which satisfy condition $G(\mathbf{n})$ form an FK-subspace $W_\mathbf{n}$ of W_Γ. By the hypothesis $W_\mathbf{n}$ contains no bounded divergent vectors. Thus by Theorem 18.1 the convergent vectors in $W_\mathbf{n}$ form a closed set. By the hypothesis that Γ is $\mathbf{n}$ gap-perfect, every vector in $W_\mathbf{n}$ can be approximated by convergent vectors. Hence there can be no divergent vectors in $W_\mathbf{n}$.

Theorem 20.4. *A regular matrix method Γ with sectional convergence is gap-perfect.*

Proof. Let $\mathbf{n}$ be any sequence $\{n_k\}$ as above, and let $W_\mathbf{n}$ be the subspace of W_Γ consisting of the elements $\mathbf{x}$ which satisfy the gap condition $G(\mathbf{n})$. These vectors have the form

$$\mathbf{x} = \{\overbrace{0, \cdots, 0}^{n_0}, \overbrace{x_{n_0}, \cdots, x_{n_0}}^{n_1-n_0}, \overbrace{x_{n_1}, \cdots, x_{n_1}}^{n_2-n_1}, x_{n_2}, \cdots\}.$$

It now follows from the sectional convergence property that the vectors in $W_\mathbf{n}$ can be approximated in W_Γ by eventually constant vectors $\mathbf{x}^{(m)}$ which also satisfy condition $G(\mathbf{n})$; cf. (17.4). Thus Γ is $\mathbf{n}$ gap-perfect: see Definition 20.2. Since $\mathbf{n} = \{n_k\}$ was arbitrary, the result follows.

Example 20.5. The Cesàro method $(C, 1)$ is regular and has sectional convergence (Example 17.3), hence it is gap-perfect.

The higher-order Cesàro methods (C, k) do not have sectional convergence, but they *are* gap-perfect; cf. Zeller [1953c], Meyer-König and Zeller [1956].

Definition 20.2 and Theorem 20.3 can be extended as follows.

Definition 20.6. Let Γ be regular and let $\mathbf{n}$ be given. The $\mathbf{n}$ *gap-perfect* part of W_Γ consists of those vectors in W_Γ, which can be approximated in W_Γ by *eventually constant* vectors $\mathbf{x}$ that satisfy condition $G(\mathbf{n})$.

Every vector in the $\mathbf{n}$ gap-perfect part of W_Γ will satisfy condition $G(\mathbf{n})$.

Theorem 20.7. *Let the method Γ be regular, and let the gap condition $G(\mathbf{n})$ be a Tauberian condition for bounded vectors in W_Γ. Then it is also a Tauberian condition for arbitrary vectors in the $\mathbf{n}$ gap-perfect part of W_Γ.*

The proof is similar to the proof of Theorem 20.3. One need only replace $W_\mathbf{n}$ by the $\mathbf{n}$ gap-perfect part of W_Γ.

21 The Abel Method

So far we have only considered sequence to sequence transformations, transformations defined by matrices $\Gamma = [c_{kn}]$, $k, n = 0, 1, \ldots$. However, the theory is also valid for certain transformations which correspond to the case of a continuous parameter k. Theorem 20.3 can be extended to every method whose effective domain is an FK-space; cf. Sections 18, 19. An important example is provided by the *Abel transformation*. Letting $\mathbf{x}$ denote the sequence $\{x_n\}_0^\infty$ and setting $x_{-1} = 0$, the transformation is given by the function

$$F(\mathbf{x}, \rho) = \sum_{n=0}^{\infty} \rho^n (x_n - x_{n-1}), \quad 0 \le \rho < 1. \tag{21.1}$$

The effective domain W_F consists of the Abel limitable vectors $\mathbf{x}$: the vectors $\mathbf{x}$ for which $F(\mathbf{x}, \rho)$ is well-defined, and tends to a limit A as $\rho \nearrow 1$. If we set

$$x_n - x_{n-1} = a_n, \tag{21.2}$$

the associated series $\sum_0^\infty a_n$ is Abel summable to A:

$$F(\mathbf{x}, \rho) = \sum_{n=0}^{\infty} a_n \rho^n \to A \quad \text{as } \rho \nearrow 1. \tag{21.3}$$

We know that the Abel method is regular. Its effective domain W_F can be made into an FK-space with the aid of the seminorms (norms, actually)

$$\|\mathbf{x}\|_0 = \sup_{0 \le \rho < 1} |F(\mathbf{x}, \rho)|, \quad \|\mathbf{x}\|_k = \sum_{n=0}^{\infty} |a_n| e^{-n/k}, \quad k = 1, 2, \ldots. \tag{21.4}$$

Cf. Zeller [1953a], Włodarski [1955].

Theorem 21.1. (Meyer-König and Zeller [1956]) *The Abel method is gap-perfect.*

Proof. Let $\mathbf{x}$ be in W_F and satisfy the gap condition $G(\mathbf{n})$ for some sequence $\mathbf{n}$ (Definition 20.1). Taking $0 \le r < 1$ and a_n as in (21.2), we define

$$\mathbf{x}(r) = \{x_0(r), x_1(r), x_2(r), \cdots\} = \{a_0, a_0 + a_1 r, a_0 + a_1 r + a_2 r^2, \cdots\}. \tag{21.5}$$

Notice that $\mathbf{x}(r)$ is in W_F and also satisfies the gap condition $G(\mathbf{n})$:

$$x_n(r) - x_{n-1}(r) = a_n r^n = 0 \quad \text{when } n \notin \{n_k\}. \tag{21.6}$$

Assuming, as we may, that the Abel limit A of $\mathbf{x}$ is equal to 0, we will show below that under the seminorms (21.4)

$$\mathbf{x}^{(m)}(r) = \{x_0(r), \cdots, x_m(r), 0, 0, \cdots\} \to \mathbf{x}(r) \quad \text{as } m \to \infty, \tag{21.7}$$

$$\mathbf{x}(r) \to \mathbf{x} \quad \text{as } r \nearrow 1. \tag{21.8}$$

It will follow that $\mathbf{x}$ can be approximated in the domain W_F by 'finite' vectors $\mathbf{x}^{(m)}(r)$ that satisfy the gap condition $G(\mathbf{n})$. The conclusion will be that the Abel method is $\mathbf{n}$ gap-perfect for arbitrary sequences $\mathbf{n}$, hence gap-perfect.

We first establish (21.8). One has

$$\|\mathbf{x}-\mathbf{x}(r)\|_0 = \sup_{0\le\rho<1} |F(\mathbf{x},\rho) - F(\mathbf{x}(r),\rho)|$$

$$= \sup_\rho \left| \sum_{n=0}^{\infty} a_n\rho^n - \sum_{n=0}^{\infty} a_n r^n\rho^n \right| = \sup_\rho |F(\mathbf{x},\rho) - F(\mathbf{x}, r\rho)|.$$

Setting $F(\mathbf{x},1) = \lim_{\rho\nearrow 1} F(\mathbf{x},\rho) = A = 0$, the function $F(\mathbf{x},\rho)$ becomes continuous for $0\le\rho\le 1$, hence uniformly continuous. Thus $\|\mathbf{x}-\mathbf{x}(r)\|_0 \to 0$ as $r\nearrow 1$. The case $k\ge 1$ is easier: by dominated convergence,

$$\|\mathbf{x}-\mathbf{x}(r)\|_k = \sum_{n=0}^{\infty} |a_n(1-r^n)| e^{-n/k} \to 0 \quad \text{as } r\nearrow 1.$$

Finally, the proof of (21.7). It will be sufficient to consider the 0-norm:

$$\|\mathbf{x}(r) - \mathbf{x}^{(m)}(r)\|_0 = \sup_\rho |F(\mathbf{x}(r),\rho) - F(\mathbf{x}^{(m)}(r),\rho)|$$

$$= \sup_\rho \left| \sum_{n=m+1}^{\infty} a_n r^n \rho^n \right| \le \sum_{n=m+1}^{\infty} |a_n| r^n.$$

For fixed $r\in[0,1)$, the final sum tends to 0 as $m\to\infty$.

The Tauberian theorem for Abel summable power series with Hadamard gaps (which is contained in the high-indices theorem of Hardy and Littlewood, Theorem I.23.1) is an easy consequence:

Theorem 21.2. *Let the increasing sequence $\mathbf{n} = \{n_k\}_{k=0}^{\infty}$ of nonnegative integers be such that for some number $\alpha > 0$,*

$$n_{k+1} - n_k \ge \alpha n_k, \quad \forall k. \tag{21.9}$$

Suppose that $\sum_0^\infty a_n$ is Abel summable and that $a_n = 0$ for $n\notin\{n_k\}$. Then the series $\sum a_n$ is convergent.

Proof. By the preceding we may apply an analog of Theorem 20.3 to Abel limitable vectors such as the vector $\mathbf{x}$, given by $x_n = a_0 + \cdots + a_n$, $n = 0, 1, \ldots$. Thus it is sufficient to show that the gap condition $G(\mathbf{n})$, in which $\mathbf{n}$ satisfies (21.9), is a Tauberian condition for *bounded* Abel limitable vectors $\mathbf{s}$. Now such vectors $\mathbf{s}$ are also Cesàro limitable; cf. Theorem I.7.3. We have to show that they converge.

Suppose then that $\mathbf{s} = \{s_n\}$ satisfies condition $G(\mathbf{n})$ and is Cesàro limitable to A. Subtracting A from every s_n we may assume that $A = 0$. Then

$$s_n^{(-1)} = s_0 + s_1 + \cdots + s_n = o(n).$$

Now for $h_k = [\alpha n_k]$, the integral part of αn_k, it follows from (21.9) that

$$s_{n_k} = s_{n_k+1} = \cdots = s_{n_k+h_k-1},$$

hence

$$h_k s_{n_k} = s^{(-1)}_{n_k+h_k-1} - s^{(-1)}_{n_k-1} = o(n_k).$$

Conclusion: $s_{n_k} = o(1)$, so that the sequence $\{s_n\}$ converges to 0.

Remarks 21.3. A recent paper involving Theorem 21.2 is Tietz and Zeller [1999c]. Since Cesàro limitability of any order implies Abel limitability, it follows from Theorem 21.2 that the gap condition $G(\mathbf{n})$, with $\mathbf{n}$ as in (21.9), is a Tauberian condition for (C, k) limitability. Several authors have given independent proofs and obtained refinements; cf. Meyer-König [1939]. In an 'opposite direction', Krishnan [1975] showed (by Wiener–Pitt theory) that the same gap condition works for the generalized Abel summability introduced by D. Borwein [1957].

In [1956] Meyer-König and Zeller also gave a functional-analytic proof of the general high-indices theorem for Dirichlet series (Theorem I.23.1); a further extension is in their paper [1960a]. In the 1956 article they showed, furthermore, that the Euler method (Examples 16.3) is gap-perfect. Thus they could prove a gap Tauberian theorem for Euler summability, in which the 'right' gaps are *square-root gaps*:

$$n_{k+1} - n_k \geq \alpha\sqrt{n_k} \quad \text{with} \quad \alpha > 0. \tag{21.10}$$

We will derive this result in a different manner; see Section VI.15. Using Theorem 20.7 and appropriate approximation, Meyer-König and Zeller also obtained a restricted analog for Borel summability; cf. Section VI.15.

Certain classical Tauberian theorems, such as Littlewood's theorem, do not seem to lend themselves well to treatment by the present methods of functional analysis. Many authors have investigated what Tauberian conditions are amenable to the functional-analytic approach. Here we mention articles by Meyer-König and Zeller [1978], [1981], Jakimovski, Meyer-König and Zeller [1981], [1987], Connor [1993], and Tietz and Zeller [1998c], [1999a], [1999b]. Additional references for circle methods will be given in Chapter VI.

22 Recurrent Events

We end the chapter with some Tauberian theorems of different character. The first result concerns so-called recurrent events, which we introduce by a classical example.

Let us consider the 'events E', described by the (failure and prompt) replacement of a lightbulb, say the bulb at the front door (which is supposed to be left on all the time). Our informal discussion is based on a model for the time-pattern $\mathcal{E}$ of these events E. We assume that our lightbulbs have a finite (but variable) positive lifetime, measured in suitable units ('days'). A new bulb is installed on day $n = 0$ and again on every day that a bulb fails. Let p_n be the probability that a bulb fails after n days, so that

$$\sum_{n=0}^{\infty} p_n = 1, \quad p_n \geq 0, \quad p_0 = 0. \tag{22.1}$$

(We assume the same probability distribution for all bulbs.) Now let q_n denote the probability that day n is a day of replacement, the day of an 'event'; by our earlier assumption, $q_0 = 1$. It is assumed that the life-times of the bulbs are 'independent random variables'. Hence if there is an event on day k, the probability that there is an event on day $k + n$ is always equal to q_n.

An important role is played by the generating functions

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, \quad Q(z) = \sum_{n=0}^{\infty} q_n z^n \qquad (|z| < 1). \tag{22.2}$$

Now let $n \geq 1$. The probability that the event E occurs for the first time on day k, $1 \leq k < n$ and again on day $n > k$ is (by our assumption) equal to $p_k q_{n-k}$. The probability that E occurs for the first time on day n is $p_n = p_n q_0$. We have mutually exclusive cases here for $1 \leq k \leq n$, hence

$$q_n = p_1 q_{n-1} + p_2 q_{n-2} + \cdots + p_n q_0, \quad \forall n \geq 1. \tag{22.3}$$

The convolution on the right has generating function $P(z)Q(z)$ because we supposed $p_0 = 0$. Since $q_0 = 1$, the generating function for the left-hand side equals $Q(z) - 1$. Equating the two and solving for Q, one obtains the basic relation

$$Q(z) = \frac{1}{1 - P(z)}. \tag{22.4}$$

The life expectancy of a bulb, or the mean 'recurrence time' in the pattern $\mathcal{E}$ of the events E, is equal to

$$\mu = \sum_{n=1}^{\infty} n p_n. \tag{22.5}$$

For our lightbulbs we may assume that this quantity is finite, but in general μ may be infinite. In our model we exclude the so-called periodic case, where events take place only on days $0, m, 2m, 3m, \ldots$ for some $m \geq 2$. The following important theorem of *integer renewal theory* was obtained by Erdős, Feller and Pollard [1949].

Theorem 22.1. *Let the recurrent pattern of events E with the properties described by the formulas above be non-periodic, and let μ be the mean recurrence time* (22.5). *Then the probability q_n that an event occurs at time n has limit $1/\mu$ as $n \to \infty$, where $1/\mu$ must be read as* 0 *when $\mu = \infty$.*

There are many proofs for this theorem. An equivalent result occurred in work of Kolmogorov [1936] on Markov chains; the proof in Section 23 is based on 'Wiener's Lemma'. Blackwell [1948], [1953] treated the renewal problem for the non-lattice case. Other early contributions are Chung and Wolfowitz [1952], and Chung and Pollard [1952]. See also Feller [1950/68] (chapter 13), [1966/71] (chapter 11), and Bingham, Goldie and Teugels [1987] (chapter 8). It is interesting that Wiener's Tauberian theorems have played an important role in the analytic approach to general renewal theory; cf. Smith [1954], Karlin [1955], and Bingham [1989].

23 The Theorem of Erdős, Feller and Pollard

We begin by stating Theorem 22.1 in its original 'Tauberian form', which is independent of probability theory.

Theorem 23.1. *Let $\{p_n\}$ be any sequence of nonnegative numbers with the property that $\sum_0^\infty p_n = 1$. Set $\sum_1^\infty np_n = \mu \le \infty$. Suppose that*

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \tag{23.1}$$

is not a power series in z^m for any integer $m \ge 2$; in particular $P(z) \not\equiv 1$. Then $1 - P(z)$ has no zeros in the unit disc $\{|z| < 1\}$, and the coefficients in the series

$$Q(z) = \frac{1}{1 - P(z)} = \sum_{n=0}^{\infty} q_n z^n \tag{23.2}$$

satisfy the limit relation

$$\lim q_n = 1/\mu \qquad (= 0 \ \textit{if} \ \mu = \infty). \tag{23.3}$$

Proof. Following Erdős, Feller and Pollard [1949], we treat *the case $\mu < \infty$* as a very nice application of 'Wiener's Lemma' on absolutely convergent Fourier series (Theorem 5.3). Let

$$r_n = \sum_{k=n+1}^{\infty} p_k \quad (n \ge -1), \qquad R(z) = \sum_{n=0}^{\infty} r_n z^n. \tag{23.4}$$

Then $\sum_{n=0}^\infty r_n = \sum_{k=1}^\infty kp_k = \mu$ and

$$(1 - z)R(z) = r_{-1} + \sum_{n=0}^{\infty} (r_n - r_{n-1})z^n = 1 - P(z). \tag{23.5}$$

Since $\mu < \infty$ the power series for $R(z)$ converges absolutely and uniformly for $|z| \le 1$. We will show that $R(z)$ has *no zeros* for $|z| \le 1$.

Since $P(z) \not\equiv 1$ one has $p_k > 0$ for at least one number $k \ge 1$; it follows that $|P(z)| = |\sum p_n z^n| < \sum p_n = 1$ for $|z| < 1$. Thus any zeros of $R(z)$ in the closed unit disc must occur on the circumference. Now $R(1) = \mu > 0$. Suppose for a moment that $R(e^{i\alpha}) = 0$, where $0 < \alpha < 2\pi$. Then by (23.5)

$$\sum_0^\infty p_k = 1 = P(e^{i\alpha}) = \sum_0^\infty p_k e^{ik\alpha} = \sum_0^\infty p_k \cos k\alpha.$$

Thus $\cos k\alpha = 1$ whenever $p_k > 0$. For such $k \ge 1$ one must have $k\alpha = \nu_k \cdot 2\pi$, where ν_k is a positive integer. Hence α is a rational multiple of 2π, $\alpha = (h/m)2\pi$, with

relatively prime positive integers h, m and $m \geq 2$. Thus $m\nu_k = mk\alpha/(2\pi) = kh$, so that m must divide k. That is, every number $k \geq 1$ for which $p_k > 0$ is a multiple of m, contradicting the hypothesis that $P(z)$ cannot be written as a power series in z^m with $m \geq 2$.

By the preceding, the reciprocal $1/R(z)$ is analytic for $|z| < 1$ and continuous for $|z| \leq 1$. It can be expanded in a power series for $|z| < 1$:

$$\frac{1}{R(z)} = \frac{1-z}{1-P(z)} = \sum_{n=0}^{\infty} b_n z^n. \tag{23.6}$$

Here the coefficients b_n are given by Cauchy's formula,

$$b_n = \frac{1}{2\pi i} \int_{C(0,\rho)} \frac{1}{R(z)} z^{-n-1} dz,$$

where one would normally take $\rho < 1$. However, by the continuity of $1/R(z)$ for $|z| \leq 1$, the circle of integration may be changed to $C(0, 1)$. Writing $z = e^{it}$, the conclusion is that the numbers b_n are the *Fourier coefficients* of $1/R(e^{it})$:

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R(e^{it})} e^{-int} dt. \tag{23.7}$$

We have seen already that the Fourier series $R(e^{it}) = \sum_0^\infty r_n e^{int}$ is absolutely convergent and that $R(e^{it}) \neq 0$ for all real t. Hence by Wiener's Lemma, the Fourier series of the reciprocal $1/R(e^{it})$ is absolutely convergent, $\sum_0^\infty |b_n| < \infty$. Relation (23.6) shows that $\sum_0^\infty b_n = 1/R(1) = 1/\mu$. Now by (23.6) and (23.2)

$$\sum_0^\infty b_n z^n = (1-z) \sum_0^\infty q_n z^n,$$

so that $b_n = q_n - q_{n-1}$ for $n \geq 1$ and $b_0 = q_0$. Hence

$$q_n = \sum_{k=0}^{n} b_k \to 1/\mu \quad \text{as} \quad n \to \infty.$$

This completes the proof for the case $\mu < \infty$.

The case $\mu = \infty$ requires a separate argument, for which we refer to the literature; the original paper by Erdős, Feller and Pollard includes an 'elementary proof' for both cases. It is an *open problem* if one could extend the complex-analytic argument to the case $\mu = \infty$. Of course, the interpretation of q_n in Section 22, as the probability that day n is a day on which an event occurs, makes it plausible that $q_n \to 0$ if the mean recurrence time μ of the events is infinite!

Probabilistic proofs of Theorem 23.1 may be found in Feller [1950/68] (chapter 15), and Grimmett and Stirzaker [1992].

Several authors have studied the speed with which $q_n \to 0$ under various 'moment conditions'. We mention C. Stone [1965], Ganelius [1971] (section 8.3), and refer to Postnikov [1980] (section 24) for additional information.

24 Milin's Theorem

Milin [1970], [1971] has found an unusual Tauberian theorem which involves exponentiation of power series. He used the result to give a simpler proof for the beautiful regularity theorem of Hayman [1953], [1955] which asserts the following: If $F(z) = z + c_2 z^2 + \cdots$ is a univalent or one-to-one holomorphic function in the unit disc, then $|c_n|/n$ tends to a limit γ (≤ 1) as $n \to \infty$; cf. Duren [1983]. Hayman's theorem is related to, but not implied by, the famous inequality $|c_n| \leq n$, which was conjectured by Bieberbach in 1916 and established by de Branges [1985].

For Milin's theorem, let

$$f(z) = \sum_0^\infty a_n z^n = e^{g(z)}, \quad g(z) = \sum_1^\infty b_n z^n, \quad |z| < 1. \tag{24.1}$$

As usual we write

$$s_n = \sum_{k=0}^n a_k, \quad \sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n}. \tag{24.2}$$

Theorem 24.1. (Milin) *Let $\sum_1^\infty n|b_n|^2 < \infty$ and let f and g be as above. Suppose that $|f(x)| \to \alpha \geq 0$ as $x \nearrow 1$. Then*

$$|s_n| \to \alpha \ \textit{and} \ |\sigma_n| \to \alpha \ \textit{as} \ n \to \infty. \tag{24.3}$$

Using the tool of 'vanishing mean oscillation', Holland [1988] has shown that the 'area condition' $\sum_1^\infty n|b_n|^2 < \infty$ can be relaxed to $\sum_1^n k|b_k|^2 = o(n)$ as $n \to \infty$. For the proof of Theorem 24.1, which will be completed in Section 26, we need one of the powerful inequalities which Lebedev and Milin developed for the theory of univalent functions; cf. Milin or Duren (loc. cit.).

Proposition 24.2. *Let f and g be as in* (24.1). *Then*

$$|a_n|^2 \leq \exp\left\{\sum_{k=1}^n \left(k|b_k|^2 - \frac{1}{k}\right)\right\}. \tag{24.4}$$

Proof. Differentiation of the relation $f(z) = e^{g(z)}$ gives $f'(z) = f(z)g'(z)$, which implies

$$na_n = \sum_{j=0}^{n-1} a_j(n-j)b_{n-j}, \quad \text{where } a_0 = 1. \tag{24.5}$$

It is convenient to set

$$A_n = \sum_{j=0}^n |a_j|^2, \quad B_n = \sum_{k=1}^n k^2|b_k|^2 \quad (B_0 = 0). \tag{24.6}$$

Cauchy-Schwarz applied to (24.5) now gives

$$n^2|a_n|^2 \leq \sum_{j=0}^{n-1} |a_j|^2 \sum_{k=1}^n k^2|b_k|^2 = A_{n-1}B_n. \tag{24.7}$$

It follows that

$$\begin{aligned}
A_n &= A_{n-1} + |a_n|^2 \le \left(1 + \frac{1}{n^2} B_n\right) A_{n-1} \\
&= \frac{n+1}{n}\left(1 + \frac{B_n - n}{n(n+1)}\right) A_{n-1} \le \frac{n+1}{n} \exp\left(\frac{B_n - n}{n(n+1)}\right) A_{n-1} \\
&\le (n+1)\exp\left(\sum_{k=1}^{n} \frac{B_k - k}{k(k+1)}\right), \quad \text{since } A_0 = 1. \qquad (24.8)
\end{aligned}$$

Applying partial summation to the exponent, one finds that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{B_k - k}{k(k+1)} &= \sum_{k=1}^{n} (B_k - k)\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \sum_{k=1}^{n} (B_k - k)\frac{1}{k} - \sum_{k=2}^{n+1} \{B_{k-1} - (k-1)\}\frac{1}{k} \\
&= \sum_{k=1}^{n} \left(k|b_k|^2 - \frac{1}{k}\right) - \frac{B_n - n}{n+1} \\
&= \sum_{k=1}^{n} \left(k|b_k|^2 - \frac{1}{k}\right)\left(1 - \frac{k}{n+1}\right) = \sum_{k=1}^{n+1} \cdots. \qquad (24.9)
\end{aligned}$$

For the next step we combine (24.8) and (24.9), but with $n-1$ instead of n:

$$\log \frac{A_{n-1}}{n} \le \sum_{k=1}^{n} \left(k|b_k|^2 - \frac{1}{k}\right)\left(1 - \frac{k}{n}\right) = \sum_{k=1}^{n} \left(k|b_k|^2 - \frac{1}{k}\right) - \frac{B_n}{n} + 1. \quad (24.10)$$

Inequalities (24.7) and (24.10) thus give

$$|a_n|^2 \le \frac{A_{n-1}}{n}\frac{B_n}{n} \le \exp\left\{\sum_{k=1}^{n} \left(k|b_k|^2 - \frac{1}{k}\right)\right\} \frac{B_n}{n} \exp\left(-\frac{B_n}{n} + 1\right).$$

Observing that $ye^{-y} \le 1/e$ one obtains (24.4).

25 Some Propositions

We continue with the notation introduced in (24.1), (24.2) and suppose that the hypotheses of Theorem 24.1 are satisfied. For the proof of the Theorem we need additional auxiliary results.

Proposition 25.1. *Let f and g be as in Theorem 24.1. Then for $N \to \infty$,*

$$\sum_{n=1}^{N} \operatorname{Re} b_n \to \log \alpha \quad (\text{to be read as } -\infty \text{ if } \alpha = 0). \qquad (25.1)$$

Proof. Take $0 < x < 1$. Then

$$\operatorname{Re} g(x) = \sum_{n=1}^{\infty} (\operatorname{Re} b_n) x^n = \log |f(x)| \to \log \alpha \quad \text{as } x \nearrow 1.$$

Also, $\sum_{n=1}^{\infty} n |\operatorname{Re} b_n|^2 < \infty$. Conclusion (25.1) now follows from Fejér's Tauberian theorem (Remarks I.5.5) or Tauber's 'second theorem', Theorem I.5.3.

Proposition 25.2. *Let f and g be as in Theorem 24.1, and let s_n and σ_n be as in* (24.2). *Then the sequence $\{s_n\}$ is bounded and*

$$s_n - \sigma_n \to 0 \quad \text{as } n \to \infty. \tag{25.2}$$

Proof. One has

$$s_n - \sigma_n = \frac{n s_n - (s_0 + s_1 + \cdots + s_{n-1})}{n} = \frac{1}{n} \sum_{k=0}^{n} k a_k. \tag{25.3}$$

Using (24.5) in the form $k a_k = \sum_{j=0}^{k} a_{k-j} j b_j$, one derives that

$$\begin{aligned} s_n - \sigma_n &= \frac{1}{n} \sum_{k=0}^{n} \sum_{j=0}^{k} a_{k-j} j b_j = \frac{1}{n} \sum_{j=0}^{n} j b_j \sum_{k=j}^{n} a_{k-j} \\ &= \frac{1}{n} \sum_{j=0}^{n} j b_j s_{n-j}. \end{aligned}$$

Hence by Cauchy-Schwarz and the convergence of $\sum_{j=0}^{\infty} j |b_j|^2$,

$$|s_n - \sigma_n|^2 \le \frac{1}{n^2} \sum_{j=0}^{n} j^2 |b_j|^2 \sum_{k=0}^{n} |s_k|^2 = o\left(\frac{1}{n}\right) \sum_{k=0}^{n} |s_k|^2. \tag{25.4}$$

(To verify the final step one may take $n > m$ and write

$$\sum_{j=0}^{n} j^2 |b_j|^2 \le \sum_{j \le m} j^2 |b_j|^2 + n \sum_{m < j \le n} j |b_j|^2.)$$

We will now apply the Lebedev-Milin inequality of Proposition 24.2 to the series

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} \sum_{n=0}^{\infty} a_n z^n = \exp\left\{ \sum_{n=1}^{\infty} \left(b_n + \frac{1}{n} \right) z^n \right\}.$$

That is, we have to replace a_n in (24.4) by s_n and b_k by $b_k + 1/k$. The result is

$$\begin{aligned} |s_n|^2 &\le \exp\left\{ \sum_{k=1}^{n} \left(k \left| b_k + \frac{1}{k} \right|^2 - \frac{1}{k} \right) \right\} \\ &\le \exp\left\{ \sum_{k=1}^{n} k |b_k|^2 + 2 \sum_{k=1}^{n} \operatorname{Re} b_k \right\}. \end{aligned} \tag{25.5}$$

Thus by the hypotheses and Proposition 25.1, the sequence $\{s_n\}$ is bounded. Relation (25.2) now follows from (25.4).

Proposition 25.3. *Let f and g be as in Theorem 24.1 and let $\{s_n\}$ be as in* (24.2). *Then the sequence $\{s_n\}$ is very slowly oscillating in the following sense. For every $\varepsilon > 0$, there is a constant B such that*

$$|s_q - s_p| < \varepsilon \quad \textit{whenever} \quad B \le p \le q \le 2p. \tag{25.6}$$

Proof. By Proposition 25.2 the sequence $\{s_n\}$ is bounded, $|s_n| \le M$, say. The averages σ_n of (24.2) will have the same bound. For given $\varepsilon > 0$, we take B so large that $|s_n - \sigma_n| < \varepsilon$ for $n \ge B$; cf. Proposition 25.2.

Now let $B \le p \le q \le 2p$. Then for some number $\theta = \theta(p, \varepsilon)$ with $|\theta| \le 1$,

$$\sigma_q = \frac{s_0 + \cdots + s_{p-1} + s_p + \cdots + s_{q-1}}{q} = \sigma_p \frac{p}{q} + \theta M \frac{q-p}{q}. \tag{25.7}$$

Hence

$$|\sigma_q - \sigma_p| \le \frac{q-p}{q}(|\sigma_p| + M) \le \frac{q-p}{p} 2M,$$
$$|s_q - \sigma_p| \le \frac{q-p}{p} 2M + \varepsilon.$$

Using the final inequality, but with q replaced by n, in (25.7), one finds that for certain numbers θ with $|\theta| \le 1$,

$$\begin{aligned}\sigma_q &= \sigma_p \frac{p}{q} + \frac{1}{q}\sum_{n=p}^{q-1} s_n = \sigma_p + \frac{1}{q}\sum_{n=p}^{q-1}(s_n - \sigma_p)\\ &= \sigma_p + \frac{\theta}{q}\sum_{n=p}^{q-1}\left(\frac{n-p}{p}2M + \varepsilon\right)\\ &= \sigma_p + \frac{\theta}{q}\left\{\frac{(q-p)^2}{2p}2M + (q-p)\varepsilon\right\},\end{aligned}$$

so that

$$|s_q - \sigma_p| \le \frac{(q-p)^2}{2p^2}2M + \frac{q-p}{p}\varepsilon + \varepsilon.$$

Continuing in this manner, one arrives at the inequalities

$$\begin{aligned}|s_q - \sigma_p| &\le \frac{(q-p)^r}{r!p^r}2M + \frac{(q-p)^{r-1}}{(r-1)!p^{r-1}}\varepsilon + \cdots + \frac{q-p}{p}\varepsilon + \varepsilon\\ &\le \frac{2M}{r!} + e\varepsilon,\end{aligned}$$
$$|s_q - s_p| \le \frac{2M}{r!} + (e+1)\varepsilon. \tag{25.8}$$

For sufficiently large r, one thus obtains (25.6), with 5ε instead of ε.

26 Proof of Milin's Theorem

Let f and g as in (24.1) satisfy the hypotheses of Theorem 24.1 and let $s_n = \sum_{k=0}^{n} a_k$. Then

$$|f(x)| = \left|\sum_0^\infty a_n x^n\right| = \left|\sum_0^\infty s_n(x^n - x^{n+1})\right| \to \alpha \quad \text{as } x \nearrow 1. \tag{26.1}$$

By Propositions 25.2 and 25.3, the sequence $\{s_n\}$ is bounded and s_n is nearly constant on intervals $p \le n \le q = mp$. More precisely, $|s_n| \le M$, say, and for given $\varepsilon \in (0, 1)$ and integer $m > 1$, there is a number $B = B(\varepsilon, m)$ such that

$$|s_n - s_p| < \varepsilon \quad \text{whenever} \quad B \le p \le n \le q = mp.$$

For our ε we fix m such that $(1-\varepsilon)^m \le \varepsilon$.

For $p \ge B$ we now take x such that $x^p = 1 - \varepsilon$, so that $x^q = x^{mp} \le \varepsilon$. Then

$$\begin{aligned}
&|f(x) - s_p(x^p - x^q)| \\
&= \left|\sum_{n=0}^{p-1} s_n(x^n - x^{n+1}) + \sum_{n=p}^{q-1}(s_n - s_p)(x^n - x^{n+1}) + \sum_{n=q}^{\infty} s_n(x^n - x^{n+1})\right| \\
&\le \sum_{n=0}^{p-1} M(x^n - x^{n+1}) + \sum_{n=p}^{q-1} \varepsilon(x^n - x^{n+1}) + \sum_{n=q}^{\infty} M(x^n - x^{n+1}) \\
&= M(1 - x^p + x^q) + \varepsilon(x^p - x^q) \le (2M+1)\varepsilon.
\end{aligned}$$

It follows that for some number $\theta \in (0, 1)$,

$$|s_p(1 - 2\theta\varepsilon) - f(x)| \le (2M+1)\varepsilon. \tag{26.2}$$

For given ε this holds for all large p and corresponding $x = (1-\varepsilon)^{1/p}$. Note that x is close to 1 when ε is small. Hence (26.1) and (26.2) imply that $|s_p|$ is close to α for all large p. □

VI

Borel Summability and General Circle Methods

1 Introduction

Émile Borel published extensively on summability theory, emphasizing applications to analytic continuation. We mention a few of his papers: [1895], [1899], [1901] and his book [1901/28]. Hardy contributed to the development of Borel's methods in [1904a], [1904b] and [1911b].

For Tauberian theory, Borel's most important method is the so-called *exponential method*. Accordingly, we will say that a series $\sum_{n=0}^{\infty} a_n$, with partial sums $s_n = \sum_{k=0}^{n} a_k$, is *Borel-* or B-summable to A if the corresponding series $\sum_{n=0}^{\infty} s_n x^n/n!$ converges for all x and

$$\frac{\sum\limits_{n=0}^{\infty} s_n x^n/n!}{\sum\limits_{n=0}^{\infty} x^n/n!} = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} e^{-x} \to A \quad \text{as } x \to \infty. \tag{1.1}$$

The *Tauberian theory* for Borel summability started with Hardy and Littlewood [1913a]. Parallel to Littlewood's Tauberian condition $|a_n| \le C/n$ for the case of Abel summability, Hardy and Littlewood soon [1916] found the optimal order condition $|a_n| \le C/\sqrt{n}$ for the implication

$$\textit{Borel summability} \;\Rightarrow\; \textit{convergence};$$

cf. Theorem I.9.1. It was followed by a broader condition of Valiron [1917] on the partial sums s_n (Section 11). The one-sided condition for convergence which is now standard in 'Borel Tauberian theory' is

$$\liminf\,(s_m - s_n) \ge 0 \quad \text{as } n \to \infty \text{ and } 0 \le \sqrt{m} - \sqrt{n} \to 0 \tag{1.2}$$

(special case: $a_k \ge -C/\sqrt{k}$). This is R. Schmidt's analog [1925b] of his earlier condition of 'slow decrease' for the case of Abel summability, where the requirement on m and n was $0 \le (m-n)/n \to 0$ instead of (what is essentially) $0 \le (m-n)/\sqrt{n} \to 0$.

Schmidt's complicated proof was simplified by Vijayaraghavan [1928], who used his 'method of the monotone minorant' (Section I.19) to derive boundedness of the partial sums; see Section 8.

A few years later, Wiener [1932] would incorporate the 'Borel Tauberian' in his general theory; cf. Section 12. As an interesting alternative to Wiener's method, Hardy and Littlewood [1943] used Vitali's theorem for the Borel Tauberian; see Section 11. Both their 1916 and their 1943 treatment made use of related summability methods, now called T_α ('Taylor method') and V_α or F_α ('Valiron method'); see Section 21. For T_α they used the name circle method; cf. Hardy's book *Divergent Series* [1949]; there appears to be no connection with the Hardy–Littlewood circle method of number theory. Starting with Knopp's work [1922–23] on Euler summability (Section 20), so-called *circle methods* were extensively studied, notably in Germany; cf. Meyer-König [1949] and the account in Zeller and Beekmann [1958/70]. The name circle methods is appropriate in connection with analytic continuation; in each step, such a method typically provides the analytic continuation throughout a circular disc.

In the meantime, Zygmund [1931] had found an interesting special Borel Tauberian theorem for series with 'Ostrowski gaps'. As applications he obtained a new proof for Ostrowski's theorem on overconvergence, and for Hadamard's theorem on the non-continuability of functions with lacunary power series; see Section 7.

Since 1965 there is also a Borel *high-indices theorem*, that is, a Tauberian theorem for lacunary series in which Borel summability implies convergence without any order condition on the terms. Whereas in the high-indices theorem for Abel summability (Theorem I.23.1) the optimal gaps are Hadamard gaps, the right gaps in the Borel case are *square-root gaps*: $a_n \neq 0$ only for n belonging to a sequence of positive integers $\{p_k\}$ such that

$$\sqrt{p_{k+1}} - \sqrt{p_k} \geq \delta \quad \text{for some number } \delta > 0, \tag{1.3}$$

or equivalently, $p_{k+1} - p_k \geq \varepsilon\sqrt{p_k}$ for some number $\varepsilon > 0$. A restricted form of the Borel high-indices theorem, one involving the order condition

$$a_n = \mathcal{O}(e^{b\sqrt{n}}) \quad \text{for some constant } b, \tag{1.4}$$

had been obtained by Pitt [1938a], cf. his correction in [1958] (p. 92). The order condition was relaxed by Meyer-König and Zeller [1956]. The final step to a true high-indices theorem was taken by Gaier [1965], [1966] and Mel'nik [1965]; see Sections 15–17.

One of the classical Hardy–Littlewood theorems [1914a] dealt with Abel summable series $\sum_{n=0}^{\infty} a_n$ whose partial sums s_n are *positive*: the relations

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} s_n x^n \to A \quad \text{as } x \nearrow 1$$

and $s_n > 0$ (or $s_n \geq -C$) imply

$$s_n^{(-1)} = \sum_{k\leq n} s_k \sim An \quad \text{as } n \to \infty.$$

In other words, $\sum_{n=0}^{\infty} a_n$ will be Cesàro summable to A; cf. Theorem I.7.3. Not so long ago, Tenenbaum [1980] obtained a Tauberian theorem for Borel summable series with positive partial sums s_n. His conclusion may be expressed in the form

$$s^{(-1)}_{[n+c\sqrt{n}]} - s^{(-1)}_n \sim Ac\sqrt{n} \quad \text{for every number } c > 0. \tag{1.5}$$

Equivalent conclusions include a certain Riesz-type summability; see Section 13 and Bingham [1984a], [1984b], [1984c].

Comparing the Tauberian results for Borel summability with the earlier results for Abel summability, one may observe that in all cases, important parameters have to be replaced by their square root. An explanation for this fact may be given in the context of probability theory; see Kosambi [1958] and cf. Diaconis and Stein [1978], Bingham, loc. cit. and [1981], [1985].

We begin this chapter with the standard results on analytic continuation involving Borel summability.

For the Tauberian theory we choose a *broader approach* which applies to what we call *general circle methods* $\Gamma = \Gamma_\lambda$ of index $\lambda > 0$. Such methods involve nonnegative functions u_n, $n = 0, 1, \ldots$, defined on a suitable unbounded subset $X \subset \mathbb{R}^+$. (One may think of a half-line or an arithmetic progression.) A key property is that for some number $\delta > 0$ and $|n - x| \le \delta x$,

$$\log u_n(x) = -\lambda \frac{(n-x)^2}{x} + \frac{1}{2}\log\left(\frac{\lambda}{\pi x}\right) + \mathcal{O}\left(\frac{|n-x|^3}{x^2} + \frac{|n-x|+1}{x}\right). \tag{1.6}$$

The Γ_λ*-limitability* of $\{s_n\}$ to A is defined by the relation

$$\sum_{n=0}^{\infty} s_n u_n(x) \to A \quad \text{as } x \ (\in X) \to \infty; \tag{1.7}$$

see Section 5. The best known special cases are Borel and Euler summability; for the Euler method and other examples, see Sections 20, 21. Under Schmidt's Tauberian condition (1.2) and under other conditions such as $s_n \ge 0$, all circle methods Γ_λ of fixed index λ turn out to be equivalent to a single integral method (Sections 9, 10). This confirms the well-known fact that the special circle methods share a great deal of Tauberian theory; see Section 22 and cf. Hardy's book, Meyer-König (loc. cit.), Zeller and Beekmann (loc. cit.), Bingham (loc. cit.), and Tietz and Zeller [2000].

The common theory presented here for the methods Γ_λ includes restricted high-indices theorems of Zygmund and Pitt type. We derive the unrestricted Borel high-indices theorem from the restricted version which involves condition (1.4). The same approach may be used to verify known high-indices results for special circle methods such as T_α (see Section 23). It also leads to a new growth result for power series with square-root gaps; see Section 19 and Korevaar [2001b], [2001c]. In Section 18 we discuss the optimality of various Tauberian conditions. Finally we observe that Borel and Abel summability can be treated together as special cases of so-called power series methods. This area has become very active notably since 1980; we present a sketch in Sections 24, 25.

There is a recent book by Shawyer and Watson [1994] which is devoted entirely to Borel-type and circle methods of summability. Besides classical results it contains much work of 'David Borwein's school', as well as some applications and an extensive bibliography. The present chapter includes more (and partly new) Tauberian theory.

2 The Methods B and B'

The results in Sections 2–4 go back to Borel [1895], [1899], [1901] and show the influence of Hardy's work [1904a], [1904b], [1911b]. For the exposition in these sections, cf. Hardy's book [1949] (sections 8.5–8.7).

In the following we require of the series $\sum_{n=0}^{\infty} a_n$ that $|a_n|^{1/n}/n \to 0$ as $n \to \infty$ or equivalently, that the series

$$\alpha(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad \beta(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} \tag{2.1}$$

(with $s_n = \sum_{k=0}^{n} a_k$) converge for all values of x. There are two Borel methods, which will be denoted by B and B', respectively.

Definitions 2.1. The series $\sum_{n=0}^{\infty} a_n$ is said to be B-summable to A if the sequence of partial sums $\{s_n\}$ is B-limitable to A, that is, the series for $\beta(x)$ converges for all x and

$$F(x) \stackrel{\text{def}}{=} e^{-x}\beta(x) = \sum_{n=0}^{\infty} s_n (x^n/n!)e^{-x} \to A \quad \text{as } x \to \infty \tag{2.2}$$

along $\mathbb{R}^+$. The series $\sum_{n=0}^{\infty} a_n$ is said to be B'-summable to A' if the series for $\alpha(x)$ converges for all x and

$$\phi(x) \stackrel{\text{def}}{=} \int_0^x e^{-y}\alpha(y)dy = \int_0^x \sum_{n=0}^{\infty} a_n (y^n/n!)e^{-y}dy \to A' \quad \text{as } x \to \infty \tag{2.3}$$

along $\mathbb{R}^+$.

We will see below that the more complicated looking method B' is the most convenient for analytic continuation.

The two methods are nearly equivalent:

Theorem 2.2. *The methods B and B' are equivalent precisely for those series $\sum_{n=0}^{\infty} a_n$ for which $e^{-x}\alpha(x) \to 0$ as $x \to \infty$. Every B-summable series is B'-summable to the same sum. The series $\sum_{n=0}^{\infty} a_n$ is B-summable to A if and only if the series $\sum_{n=0}^{\infty} a_{n+1}$ is B'-summable to $A - a_0$.*

Proof. Differentiation of (2.1) gives

$$\alpha'(x) = \sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!}, \qquad \beta'(x) = \sum_{n=0}^{\infty} s_{n+1}\frac{x^n}{n!},$$

so that

$$\begin{aligned} e^{-x}\beta(x) - a_0 &= \int_0^x d\{e^{-y}\beta(y)\} = \int_0^x e^{-y}\{\beta'(y) - \beta(y)\}dy \\ &= \int_0^x e^{-y}\sum_{n=0}^{\infty} a_{n+1}\frac{y^n}{n!}dy = \int_0^x e^{-y}\alpha'(y)dy \\ &= e^{-x}\alpha(x) - a_0 + \int_0^x e^{-y}\alpha(y)dy. \end{aligned} \tag{2.4}$$

Thus by (2.2)–(2.4)

$$F(x) = e^{-x}\beta(x) = e^{-x}\alpha(x) + \phi(x) = \phi'(x) + \phi(x). \tag{2.5}$$

The first statement in the Theorem is an immediate consequence: the condition $e^{-x}\alpha(x) \to 0$ is necessary and sufficient in order that $F(x) \to A$ imply $\phi(x) \to A$ and vice versa. To verify the second statement, suppose that $\sum_0^\infty a_n$ is B-summable to A. Then by (2.5)

$$\phi'(x) + \phi(x) \to A \quad \text{or} \quad (d/dx)\{e^x\phi(x)\} \sim Ae^x \quad \text{as } x \to \infty.$$

By integration from large $C > 0$ to x this is seen to imply that $\phi(x) \to A$. [Alternatively, one may apply l'Hospital's rule to the quotient of $e^x\phi(x)$ and e^x.] The third statement follows from the second line of (2.4). □

Convergence $s_n \to A$ implies B-summability of $\sum_0^\infty a_n$ to A; see formula (1.1). Hence by Theorem 2.2, convergence of $\sum_0^\infty a_n$ to A also implies its B'-summability to A. Thus the Borel methods are regular.

There *are* series which are B'-summable but not B-summable:

Example 2.3. The series $\sum_0^\infty a_n$ given by $\alpha(x) = e^x(\sin e^x)$ is B'-summable:

$$\int_0^x e^{-y}\alpha(y)dy = \int_0^x (\sin e^y)dy = \int_1^{e^x} \frac{\sin v}{v}dv \to \frac{1}{2}\pi - \int_0^1 \frac{\sin v}{v}dv$$

as $x \to \infty$. However, the series cannot be B-summable: $e^{-x}\alpha(x) = \sin e^x$ does not have a limit, hence by (2.4), neither does $e^{-x}\beta(x)$.

Gaier [1953b] has shown that the methods B and B' are equivalent whenever $a_n = \mathcal{O}(e^{cn})$ for some constant c. Cf. Karamata [1939b] for an early result in the same direction, and D. Borwein and Smet [1974] for an extension.

3 Borel Summability of Power Series

In the case of power series $\sum_{n=0}^\infty a_n z^n$ it is convenient to work with summability B'; if desired, one can pass to summability B with the aid of Theorem 2.2. In this section it is not assumed that the power series has positive radius of convergence.

Theorem 3.1. *Suppose that the (formal) power series $\sum_{n=0}^{\infty} a_n z^n$ is B'-summable for $z = z_0 \neq 0$. Then it is B'-summable for z on the segment* $[0, z_0]$, *and uniformly so on every segment* $[\delta z_0, z_0]$ *with* $\delta \in (0, 1)$.

Proof. It may be assumed without loss of generality that $z_0 = 1$. The hypothesis implies that the series for $\alpha(x)$ in (2.1) converges for all x. Accordingly one may write

$$\alpha(zx) = \sum_{n=0}^{\infty} a_n (zx)^n / n!. \tag{3.1}$$

Given relation (2.3), one now has to show that for $x \to \infty$, the transform

$$g_x(z) = \int_0^x e^{-y} \alpha(zy) dy \quad \text{tends to a function } \; g(z) \tag{3.2}$$

if $0 \leq z \leq 1$, with uniform convergence for $\delta \leq z \leq 1$. It is clear that one has summability for $z = 0$, hence let $z > 0$. Treating real and imaginary part separately, we may assume that $a(y)$ is real. Then for $0 < x < x'$, by the standard mean value theorem (of Bonnet) involving a positive nonincreasing function,

$$\begin{aligned} g_{x'}(z) - g_x(z) &= \int_x^{x'} e^{-y} \alpha(zy) dy = \frac{1}{z} \int_{zx}^{zx'} e^{(1-1/z)v} e^{-v} \alpha(v) dv \\ &= \frac{1}{z} e^{(1-1/z)zx} \int_{zx}^{w} e^{-v} \alpha(v) dv, \end{aligned} \tag{3.3}$$

where $zx < w \leq zx'$. Since the improper integral $\int_0^{\infty -} e^{-v} \alpha(v) dv$ is convergent, the final member in (3.3) tends to 0 as $x, \ x' \to \infty$, uniformly for $\delta \leq z \leq 1$.

Remarks 3.2. More elaborate analysis shows that the B'-summability in Theorem 3.1 is uniform on $[0, z_0]$; cf. Hardy's book [1949] (section 8.7).

A power series with zero radius of convergence may be B'-summable on a segment $[0, z_0]$ and *nowhere else.* This can be derived from examples given by Hardy [1914b]; cf. Hardy's book (sections 8.7, 8.9). However, the B'-sum can always be continued analytically to a neighborhood of the open interval $(0, z_0)$:

Theorem 3.3. *Suppose that the (formal) power series $\sum_{n=0}^{\infty} a_n z^n$ is B'-summable for $z = z_0 \neq 0$. Then its B'-sum $g(\cdot)$ on the segment* $[0, z_0]$ *has an analytic extension (also called g) to the open disc which has this segment as a diameter.*

Proof. We may again take $z_0 = 1$. By the preceding proof, the B'-sum $g(\cdot)$ on $[0, 1]$ can be represented as

$$g(z) = \int_0^{\infty -} e^{-y} \alpha(zy) dy = \frac{1}{z} \int_0^{\infty -} e^{(1-1/z)v} e^{-v} \alpha(v) dv. \tag{3.4}$$

By the summability for $z = 1$ the function $\phi(v) = \int_0^v e^{-y} \alpha(y) dy$ is bounded. Hence, replacing $1/z$ by s and taking $s > 1$, one finds

$$g(1/s) = s \int_0^{\infty -} e^{-(s-1)v} d\phi(v) = s(s-1) \int_0^{\infty} e^{-(s-1)v} \phi(v) dv. \tag{3.5}$$

The Laplace integral on the right is absolutely convergent for $\mathrm{Re}\, s > 1$ and uniformly convergent for $\mathrm{Re}\, s \geq 1 + \delta > 1$. Thus the right-hand side represents an analytic function of s in the half-plane $\{\mathrm{Re}\, s > 1\}$ which agrees with $g(1/s)$ for real $s > 1$. We have thereby obtained an analytic extension of $g(1/s)$ to the half-plane $\{\mathrm{Re}\, s > 1\}$, and of $g(z)$ to the disc

$$\left\{\mathrm{Re}\,\frac{1}{z} > 1\right\} \quad \text{or} \quad \left\{\left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}.$$

4 The Borel Polygon

We now consider Borel summability of power series with finite positive radius of convergence R,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R < \infty. \tag{4.1}$$

In particular the power series will be B'-summable to the value $f(z)$ for real z in $(0, R)$. Thus

$$\begin{aligned} f(z) = \sum_{0}^{\infty} a_n z^n &= \lim_{x \to \infty} \int_0^x e^{-y} \alpha(zy) dy = \int_0^{\infty -} e^{-y} \alpha(zy) dy \\ &= \frac{1}{z} \int_0^{\infty -} e^{-v/z} \alpha(v) dv, \quad 0 < z < R; \end{aligned} \tag{4.2}$$

cf. the preceding section. By change of scale one may assume that $R < 1$ and we now suppose that the series $\sum_0^\infty a_n z^n$ is B'-summable for $z = 1$. Then the right-hand side of (4.2) represents the B'-sum $g(z)$ of $\sum_0^\infty a_n z^n$ for $0 \leq z \leq 1$. By Theorem 3.3, g has an analytic extension from $[0, 1]$ to the open disc with diameter $(0, 1)$. Since g agrees with f on $(0, R)$, the extension of g provides an analytic continuation of f from the disc $\Delta(0, R) = \{|z| < R\}$ to the disc $\Delta(1/2, 1/2) = \{|z - 1/2| < 1/2\}$.

By another change of scale and rotation the result may be formulated as follows:

Theorem 4.1. *Let $f(z) = \sum_0^\infty a_n z^n$ have finite positive radius of convergence and suppose that the power series is B'-summable at $z_0 \neq 0$. Then f has an analytic continuation $\tilde{f}$ to the disc* $\{|z - z_0/2| < |z_0|/2\}$.

MITTAG-LEFFLER STAR AND BOREL POLYGON. Let $f(\cdot)$ be analytic in a disc $\Delta(0, R)$. One considers analytic continuation of f from the origin along each ray $\arg z = \theta$. If there is a singular point on such a ray, the one closest to the origin is denoted by $P_\theta = R_\theta e^{i\theta}$. The *Mittag-Leffler star* for f (with base-point 0) is obtained from the z-plane by omitting the part of each ray $\arg z = \theta$ from the first singular point P_θ to ∞. It is clear that f has a (single-valued) analytic extension $\tilde{f}$ to its Mittag-Leffler star. Borel summability provides a convenient way to obtain at least part of this extension.

The *Borel polygon* Π for f (with base point 0) is obtained as follows. For each ray $\arg z = \theta$ which contains a (first) singular point P_θ, one draws the line L_{P_θ} through

P_θ perpendicular to the ray. One now considers the open half-plane H_θ with boundary L_{P_θ} which contains the origin and hence the disc $\Delta(0, R)$. (The points $z \in H_\theta$ are given by the inequality $\mathrm{Re}\{ze^{-i\theta}\} < R_\theta$.) The intersection of the half-planes H_θ, $0 \le \theta < 2\pi$ is a convex open set and this is the Borel polygon Π. It contains the disc $\Delta(0, R)$ and is contained in the Mittag-Leffler star. The function f has a unique analytic extension from $\Delta(0, R)$ to Π; it is given by $\tilde{f}$.

Examples 4.2. For $f(z) = 1/(1-z)$ on $\Delta(0, 1)$ the Mittag-Leffler star is the plane minus the half-line $[1, \infty)$. The Borel polygon is the half-plane given by $\{\mathrm{Re}\, z < 1\}$. For $f(z) = 1/(1-z^2)$ the Borel polygon is a strip, for $f(z) = 1/(1-z^3)$ it is a triangle.

Theorem 4.3. *Let $f(z) = \sum_0^\infty a_n z^n$ have finite positive radius of convergence R. Then the power series $\sum_0^\infty a_n z^n$ is B'-summable at the points of the Borel polygon Π for f and it is uniformly summable on every compact subset of Π. The B'-sums provide the analytic continuation $\tilde{f}$ of f from $\Delta(0, R)$ to Π. The power series is not B'-summable at any point outside the closure of Π.*

Proof. We continue with the notation P_θ, H_θ introduced above. Let $P = \zeta_0$ be a point of Π. For the proof that the series $\sum_0^\infty a_n \zeta_0^n$ is B'-summable one may assume that $|\zeta_0| \ge R$, otherwise the series is convergent and *a fortiori* B'-summable. We first show that Π must contain the closed disc

$$\overline{\Delta}_0 = \{|z - \zeta_0/2| \le |\zeta_0|/2\}$$

with diameter OP; see Figure VI.4. Indeed, a ray $\arg z = \theta$ which makes an angle less than $\pi/2$ with OP cannot contain a singular point P_θ in $\overline{\Delta}_0$. Otherwise the boundary line of H_θ would intersect the segment OP and this would put P outside the open set Π. It follows that Π contains the union D of the disc $\Delta(0, R)$ and a (small) neighborhood of $\overline{\Delta}_0$. In particular the analytic continuation $\tilde{f}$ of f to Π is analytic in D.

We now let Γ be a positively oriented circle in D which contains the closed disc $\overline{\Delta}_0$ in its interior. Then there are positive numbers δ and η such that for every point $z \in \Gamma$,

$$\left|z - \frac{1}{2}\zeta_0\right|^2 \ge \frac{1}{4}|\zeta_0|^2 + \delta, \qquad \left|1 - \frac{\zeta_0}{2z}\right|^2 \ge \left|\frac{\zeta_0}{2z}\right|^2 + \eta.$$

As a result $\mathrm{Re}\,(1 - \zeta_0/z) \ge \eta > 0$. By Cauchy's formula,

$$\begin{aligned} \tilde{f}(\zeta_0) &= \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{f}(z)}{z - \zeta_0}dz = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{f}(z)}{z}dz \int_0^\infty e^{-(1-\zeta_0/z)t}dt \\ &= \frac{1}{2\pi i}\int_0^\infty e^{-t}dt \int_\Gamma \frac{\tilde{f}(z)}{z}e^{\zeta_0 t/z}dz. \end{aligned} \tag{4.3}$$

Here the inversion of the order of integration is justified by absolute convergence: the first repeated integral is majorized by

$$\int_\Gamma \left|\frac{\tilde{f}(z)}{z}\right| |dz| \int_0^\infty e^{-\eta t}dt < \infty. \tag{4.4}$$

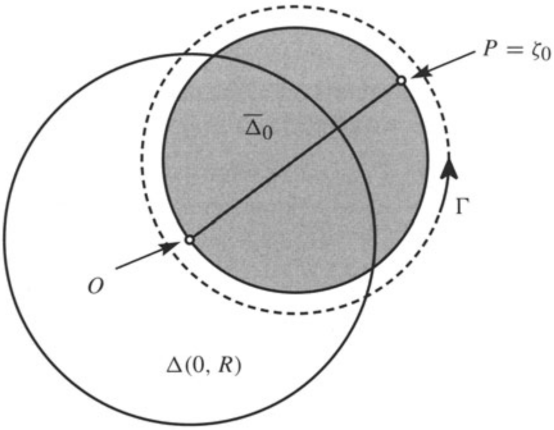

Fig. VI.4. $\Delta(0, R)$, $\overline{\Delta}_0$ and Γ

To evaluate the final inner integral in (4.3) we change the path of integration to a circle $C = C(0, r)$ inside $\Delta(0, R)$ without crossing the origin. By Cauchy's theorem,

$$\int_\Gamma \frac{\tilde{f}(z)}{z} e^{\zeta_0 t/z} dz = \int_C \frac{f(z)}{z} e^{\zeta_0 t/z} dz = \sum_{n=0}^{\infty} a_n \int_C z^{n-1} e^{\zeta_0 t/z} dz,$$

$$\int_C z^{n-1} e^{\zeta_0 t/z} dz = \int_C z^{n-1} \sum_{k=0}^{\infty} \frac{(\zeta_0 t)^k}{k! z^k} dz = 2\pi i \frac{(\zeta_0 t)^n}{n!}.$$

Hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{\tilde{f}(z)}{z} e^{\zeta_0 t/z} dz = \sum_{n=0}^{\infty} a_n \frac{(\zeta_0 t)^n}{n!} = \alpha(\zeta_0 t);$$

cf. (2.1), so that by (4.3)

$$\tilde{f}(\zeta_0) = \int_0^\infty e^{-t} \alpha(\zeta_0 t) dt. \tag{4.5}$$

The integral here is (absolutely) convergent; this follows from the absolute convergence of the final repeated integral in (4.3). Thus by Definitions 2.1, the series $\sum_{n=0}^{\infty} a_n \zeta_0^n$ is B'-summable to $\tilde{f}(\zeta_0)$.

The proof shows also that the series $\sum_0^\infty a_n \zeta^n$ is uniformly B'-summable to $\tilde{f}(\zeta)$ in a small neighborhood of ζ_0; cf. (4.3) and (4.5) with ζ instead of ζ_0 and (4.4) with an adjusted value η.

The series $\sum_0^\infty a_n z^n$ cannot be B'-summable at a point Q outside clos Π. Indeed, if it were, f would have an analytic continuation to the open disc with diameter OQ. It would follow that on rays which make an angle less than $\pi/2$ with OQ, f would have no singular point inside this disc. Thus by the definition of Π, the point Q would belong to clos Π. $\square$

Corollary 4.4. *The power series $\sum_0^\infty a_n z^n$ for $f(z)$ is B'-summable at every regular point z_0 of f on the circle of convergence, and uniformly so in a neighborhood of z_0.*

For if f is analytic in a neighborhood of the point z_0 on the circle of convergence, then z_0 (and some neighborhood of it) will lie inside the Borel polygon for f.

Remarks 4.5. In Theorem 4.3 and Corollary 4.4 one can also use B-summability. Indeed, the function $f(z) - a_0$ with power series $\sum_{n=1}^\infty a_n z^n = \sum_{n=0}^\infty a_{n+1} z^{n+1}$ has the same Borel polygon Π as f, hence the latter series is B'-summable to $\tilde{f}(z) - a_0$ in Π. By Theorem 2.2 the series $\sum_{n=0}^\infty a_n z^n$ is then B-summable to $\tilde{f}(z)$ in Π.

Many other so-called circle methods can be used for analytic continuation; cf. Sections 20, 21 and Meyer-König [1949], Shawyer and Watson [1994]. Although easy to apply, the Borel and circle methods usually do not give maximal continuations. There exist more powerful summability methods, such as those of Lindelöf, Mittag-Leffler and Le Roy, which give the analytic continuation of $f(z) = \sum_0^\infty a_n z^n$ throughout the Mittag-Leffler star. See for example Hardy [1949] (sections 4.11 and 8.10) and cf. Zeller and Beekmann [1958/70] (section 69).

There is an important *principle* for analytic continuation by summability methods, sometimes called the Borel–Okada principle; cf. Okada [1925–28]. If a regular method sums the geometric series $\sum_0^\infty z^n$ to $1/(1-z)$ throughout a certain starlike domain S with base-point 0, then it provides the analytic continuation for $f(z) = \sum_0^\infty a_n z^n$ throughout the intersection of S with the Mittag-Leffler star for f. See Hardy (loc. cit.), Hille [1962] (section 11.4), Boos [2000] (section 5.2). Thus it is important to know which methods sum the geometric series in prescribed domains. This question and related problems of overconvergence (Section 7) have been extensively studied by Luh and collaborators; see Luh [1971], [1974], Luh and Trautner [1976], [1978], Faulstich (= K. Stadtmüller) and Luh [1982], Faulstich, Luh and Tomm [1981/82], [1983]. There are many other approaches to these problems; see for example Jakimovski [1964], Grosse-Erdmann [1993a], [1993b], Luh, Martirosian and Müller [2002].

5 General Circle Methods Γ_λ

Borel limitability of a sequence $\{s_n\}$ to A is defined by the relation

$$\sum_{n=0}^{\infty} s_n u_n(x) \to A \quad \text{as } x \to \infty, \tag{5.1}$$

where

$$u_n(x) \stackrel{\text{def}}{=} \frac{x^n}{n!} e^{-x}, \quad \text{so that } \sum_{n=0}^{\infty} u_n(x) = 1. \tag{5.2}$$

We will verify that the 'Borel functions' u_n have the following

Basic Properties 5.1. *The functions u_n, $n = 0, 1, 2, \ldots$ are defined on a subset X of $\mathbb{R}^+$ which contains an increasing sequence of points $x_k \nearrow \infty$ for which the difference*

sequence $\{x_{k+1} - x_k\}$ *is bounded. They are nonnegative and uniformly bounded. For fixed small* $\delta > 0$ *and* $-\delta x \le h = n - x \le \delta x$, *their logarithms have a uniformly convergent development*

$$\begin{aligned}\log u_n(x) &= -\lambda\frac{h^2}{x} + \frac{1}{2}\log\left(\frac{\lambda}{\pi x}\right) + a_3\frac{h^3}{x^2} + a_4\frac{h^4}{x^3} + \cdots \\ &\quad + b_1\frac{h}{x} + b_2\frac{h^2}{x^2} + \cdots + \mathcal{O}\left(\frac{1}{x}\right) \\ &= -\lambda\frac{h^2}{x} + \frac{1}{2}\log\left(\frac{\lambda}{\pi x}\right) + \mathcal{O}\left(\frac{|h|^3}{x^2}\right) + \mathcal{O}\left(\frac{|h|+1}{x}\right), \end{aligned} \tag{5.3}$$

with $\lambda > 0$. *Moreover, for any number* $\delta \in (0, 1)$, *there is a positive constant* $c = c(\delta, \cdot)$ *such that for* $|n - x| \ge \delta x$, *the functions* u_n *satisfy a uniform estimate*

$$u_n(x) = \mathcal{O}(e^{-c|n-x|}). \tag{5.4}$$

Verification for the Borel Functions. We recall Stirling's formula; cf. Whittaker and Watson [1927/96]:

$$\log(n!) = n\log n - n + (1/2)\log(2\pi n) + \mathcal{O}(1/n), \quad n \ge 1.$$

Applied to the Borel functions in (5.2), it shows that for $-\delta x \le h = n - x \le \delta x$ with $0 < \delta < 1$ (so that $n \ge 1$ when $x > 0$),

$$\begin{aligned}\log u_{x+h}(x) &= -\frac{h^2}{1\cdot 2x} + \frac{h^3}{2\cdot 3x^2} - \frac{h^4}{3\cdot 4x^3} + \cdots \\ &\quad -\frac{1}{2}\log\{2\pi(x+h)\} + \mathcal{O}\left(\frac{1}{x}\right).\end{aligned} \tag{5.5}$$

This formula implies (5.3) with $\lambda = 1/2$.

For the proof of (5.4) we first take $n = p = [x + \delta x]$. Then by (5.5) for $x \ge x_0$,

$$\log u_p(x) \le -\delta^2 x/2 + \delta^3 x/6 \le -\delta^2 x/3 \le -\delta(p - x)/3.$$

For $n > p$ one has

$$\frac{u_n(x)}{u_p(x)} = \frac{x^{n-p}}{n(n-1)\dots(p+1)} \le \left(\frac{x}{p+1}\right)^{n-p},$$

so that

$$\log\{u_n(x)/u_p(x)\} \le -(n-p)\log(1+\delta) < -(n-p)\delta/3.$$

Combining the above with the inequality for $\log u_p(x)$ one obtains (5.4) for the case $n - x \ge \delta x$ with $c = \delta/3$, provided $x \ge x_0$. The result is also true for $0 < x < x_0$ and the proof for $n - x \le -\delta x$ is similar.

The Basic Properties 5.1 will suffice to develop the Tauberian theory for Borel summability. These properties are shared by a large class of methods which accordingly have a corresponding Tauberian theory.

Definition 5.2. Let the functions u_n ($n \in \mathbb{N}_0$) on a domain $X \subset \mathbb{R}^+$ have the Basic Properties 5.1 with parameter λ; in particular X must contain a sequence $x_k \nearrow \infty$ with bounded difference sequence $\{x_{k+1} - x_k\}$. A limitation method (5.1) involving such functions u_n will be called a *general circle method* Γ_λ *of index* λ.

A sequence $\{s_n\}$ is said to be *limitable* to A by the method Γ_λ if

$$F(x) = \sum_{n=0}^{\infty} s_n u_n(x) \quad \text{is well-defined for } x \in X$$
$$\text{and } F(x) \to A \quad \text{as } x\ (\in X) \to \infty. \tag{5.6}$$

[It will follow from (6.7) that $U(x) = \sum_{n=0}^{\infty} u_n(x) \to 1$ as $x \to \infty$.]

A series $\sum_{n=0}^{\infty} a_n$ is called Γ_λ*-summable* to A if its partial sums s_n are Γ_λ-limitable to A.

Explicit examples of circle methods other than the Borel method B will be discussed in Sections 20, 21. For the classical matrix circle methods, a prototype of formula (5.3) was given by Meyer-König [1949]; cf. also Meir [1963] and Shawyer and Watson [1994] (p. 51).

6 Auxiliary Estimates

We will derive several consequences of the Basic Properties 5.1. Here the following notation is convenient; we take $h \geq 0$:

$$r_1(x,h) = \sum_{n<x-h} u_n(x), \quad r_2(x,h) = \sum_{n>x+h} u_n(x),$$
$$R_1(x,h) = \sum_{n<x-h} (n+1)u_n(x), \quad R_2(x,h) = \sum_{n>x+h} (n+1)u_n(x),$$
$$r(x,h) = r_1 + r_2, \quad R(x,h) = R_1 + R_2 = \sum_{|n-x|>h} (n+1)u_n(x). \tag{6.1}$$

Proposition 6.1. *Let the functions u_n belong to a circle method Γ_λ as in Definition 5.2, let $1/2 < \gamma < 2/3$ and let $x \in X$ be ≥ 1. Then for $|n-x| \leq x^\gamma$,*

$$u_n(x) = \sqrt{\frac{\lambda}{\pi x}} e^{-\lambda(n-x)^2/x} \left\{ 1 + \mathcal{O}\left(\frac{|n-x|^3}{x^2} + \frac{|n-x|+1}{x} \right) \right\}, \tag{6.2}$$

$$u_n(x) - u_{n+1}(x) = \left\{ 2\lambda \frac{n-x}{x} + \mathcal{O}\left(\frac{(n-x)^2}{x^2} + \frac{1}{x} \right) \right\} u_n(x), \tag{6.3}$$

while for $|n-x| \leq \delta x$ with sufficiently small $\delta > 0$,

$$u_n(x) \leq \frac{C}{\sqrt{x}} e^{-\lambda(n-x)^2/(2x)}. \tag{6.4}$$

For $\delta \in (0, 1)$, γ *as before and* $x \to \infty$, *one has*

$$r(x, \delta x) \leq R(x, \delta x) = \mathcal{O}(e^{-ax}) \quad \textit{for some} \;\; a = a(\delta, \lambda) > 0, \tag{6.5}$$

$$r(x, x^\gamma) \leq R(x, x^\gamma) = \mathcal{O}(e^{-x^\eta}) \quad \textit{for some} \;\; \eta > 0. \tag{6.6}$$

Also, $\max_n u_n(x) = \mathcal{O}(1/\sqrt{x})$ *and* $\max_x u_n(x) = \mathcal{O}(1/\sqrt{n})$. *Finally, for any* $\alpha \in \mathbb{R}$ *and* $b \geq 0$, *and* $x \to \infty$,

$$U(x) = \sum_{n=0}^{\infty} u_n(x) \to 1, \quad \sum_{n=0}^{\infty} |n^\alpha - x^\alpha|\, u_n(x) = \mathcal{O}(x^{\alpha - 1/2}), \tag{6.7}$$

$$r_j(x, b\sqrt{x}) \to \omega(b) = \frac{1}{\sqrt{\pi}} \int_{b\sqrt{\lambda}}^{\infty} e^{-v^2} dv. \tag{6.8}$$

(The limit $\omega(b)$ *ranges over the interval* $(0, 1/2]$.*)*

Proof. We begin with the first $\mathcal{O}$-term in the last line of (5.3). For $|h| \leq \delta_j x$ with sufficiently small $\delta_j > 0$, one has inequalities of the form

$$\left| \mathcal{O}\left(\frac{|h|^3}{x^2}\right) \right| \leq \begin{cases} M|h|^3/x^2 \text{ if } \delta \leq \delta_1, \\ \lambda h^2/(2x) \text{ if } \delta \leq \delta_2. \end{cases} \tag{6.9}$$

For $x \to \infty$, the first estimate shows that this $\mathcal{O}$-term tends to 0 uniformly when $|h| \leq x^\gamma$ since $3\gamma < 2$. The exponential of that remainder will then be equal to $1 + \mathcal{O}(|h|^3/x^2)$. The exponential of the final $\mathcal{O}$-term in (5.3) can be dealt with in the same way to complete a proof of (6.2). For the proof of (6.4) one combines (5.3) with the second remainder estimate in (6.9). Formula (6.3) requires the first expansion in (5.3).

The proof of (6.5) follows from (5.4) by simple computation. For the proof of (6.6) one may combine (6.5) with an estimate for $R(x, \delta x) - R(x, x^\gamma)$. If δ is small, (6.4) provides a good bound on the terms $u_n(x)$ with $x^\gamma \leq |n - x| \leq \delta x$. Their sum may be estimated with the aid of an integral of the form

$$\frac{C_1}{\sqrt{x}} \int_{x^\gamma}^{\delta x} e^{-\lambda t^2/(2x)} dt \leq C_2 \int_{cx^{\gamma - 1/2}}^{\infty} e^{-v^2} dv = \mathcal{O}(e^{-c' x^{2\gamma - 1}})$$

with $c' > 0$. Since $\gamma > 1/2$ this gives (6.6).

The results on the maxima of $u_n(x)$ will follow from (6.2) since by (6.6) the maxima must occur for relatively small $|n - x|$; cf. also (6.3). The remaining results for sums may be based on additional comparisons with integrals. In these sums one may limit oneself to $|n - x| \leq x^\gamma$ because of (6.6). The pattern of the proofs will become clear from a discussion of $r_2(x, b\sqrt{x})$ with $b \geq 0$.

Let x be ≥ 1. For $|v - n| \leq 1$,

$$e^{-\lambda(n-x)^2/x} = e^{-\lambda(v-x)^2/x} \left\{ 1 + \mathcal{O}\left(\frac{|v - x| + 1}{x} \right) \right\}.$$

Thus by (6.2) for $|v-n| \le 1$ and $|v-x| \le x^\gamma$,

$$u_n(x) = \sqrt{\frac{\lambda}{\pi x}} e^{-\lambda(v-x)^2/x} \left\{1 + \mathcal{O}\left(\frac{|v-x|^3}{x^2} + \frac{|v-x|+1}{x}\right)\right\}. \tag{6.10}$$

This estimate will be important when we derive integral forms of Γ_λ-limitability. For $h = b\sqrt{x}$ and $x \to \infty$ formulas (6.6) and (6.10) imply

$$\begin{aligned} r_2(x,h) &= \sum_{x+h<n\le x+x^\gamma} u_n(x) + r_2(x, x^\gamma) \\ &= \sum_{x+h<n\le x+x^\gamma} \int_n^{n+1} u_n(x)dv + o(1) \\ &= \sqrt{\frac{\lambda}{\pi x}} \int_h^{x^\gamma} e^{-\lambda t^2/x} \left\{1 + \mathcal{O}\left(\frac{t^3}{x^2} + \frac{t+1}{x}\right)\right\} dt + o(1) \\ &= \frac{1}{\sqrt{\pi}} \int_{h\sqrt{\lambda/x}}^{\infty} e^{-v^2} dv + o(1) \to \frac{1}{\sqrt{\pi}} \int_{b\sqrt{\lambda}}^{\infty} e^{-v^2} dv. \end{aligned} \tag{6.11}$$

For $b \ge 0$ the limit $\omega(b)$ in (6.11) can take any positive value not exceeding $1/2$.

Corollary 6.2. *It follows from the first relation in* (6.7) *and* (6.6) *that the methods* Γ_λ *are regular:*

$$\text{if } s_n \to A \quad \text{then } \sum_{n=0}^{\infty} s_n u_n(x) \to A \quad \text{as } x \to \infty. \tag{6.12}$$

Remarks 6.3. The estimates in Proposition 6.1 are extensions of those for the Borel case in Hardy [1949] (section 9.1). They can be interpreted in terms of probability theory. The formulas describe the approximation of 'circle measures' by the normal distribution. Borel summability corresponds to the Poisson measure, Euler summability to the binomial distribution; cf. Feller [1950/68] (chapter 7). Many authors have established connections with probability theory; see Schmetterer [1963] and the references in Remarks 13.2. The restriction $1/2 < \gamma < 2/3$ is the signature of so-called 'large deviations'; cf. Ibragimov and Linnik [1971]. Diaconis [2002] made interesting comments on the relation between summability theory and probability.

7 Series with Ostrowski Gaps

One says that a series $\sum_{n=0}^{\infty} a_n$ has a *gap* (p_k, q_k) if $a_n = 0$ for $p_k < n < q_k$. If there is an infinite sequence of gaps (p_k, q_k) with $q_k/p_k \ge \rho > 1$, one speaks of a series with *Ostrowski gaps*. Zygmund [1931] proved the special Tauberian theorem below for the case of Borel summable series with Ostrowski gaps. The theorem is special in that it implies convergence only of partial sums corresponding to the gaps. As an interesting application we discuss Zygmund's proof for Ostrowski's 1921 theorem on

so-called *overconvergence*: the convergence of special partial sums of lacunary power series outside the circle of convergence. Of the many publications on overconvergence we mention Ostrowski [1926], [1930] and the survey by Bourion [1937]. Ostrowski's theorem implies Hadamard's theorem on non-continuability of analytic functions with lacunary power series; see below.

We will prove a Zygmund-type Tauberian theorem for all methods Γ_λ as in Section 5.

Theorem 7.1. *Given a number $\rho > 1$ and a circle method Γ_λ, one can find numbers $\sigma = \sigma(\rho, \Gamma_\lambda) > 1$ with the following property. For series $\sum_{n=0}^{\infty} a_n$ which are Γ_λ-summable to A and have an infinite sequence of gaps (p_k, q_k) with $q_k/p_k \geq \rho$, the condition*

$$a_n = \mathcal{O}(\sigma^n), \quad \text{or} \quad s_n = \sum_{k \leq n} a_k = \mathcal{O}(\sigma^n), \tag{7.1}$$

implies that the partial sums s_{p_k} corresponding to the gaps converge to A:

$$s_{p_k} \to A \quad \text{as } k \to \infty. \tag{7.2}$$

Proof. Let $\sum_0^\infty a_n$ be Γ_λ-summable to A and have Ostrowski gaps with constant ρ. In terms of the functions u_n for the method Γ_λ, the summability means that

$$\begin{aligned} F(x) &= \sum_{(1-\delta)x \leq n \leq (1+\delta)x} s_n u_n(x) + \sum_{|n-x|>\delta x} s_n u_n(x) \\ &= S_x + T_x \quad \text{(say) tends to } A \end{aligned} \tag{7.3}$$

as $x \in X$ goes to ∞. Recall that the domain X of the functions u_n contains a sequence $x_\nu \nearrow \infty$ such that $x_{\nu+1} - x_\nu \leq \mu$ for some constant μ. We now choose $\delta = 1 - 1/\sqrt{\rho}$, so that for large k,

$$\frac{p_k}{1-\delta} \leq \sqrt{p_k q_k} < \sqrt{p_k q_k} + \mu \leq (q_k/\sqrt{\rho}) + \mu = (1-\delta) q_k + \mu < \frac{q_k}{1+\delta}.$$

From here on we *restrict* x to a subsequence $\{x_{\nu_k}\}$ of $\{x_\nu\}$ such that

$$\frac{p_k}{1-\delta} \leq x_{\nu_k} < \frac{q_k}{1+\delta}. \tag{7.4}$$

For $x = x_{\nu_k}$, all terms in the principal sum S_x in (7.3) have $p_k \leq n < q_k$, so that $s_n = s_{p_k}$. Hence by (6.7) and inequality (6.5) for the remainder $r(x, \delta x)$,

$$S_x = s_{p_k} \sum_{|n-x| \leq \delta x} u_n(x) = s_{p_k}\{U(x) - r(x, \delta x)\} = s_{p_k}\{1 + o(1)\} \tag{7.5}$$

as $x = x_{\nu_k} \to \infty$.

Suppose now that condition (7.1) holds with some number $\sigma > 1$. By (5.4) the remainder T_x may then be estimated as follows:

$$\begin{aligned} |T_x| &\leq \sum_{|n-x|>\delta x} |s_n| u_n(x) \leq C \sum_{|n-x|>\delta x} \sigma^n e^{-c|n-x|} \\ &\leq C' \sigma^{x+\delta x} e^{-c\delta x}, \quad \text{with } c = c(\delta, \Gamma_\lambda) > 0, \end{aligned} \tag{7.6}$$

provided $\sigma^{1+\delta}e^{-c\delta} < 1$. Thus if we require $1 < \sigma < \exp\{c\delta/(1+\delta)\}$, it follows from (7.6) that $T_x \to 0$ as $x = x_{\nu_k} \to \infty$. Then by (7.3) $S_x \to A$, and by (7.5) also $s_{p_k} \to A$. □

In the Borel case, Ostrowski's theorem on overconvergence is a corollary:

Theorem 7.2. *Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have an infinite sequence of gaps (p_k, q_k) with $q_k/p_k \geq \rho > 1$ and finite positive radius of convergence R. Suppose that the sum function $f(\cdot)$ is regular at a point z_0 on the circle $C(0, R)$. (More precisely: f has an analytic continuation $\tilde{f}$ to a neighborhood of z_0.) Then the partial sums $s_{p_k}(z)$ corresponding to the gaps converge at and around z_0; the convergence is uniform in a neighborhood of z_0.*

Proof. One may assume $R = 1$, so that $a_n = \mathcal{O}\{(1+\varepsilon)^n\}$ for every $\varepsilon > 0$. Hence if σ is any number greater than 1 and z_1 has absolute value less than σ, then

$$a_n z_1^n = \mathcal{O}(\sigma^n) \quad \text{as } n \to \infty. \tag{7.7}$$

By Corollary 4.4 and Remarks 4.5, the series $\sum_0^{\infty} a_n z^n$ is B-summable to $\tilde{f}(z)$ in a neighborhood $U(z_0)$ of the regular point z_0 on $C(0, 1)$. We now apply Theorem 7.1 to the series $\sum a_n z_1^n$ instead of $\sum a_n$, specializing Γ_λ to the Borel method and taking σ in inequality (7.7) equal to a number $\sigma(\rho, B)$. The conclusion is that the partial sums $s_{p_k}(z_1)$ converge to $\tilde{f}(z_1)$ whenever z_1 belongs to the intersection of $U(z_0)$ and the disc $\{|z| < \sigma\}$. More precise analysis shows that the convergence is uniform in some neighborhood of z_0.

Hadamard's theorem [1892] may be deduced from Ostrowski's:

Theorem 7.3. *Let $f(z) = \sum_{k=0}^{\infty} a_{p_k} z^{p_k}$ with exponents $p_k \in \mathbb{N}_0$ be a so-called lacunary power series with 'Hadamard gaps':*

$$\frac{p_{k+1}}{p_k} \geq \rho > 1 \quad \textit{for } k \geq k_0, \tag{7.8}$$

and with finite positive radius of convergence R. Then every point of the circle $C(0, R)$ is a singular point for the sum function f.

Proof. We again take $R = 1$ and suppose for a moment that $z_0 \in C(0, 1)$ is a regular point for f. Then by Theorem 7.2 the partial sums $s_{p_k}(z)$ will converge in a neighborhood of z_0. But then the whole sequence of partial sums $\{s_n(z)\}$ would converge at a point outside the circle of convergence, which is impossible.

Remarks 7.4. Condition (7.8) can be weakened: according to Fabry's gap theorem (1896), the condition $p_k/k \to \infty$ is sufficient for non-continuability of f beyond the circle of convergence. See for example Landau and Gaier [1986] (appendix 2, section 2).

Theorem 7.1 implies a 'restricted high-indices theorem' for Γ_λ-summability in the case of Hadamard gaps (7.8). In Sections 15–17 we will discuss high-indices theorems under a weaker gap condition.

8 Boundedness Results

This section serves as preparation for proofs of the Borel Tauberian Theorem and its extension to circle methods under Schmidt's general condition (1.2).

Simple Results. Under the Hardy–Littlewood condition $a_n = \mathcal{O}(1/\sqrt{n})$, boundedness of $F(x) = \sum_{n=0}^{\infty} s_n u_n(x)$ on X (Definition 5.2) readily implies boundedness of the partial sum function $s(x) = \sum_{k \le x} a_k$. Indeed, by (6.7)

$$\sum_{n=0}^{\infty} \{s_n - s(x)\} u_n(x) = F(x) - s(x)U(x) = F(x) - \{1 + o(1)\} s(x)$$

as $x \to \infty$. The boundedness of $s(x)$ (initially on X) then follows from the inequality

$$|s_n - s(x)| \le \sum_{\min\{n,x\} < k \le \max\{n,x\}} |a_k| = \mathcal{O}(|\sqrt{n} - \sqrt{x}| + 1),$$

together with the relations (6.7). (Incidentally, a similar argument shows that the stronger condition $a_n = o(1/\sqrt{n})$ is a Tauberian condition for Γ_λ-summability; cf. Hardy and Littlewood [1913a] for the Borel case.)

The same method shows that the conditions $a_n = o(1)$ and $F(x) = o(\sqrt{x})$ imply $s(x) = o(\sqrt{x})$, and similarly with $\mathcal{O}$ instead of o. It is more difficult to show that $a_n = \mathcal{O}(1)$ and *convergence* $F(x) \to A$ imply $s(x) = o(\sqrt{x})$ in, for example, the Borel case. (The proof indicated by Hardy and Littlewood [1916] proceeds by way of Cesàro summability; cf. Lord [1935].)

We now turn to Schmidt's Condition (1.2). It requires that for any given number $\varepsilon > 0$, there are constants B and $\delta > 0$ such that

$$s_m - s_n \ge -\varepsilon \quad \text{for } n \ge B \text{ and } 0 \le \sqrt{m} - \sqrt{n} \le \delta. \tag{8.1}$$

This implies that for the function $s(x)$, which is equal to s_n for $n \le x < n + 1$, there are positive constants c_j such that

$$s(z) - s(y) \ge -c_1(\sqrt{z} - \sqrt{y}) - c_2 \quad \text{for } z \ge y \ge 0; \tag{8.2}$$

cf. the derivation of formula (I.16.4). Below we deduce the boundedness of the sequence $\{s_n\}$ from that of $F(x)$ under Schmidt's condition. In Section 9, the condition of Γ_λ-limitability will be written in integral form for the case of bounded $\{s_n\}$ or $s_n = o(\sqrt{n})$. In Section 11 we prove the Tauberian theorem under Valiron's condition, the two-sided predecessor of (8.1). For the one-sided result we will use Wiener theory (Section 12).

Theorem 8.1. (Boundedness Theorem) *With functions u_n as in Definition 5.2 for a general circle method Γ_λ, let the sum*

$$F(x) = \sum_{n=0}^{\infty} s_n u_n(x) \tag{8.3}$$

be well-defined for x in the domain X of the functions u_n, and let the sum function $F(x)$ be bounded as $x \in X$ goes to ∞. Suppose that Schmidt's Tauberian condition (8.1) *is satisfied, or at least, that there are constants c_j such that the function $s(\cdot)$ satisfies condition* (8.2). *Then $s(\cdot)$ is bounded.*

Proof. We will again use the 'method of the monotone minorant' (Section I.19); cf. Vijayaraghavan [1928] for the case of Borel summability. The negative of the minorant,

$$\sigma(z) \stackrel{\text{def}}{=} -\min_{y\leq z} s(y) = \max_{y\leq z}\{-s(y)\}, \tag{8.4}$$

is nondecreasing and one can derive from (8.2) that

$$\sigma(z) - \sigma(y) \leq c_1(\sqrt{z} - \sqrt{y}) + c_2 \quad \text{for } z \geq y \geq 0; \tag{8.5}$$

cf. the proof of Lemma I.19.3. In the following we take $b > 0$, $x \in X$ large, and we set $v = x - b\sqrt{x}$, $w = x + b\sqrt{x}$. Then by Proposition 6.1,

$$r_1(x, b\sqrt{x}) = \sum_{n<v} u_n(x) \approx \omega(b), \quad r_2(x, b\sqrt{x}) = \sum_{n>w} u_n(x) \approx \omega(b). \tag{8.6}$$

Here the values of $\omega(b)$ range over the interval $(0, 1/2)$.

(i) As before we write $U(x) = \sum_{n=0}^{\infty} u_n(x)$, which by (6.7) is close to 1. Observe also that $\sqrt{x} - \sqrt{v} < b$, so that $\sqrt{n} - \sqrt{v} \leq |\sqrt{n} - \sqrt{x}| + b$. Thus condition (8.2) and the relations (6.7) give

$$\begin{aligned}\sum_{n\geq v} s_n u_n(x) &\geq s(v)\sum_{n\geq v} u_n(x) - \sum_{n\geq v}\{c_1(\sqrt{n} - \sqrt{v}) + c_2\}u_n(x)\\ &\geq s(v)\{U(x) - r_1(x, b\sqrt{x})\} - \mathcal{O}(1).\end{aligned} \tag{8.7}$$

On the other hand, by the boundedness of $F(x)$ for our large $x \in X$ and (8.4),

$$\sum_{n\geq v} s_n u_n(x) = F(x) - \sum_{n<v} s_n u_n(x) \leq \mathcal{O}(1) + \sigma(v) r_1(x, b\sqrt{x}). \tag{8.8}$$

Since $U(x) - r_1(x, \sqrt{x}) \approx 1 - \omega(b)$, combination of (8.7) and (8.8) gives

$$s(v) \leq \frac{r_1(x, b\sqrt{x})}{U(x) - r_1(x, b\sqrt{x})}\sigma(v) + \mathcal{O}(1) \quad \text{as } x \to \infty. \tag{8.9}$$

We will use (8.9) with a fixed value $b_1 > 0$ of b, so that $\omega(b_1) < 1/2$. Then the coefficient of $\sigma(v)$ will be less than 1 for all large $x \in X$. Thus

$$[-\sigma(v) \leq]\ s(v) \leq \sigma(v) + \mathcal{O}(1), \tag{8.10}$$

first for large $v = x - b_1\sqrt{x}$, and finally for all $v \geq 0$. Hence if $\sigma(\cdot)$ is bounded, so is $s(\cdot)$.

(ii) With the aid of (8.10) one can prove an inequality for $s(\cdot)$ which goes in the other direction. By (8.2) and (6.7),

$$\begin{aligned}
-\sum_{n\le w} s_n u_n(x) &\ge -s(w)\sum_{n\le w} u_n(x) - \sum_{n\le w}\{c_1(\sqrt{w}-\sqrt{n})+c_2\}u_n(x)\\
&\ge -s(w)\{U(x)-r_2(x,b\sqrt{x})\} - \mathcal{O}(1);
\end{aligned}$$

cf. the derivation of (8.7). On the other hand, by (8.10) and (8.5),

$$\begin{aligned}
&-\sum_{n\le w} s_n u_n(x) = -F(x) + \sum_{n>w} s_n u_n(x) \le \mathcal{O}(1) + \sum_{n>w}\sigma(n)u_n(x)\\
&\le \mathcal{O}(1) + \sigma(w)\sum_{n>w} u_n(x) + \sum_{n>w}\{c_1(\sqrt{n}-\sqrt{w})+c_2\}u_n(x)\\
&\le \sigma(w)r_2(x,b\sqrt{x}) + \mathcal{O}(1).
\end{aligned}$$

Combining these inequalities one finds that

$$-s(w) \le \frac{r_2(x,b\sqrt{x})}{U(x)-r_2(x,b\sqrt{x})}\sigma(w) + \mathcal{O}(1) \quad \text{as } x\to\infty. \tag{8.11}$$

We will use (8.11) with a value b_2 of b for which $\omega(b_2) < 1/3$. Then by (8.6) the coefficient of $\sigma(w)$ will be less than $1/2$ for all large x. It follows that

$$-s(w) \le \sigma(w)/2 + \mathcal{O}(1), \tag{8.12}$$

first for all large $w = x + b_2\sqrt{x}$, and finally, for all $w \ge 0$. Suppose now that $\sigma(\cdot)$ is unbounded. Then by (8.12), $-s(w) \le 2\sigma(w)/3$ for all large w, but this would contradict the definition of σ in (8.4). Hence $\sigma(\cdot)$ and $s(\cdot)$ are bounded.

9 Integral Formulas for Limitability

If one has a suitable boundedness result for the sequence $\{s_n\}$, the condition of Γ_λ-limitability,

$$F(x) = \sum_{n=0}^{\infty} s_n u_n(x) \to A \quad \text{as } x\in X \text{ goes to } \infty, \tag{9.1}$$

can be put into convenient integral form. For the Borel case, where $X = \mathbb{R}^+$, the form below is due to Hardy and Littlewood [1916]; cf. Hardy [1949] (section 9.10).

Theorem 9.1. *Let the functions u_n belong to a circle method Γ_λ as in Definition 5.2 and let $s_n = o(\sqrt{n})$ as $n\to\infty$. Define*

$$s(v) = s_n \quad \textit{for } n \le v < n+1, \quad n = 0,1,2,\dots. \tag{9.2}$$

Then the limit relation (9.1) *for a sum is equivalent to the integral relation*

$$\tilde{F}(x) = \sqrt{\frac{\lambda}{\pi x}}\int_0^\infty e^{-\lambda(v-x)^2/x} s(v)\,dv \to A \quad \textit{as } x\to\infty \tag{9.3}$$

in X. The validity of (9.3) *for x in X implies its validity for $x\to\infty$ along $\mathbb{R}^+$.*

Proof. Let $1/2 < \gamma < 2/3$ and $s_n = o(\sqrt{n})$.

(i) Taking $x \in X$ we set $u(x, v) = u_n(x)$ for $n \leq v < n+1$. Since $s(v) = o(\sqrt{v})$ as $v \to \infty$, it follows from the uniform estimate $u_n(x) = \mathcal{O}(1/\sqrt{n})$ and the estimate (6.10) for $u_n(x)$ that

$$\begin{aligned}
\sum_{|n-x|\leq x^\gamma} s_n u_n(x) &= \int_{|v-x|\leq x^\gamma} u(x, v)s(v)dv + o(1) \\
&= \sqrt{\frac{\lambda}{\pi x}} \int_{|v-x|\leq x^\gamma} e^{-\lambda(v-x)^2/x} \left\{1 + \mathcal{O}\left(\frac{|v-x|^3}{x^2} + \frac{|v-x|+1}{x}\right)\right\} s(v)dv \\
&\qquad +o(1) \\
&= \sqrt{\frac{\lambda}{\pi x}} \int_{|v-x|\leq x^\gamma} e^{-\lambda(v-x)^2/x} s(v)dv + o(1) \\
&\qquad +o(1) \int_{|v-x|\leq x^\gamma} e^{-\lambda(v-x)^2/x} \left\{\frac{|v-x|^3}{x^2} + \frac{|v-x|+1}{x}\right\} dv \\
&= \sqrt{\frac{\lambda}{\pi x}} \int_{|v-x|\leq x^\gamma} e^{-\lambda(v-x)^2/x} s(v)dv + o(1) \quad \text{as } x \to \infty. \qquad (9.4)
\end{aligned}$$

On the other hand, by (6.1) and (6.6)

$$\sum_{|n-x|>x^\gamma} s_n u_n(x) = \mathcal{O}\{R(x, x^\gamma)\} = o(1) \quad \text{as } x \to \infty, \qquad (9.5)$$

while the estimate $s(v) = \mathcal{O}(v+1)$ on $\mathbb{R}^+$ implies that

$$\sqrt{\frac{\lambda}{\pi x}} \int_{|v-x|>x^\gamma} e^{-\lambda(v-x)^2/x} s(v)dv = o(1). \qquad (9.6)$$

Combination of (9.4)–(9.6) shows that for $x \to \infty$ in X,

$$F(x) = \sum_{n=0}^{\infty} s_n u_n(x) = \sqrt{\frac{\lambda}{\pi x}} \int_0^\infty e^{-\lambda(v-x)^2/x} s(v)dv + o(1). \qquad (9.7)$$

It follows that under our conditions, (9.1) is equivalent to (9.3) for $x \in X$.

(ii) Having convergence for $x \in X$, one derives from Definition 5.2 that (9.3) must hold for a sequence $x = x_k \to \infty$ with $0 < x_{k+1} - x_k \leq \mu$, say. Now for $x_k \leq z < x_{k+1}$ and $|v - x_k| \leq x_k^\gamma$,

$$\frac{(v-z)^2}{z} - \frac{(v-x_k)^2}{x_k} = \mathcal{O}\left(\frac{|v-x_k|+1}{x_k}\right) \quad \text{as } k \to \infty,$$

so that by an argument as before,

$$\sqrt{\frac{\lambda}{\pi x_k}} \int_{|v-x_k|\leq x_k^\gamma} \left\{e^{-\lambda(v-z)^2/z} - e^{-\lambda(v-x_k)^2/x_k}\right\} s(v)dv = o(1) \quad \text{as } x_k \to \infty.$$

From this and (9.3) for $x = x_k \to \infty$ one concludes that

$$\sqrt{\frac{\lambda}{\pi z}} \int_0^\infty e^{-\lambda(v-z)^2/z} s(v)dv \to A$$

as $z \to \infty$ along $\mathbb{R}^+$; cf. part (i).

For Tauberian theory, one likes to have a limit relation in *convolution form.* For the Borel case and bounded $s(\cdot)$, the integral formula below is due to Wiener [1932]; cf. Hardy [1949] (section 12.15).

Theorem 9.2. *Let $s(v)$ be locally bounded for $v \geq 0$ and $o(\sqrt{v})$ as $v \to \infty$. Define*

$$S(y) = \begin{cases} s(y^2) \text{ for } y \geq 0, \\ 0 \qquad \text{for } y < 0. \end{cases} \tag{9.8}$$

Then the limit relation $\tilde{F}(x) \to A$ (9.3) for $x \to \infty$ along $\mathbb{R}^+$ is equivalent to

$$F^*(x) = \sqrt{\frac{4\lambda}{\pi}} \int_{\mathbb{R}} e^{-4\lambda(x-y)^2} S(y)dy \to A \;\; as \;\; x \to \infty \;\; along \;\; \mathbb{R}^+. \tag{9.9}$$

Proof. We take $1/2 < \gamma < 2/3$ and $|v - x| \leq x^\gamma$ with $x \geq 1$. Then

$$\begin{aligned} \frac{v-x}{\sqrt{x}} &= (\sqrt{v} - \sqrt{x})\frac{\sqrt{v}+\sqrt{x}}{\sqrt{x}} = 2(\sqrt{v}-\sqrt{x})\left(1 + \frac{\sqrt{v}-\sqrt{x}}{2\sqrt{x}}\right) \\ &= 2(\sqrt{v}-\sqrt{x}) + \mathcal{O}\left(\frac{(v-x)^2}{x\sqrt{x}}\right), \end{aligned}$$

$$e^{-\lambda(v-x)^2/x} - e^{-4\lambda(\sqrt{v}-\sqrt{x})^2} = e^{-\lambda(v-x)^2/x}\mathcal{O}\left(\frac{|v-x|^3}{x^2}\right). \tag{9.10}$$

Similarly

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{v}}\left\{1 + \mathcal{O}\left(\frac{|v-x|}{x}\right)\right\}. \tag{9.11}$$

Hence by the definition of $\tilde{F}$ and (9.6),

$$\begin{aligned} \tilde{F}(x) &= \sqrt{\frac{\lambda}{\pi x}} \int_{x-x^\gamma}^{x+x^\gamma} e^{-\lambda(v-x)^2/x} s(v)dv + o(1) \\ &= \sqrt{\frac{\lambda}{\pi}} \int_{x-x^\gamma}^{x+x^\gamma} e^{-4\lambda(\sqrt{v}-\sqrt{x})^2} \frac{s(v)}{\sqrt{v}}dv + o(1) \end{aligned} \tag{9.12}$$

as $x \to \infty$. Substituting $v = y^2$ and replacing x by x^2, one next obtains

$$\tilde{F}(x^2) = \sqrt{\frac{4\lambda}{\pi}} \int_{(x^2-x^{2\gamma})^{1/2}}^{(x^2+x^{2\gamma})^{1/2}} e^{-4\lambda(y-x)^2} S(y)dy + o(1). \tag{9.13}$$

It remains to show that $F^*(x) - \tilde{F}(x^2) \to 0$ as $x \to \infty$. This may be derived from the observation that integrals of the form

$$\int_{cx^{2\gamma-1}}^{\infty} e^{-4\lambda z^2}(x+z)dz, \qquad \int_{-x}^{-cx^{2\gamma-1}} e^{-4\lambda z^2} x\,dz$$

tend to 0 as $x \to \infty$.

Thus under the given conditions, (9.3) implies (9.9), and conversely.

Remark 9.3. In the derivation of Tauberian theorems for Borel summability, Hardy and Littlewood [1916], [1943] used an integral formula related to (9.3) and analogs of (9.3) for sums; cf. Hardy [1949] (chapter 9).

10 Integral Formulas: Case of Positive s_n

Let the functions u_n belong to a circle method Γ_λ as in Definition 5.2. In the case $s_n \geq 0$ it is convenient to introduce the sum function

$$s^{(-1)}(v) = \sum_{n \leq v} s_n. \tag{10.1}$$

Lemma 10.1. *Let $s_n \geq 0$ for all n and let $F(x) = \sum_{n=0}^{\infty} s_n u_n(x)$ remain bounded as $x \to \infty$ in X. Then for v and $n \to \infty$,*

$$\begin{aligned} &s^{(-1)}(v+\sqrt{v}) - s^{(-1)}(v) = \mathcal{O}(\sqrt{v}), \quad s_n = \mathcal{O}(\sqrt{n}), \\ &s^{(-1)}(2v) - s^{(-1)}(v) = \mathcal{O}(v), \quad s^{(-1)}(v) = \mathcal{O}(v), \\ &s^{(-1)}(v+w) - s^{(-1)}(v) = \mathcal{O}(w) \ \textit{for} \ w \geq \sqrt{v}. \end{aligned} \tag{10.2}$$

Proof. For $|n - x| \leq \sqrt{x}$ and large $x \in X$ relation (6.2) shows that

$$u_n(x) \geq \frac{c}{\sqrt{x}} \quad \text{with} \quad c = c(\Gamma_\lambda) > 0.$$

Thus by the boundedness condition on F,

$$\begin{aligned} s^{(-1)}(x+\sqrt{x}) - s^{(-1)}(x-\sqrt{x}) &= \sum_{x-\sqrt{x}<n\leq x+\sqrt{x}} s_n \\ \leq \frac{\sqrt{x}}{c} \sum_{x-\sqrt{x}<n\leq x+\sqrt{x}} s_n u_n(x) &\leq \frac{\sqrt{x}}{c} F(x) = \mathcal{O}(\sqrt{x}) \end{aligned}$$

as $x \in X$ goes to ∞. Since X contains a sequence $x_k \nearrow \infty$ with $x_{k+1} - x_k = \mathcal{O}(1)$, the first relation (10.2) follows. The other relations are easy consequences.

Theorem 10.2. *For $s_n \geq 0$ the relation $F(x) \to A$ as $x \in X$ goes to infinity* (5.6), *which expresses the Γ_λ-limitability of $\{s_n\}$ to A, is equivalent to the integral relation*

$$\tilde{F}(x) = \sqrt{\frac{\lambda}{\pi x}} \int_0^{\infty} e^{-\lambda(v-x)^2/x} ds^{(-1)}(v) \to A \quad \textit{as} \ x \to \infty \tag{10.3}$$

along X or $\mathbb{R}^+$.

Proof. Looking at the proof of Theorem 9.1 we indicate the necessary changes. In order to preclude common discontinuities of $u(x, v)$ and $s^{(-1)}(v)$, we now set $u(x, v) = u_n(x)$ for $n - 1/2 \le v < n + 1/2$. Lemma 10.1 shows that $s_n = \mathcal{O}(\sqrt{n})$ and by (6.6), $\sqrt{n}u_n(x)$ is quite small for large x when $n \approx x \pm x^\gamma$, where $1/2 < \gamma < 2/3$. Thus

$$\sum_{|n-x|\le x^\gamma} s_n u_n(x) = \int_{|v-x|\le x^\gamma} u(x, v) ds^{(-1)}(v) + o(1);$$

cf. the beginning of array (9.4). The second line of (9.4), this time with $ds^{(-1)}(v)$, follows from (6.10). The terms after the $\mathcal{O}$-sign again result in remainders $o(1)$. For example,

$$\begin{aligned} &\frac{1}{\sqrt{x}} \int_{|v-x|\le x^\gamma} e^{-\lambda(v-x)^2/x} \frac{|v-x|^3}{x^2} ds^{(-1)}(v) \\ &= \frac{1}{x} \int_{|t|\le x^\gamma} \frac{|t|^3}{x\sqrt{x}} e^{-\lambda t^2/x} d_t s^{(-1)}(x+t) = \frac{1}{x} \int_{|t|\le x^\gamma} \mathcal{O}(1) d_t s^{(-1)}(x+t) \\ &= \mathcal{O}(1) \frac{s^{(-1)}(x+x^\gamma) - s^{(-1)}(x-x^\gamma)}{x} = \mathcal{O}(x^{\gamma-1}) \end{aligned} \tag{10.4}$$

by the final inequality (10.2). Thus

$$\sum_{|n-x|\le x^\gamma} s_n u_n(x) = \sqrt{\frac{\lambda}{\pi x}} \int_{|v-x|\le x^\gamma} e^{-\lambda(v-x)^2/x} ds^{(-1)}(v) + o(1). \tag{10.5}$$

For the sum over $|n - x| > x^\gamma$ one has (9.5) and for the analog of (9.6) with $ds^{(-1)}(v)$ one has to estimate a similar sum.

This gives the analog of (9.7). It follows that for $x \to \infty$ in X, the limit relation $F(x) \to A$ is equivalent to the integral relation (10.3). The result may be extended to $x \to \infty$ along $\mathbb{R}^+$ as in part (ii) of the proof for Theorem 9.1; this time one would use (10.2).

Theorem 10.3. *For $s_n \ge 0$ and $s^{(-1)}(v) = \sum_{n\le v} s_n$ as in* (10.1)*, let*

$$d\Sigma(y) \stackrel{\text{def}}{=} \begin{cases} (2y+1)^{-1} ds^{(-1)}(y^2) & \text{for } y \ge 0, \\ 0 & \text{for } y < 0. \end{cases} \tag{10.6}$$

Suppose that the sequence $\{s_n\}$ is Γ_λ-limitable to A. Then

$$\int_n^{n+1} d\Sigma(y) = \mathcal{O}(1) \quad \text{for } n \in \mathbb{Z}. \tag{10.7}$$

Furthermore, the limit relation $\tilde{F}(x) \to A$ as $x \to \infty$ along $\mathbb{R}^+$ in (10.3) *is equivalent to the limit relation*

$$F^*(x) = \sqrt{\frac{4\lambda}{\pi}} \int_{\mathbb{R}} e^{-4\lambda(x-y)^2} d\Sigma(y) \to A \quad \text{as } x \to \infty \text{ along } \mathbb{R}^+. \tag{10.8}$$

Proof. For (10.7) we may take $n \geq 0$. Then by Lemma 10.1

$$\int_n^{n+1} d\Sigma(y) = \int_n^{n+1} \frac{ds^{(-1)}(y^2)}{2y+1} \leq \frac{s^{(-1)}(n^2+2n+1) - s^{(-1)}(n^2)}{2n+1} = \mathcal{O}(1).$$

The derivation of (10.8) from (10.3) is similar to the derivation of (9.9) from (9.3). Indeed, combination of (10.5) (and the lines following it) with formulas (9.10), (9.11) will show that

$$\begin{aligned}\tilde{F}(x) &= \sqrt{\frac{\lambda}{\pi x}} \int_{x-x^\gamma}^{x+x^\gamma} e^{-\lambda(v-x)^2/x} ds^{(-1)}(v) + o(1) \\ &= \sqrt{\frac{4\lambda}{\pi}} \int_{x-x^\gamma}^{x+x^\gamma} e^{-4\lambda(\sqrt{v}-\sqrt{x})^2} \frac{ds^{(-1)}(v)}{2\sqrt{v}+1} + o(1).\end{aligned}$$

[For the remainder estimates required in the last step one may proceed as in (10.4).]

As before, one next substitutes $v = y^2$ and replaces x by x^2:

$$\tilde{F}(x^2) = \sqrt{\frac{4\lambda}{\pi}} \int_{(x^2-x^{2\gamma})^{1/2}}^{(x^2+x^{2\gamma})^{1/2}} e^{-4\lambda(y-x)^2} d\Sigma(y) + o(1). \tag{10.9}$$

For the proof that $F^*(x) - \tilde{F}(x^2) \to 0$ as $x \to \infty$ one may set $x - y = z$ and integrate by parts, using the estimate $\Sigma(y) = \int_0^y d\Sigma(t) = \mathcal{O}(y)$ as $y \to \infty$.

11 First Form of the Tauberian Theorem

There seems to be no really simple proof for the Borel Tauberian theorem as there is for Littlewood's theorem, even under the two-sided Hardy–Littlewood condition of [1916]:

$$|a_n| \leq C/\sqrt{n}. \tag{11.1}$$

It may be argued that Wiener theory provides the best approach, as it also handles the general Schmidt condition (1.2); see Section 12. To be sure, the boundedness of the sequence $\{s_n\}$ is easier to obtain under condition (11.1) and one can give a Wiener-type proof without invoking Wiener's general Fourier theory; cf. Pitt [1957]. At this point we also mention a complex-analysis proof by Jurkat [1956b], which uses condition (11.1) to go from Borel summability to Euler summability (E, q) of arbitrarily high order q (cf. Section 20), and from there to convergence. A probabilistic treatment involving Wiener theory was given by Schmaal, Stam and de Vries [1976].

Here we pursue another alternative. Instead of Wiener theory, we use Vitali's theorem, as Hardy and Littlewood [1943] have done for an auxiliary special circle method. Along that path they obtained a new proof for the Borel Tauberian theorem under condition (11.1). We will see that such an approach also handles Valiron's generalization [1917] of condition (11.1):

$$s_m - s_n \to 0 \quad \text{as } n \to \infty \text{ while } 0 \leq \sqrt{m} - \sqrt{n} \to 0. \tag{11.2}$$

Theorem 11.1. *Suppose that the sequence* $\{s_n\}$ *is Borel- or* Γ_λ*-limitable to* A *and satisfies the Valiron condition* (11.2). *Then* $s_n \to A$.

Proof. We carry out a reduction to Theorem 11.2 below. Set

$$S(y) = \begin{cases} s_n \text{ for } \sqrt{n} \le y < \sqrt{n+1}, \quad n = 0, 1, 2, \ldots, \\ 0 \ \text{ for } y < 0. \end{cases} \tag{11.3}$$

Then for every number $\varepsilon > 0$, there are M and $\delta > 0$ such that

$$|S(y) - S(x)| = |s([y^2]) - s([x^2])| \le \varepsilon \quad \text{for } x \ge M \text{ and } |y - x| \le \delta. \tag{11.4}$$

In other words, the function $S(\cdot)$ is *slowly oscillating* on $\mathbb{R}$; cf. Definition II.2.3. From Section 8 we also know that the sequence $\{s_n\}$ is bounded, hence so is S. By Section 9 the Γ_λ-limitability then implies the integral relation (9.9):

$$F^*(x) = \sqrt{\frac{4\lambda}{\pi}} \int_{\mathbb{R}} e^{-4\lambda(x-y)^2} S(y)dy \to A \ \text{ as } \ x \to \infty. \tag{11.5}$$

The convergence of $S(x)$ and hence $\{s_n\}$ to A now follows from

Theorem 11.2. *Let S be bounded and slowly oscillating on $\mathbb{R}$, and satisfy the limit relation* (11.5). *Then $S(x)$ converges to A as $x \to \infty$.*

For the proof of Theorem 11.2 we will extend the limit relation (11.5) to a class of related integral transforms. For bounded S and $x \in \mathbb{R}$ we consider the *analytic functions* of w given by

$$G_w(x) = \sqrt{\frac{w}{\pi}} \int_{\mathbb{R}} e^{-w(x-y)^2} S(y)dy, \qquad \operatorname{Re} w = u > 0, \tag{11.6}$$

with the principal value of the square root. Observe that these functions are uniformly bounded in every angular domain of the form

$$D_C \stackrel{\text{def}}{=} \{w = u + iv \, : \, u > 0, \ |w|/u < C\}. \tag{11.7}$$

Indeed, for all $x \in \mathbb{R}$

$$|G_w(x)| \le \sqrt{\frac{|w|}{u}} \sqrt{\frac{u}{\pi}} \int_{\mathbb{R}} e^{-u(x-y)^2} |S(y)|dy \le \sqrt{C} \sup |S(y)|, \qquad \forall\, w \in D_C. \tag{11.8}$$

Proposition 11.3. *Let S be bounded and suppose that $G_a(x) \to A$ as $x \to \infty$ for some constant $a > 0$. Then $G_w(x) \to A$ as $x \to \infty$ for $0 < w < a$, and in fact, for all w with* $\operatorname{Re} w > 0$.

We first prove

Lemma 11.4. *Let $0 < b < a$ and let S be such that the integral for $G_b(x)$ is absolutely convergent. Then so is the integral for $G_a(x)$ and*

$$G_b(x) = \sqrt{\frac{c}{\pi}} \int_{\mathbb{R}} e^{-cz^2} G_a(x - z)dz, \quad \text{where } \ \frac{1}{c} = \frac{1}{b} - \frac{1}{a}. \tag{11.9}$$

Proof. For absolutely integrable S the formula may be obtained by Fourier transformation. Indeed, G_a is the convolution of $\sqrt{a/\pi}e^{-ax^2}$ and S, so that the Fourier transform $\hat{G}_a(t) = \int_{\mathbb{R}} G_a(x)e^{-itx}dx$ is equal to the product of the Fourier transforms $e^{-t^2/(4a)}$ [cf. formula (I.25.7)] and $\hat{S}(t)$. The relation

$$\hat{G}_b(t) = e^{-t^2/(4b)}\hat{S}(t) = e^{-t^2/(4c)} \cdot e^{-t^2/(4a)}\,\hat{S}(t)$$

then shows that G_b is the convolution of $\sqrt{c/\pi}e^{-cx^2}$ and G_a.

For a general proof of (11.9) one can use the formula

$$\sqrt{\frac{b}{\pi}}e^{-b(x-z)^2} = \int_{\mathbb{R}} \sqrt{\frac{c}{\pi}}e^{-c(x-t)^2}\sqrt{\frac{a}{\pi}}e^{-a(t-z)^2}dt, \tag{11.10}$$

which may be verified directly or by Fourier transformation. Having (11.10) one multiplies both sides by $S(z)$, integrates over $\mathbb{R}$ and finally inverts the order of integration.

Proof of Proposition 11.3. (i) By the hypotheses and (11.8), $G_a(x-z) \to A$ boundedly for $z \in \mathbb{R}$ as $x \to \infty$. Hence by (11.9), $G_b(x) \to A$ by dominated convergence whenever $0 < b < a$.

(ii) For the second part we use VITALI'S THEOREM: Let D be a domain (connected open set) in the complex w-plane, let $\{\phi(w;\nu)\}$, $\nu \to \nu_0$ be a uniformly bounded family of analytic functions on D, and let $\phi(w;\nu)$ converge to a limit as $\nu \to \nu_0$ for every w in a subset with at least one limit point in D. Then there is a unique analytic function $\psi(w)$ on D such that $\phi(w;\nu) \to \psi(w)$ uniformly on every compact subset of D. Cf. Titchmarsh [1939] (section 5.21).

We apply this theorem with

$$D = D_C \text{ as in } (11.7), \quad \nu = x \in \mathbb{R} \to \nu_0 = \infty, \quad \phi(w;\nu) = G_w(x).$$

The required boundedness follows from (11.8), while by part (i) of the proof there is convergence to the constant A everywhere on the real interval $0 < w \le a$ in D_C. Thus $\phi(w;\nu) \to \psi(w) \equiv A$ throughout D_C. Since this holds for every $C > 0$, one concludes that $G_w(x) \to A$ throughout the half-plane $\operatorname{Re} w > 0$.

Proof of Theorem 11.2. Let S satisfy the hypotheses of the Theorem. Then by (11.5) the hypotheses of Proposition 11.3 are satisfied with $a = 4\lambda$. Conclusion: for *every* number $u > 0$,

$$G_u(x) = \sqrt{\frac{u}{\pi}}\int_{\mathbb{R}} e^{-u(x-y)^2}S(y)dy = \sqrt{\frac{u}{\pi}}\int_{\mathbb{R}} e^{-ut^2}S(x+t)dt \to A \tag{11.11}$$

as $x \to \infty$. By the boundedness of $S(x+t)$ we also have

$$\sqrt{\frac{u}{\pi}}\int_{-x/2}^{x/2} e^{-ut^2}S(x+t)dt \to A \quad \text{as } x \to \infty. \tag{11.12}$$

Now for $|t| \le x/2$ and large x, it follows from (11.4) that

$$|S(x+t) - S(x)| < E|t| + \varepsilon, \quad \text{where } E = \varepsilon/\delta.$$

Thus for large x

$$\left|\sqrt{\frac{u}{\pi}} \int_{-x/2}^{x/2} e^{-ut^2}\{S(x) - S(x+t)\}dt\right| < \sqrt{\frac{u}{\pi}} \int_{-x/2}^{x/2} e^{-ut^2}(E|t| + \varepsilon)dt$$
$$< \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2}\left(E\frac{|y|}{\sqrt{u}} + \varepsilon\right)dy < 2\varepsilon, \quad \text{provided } u > u_0. \tag{11.13}$$

Letting $x \to \infty$ for fixed $u > u_0$, we conclude from (11.13) and (11.12) that

$$\limsup_{x\to\infty} |S(x) - A| \le 2\varepsilon.$$

This completes the proof of Theorems 11.2 and 11.1. □

We remark that Kratz [2000] gave a proof of the Borel Tauberian theorem under condition (11.1) which also used Vitali's theorem.

12 General Tauberian Theorem with Schmidt's Condition

Here we obtain Schmidt's form [1925b] of Theorem I.9.1 for Borel summability and our extension to Γ_λ-summability. Vijayaraghavan [1928] simplified Schmidt's original proof. Wiener [1932] used his own method to deal with Borel summability, but if one has already proved boundedness of the sequence of partial sums $\{s_n\}$, it is more convenient to use the Wiener–Pitt Theorem II.8.4.

Theorem 12.1. *Let $\sum_0^\infty a_n$ be Borel summable to A, or more generally, summable to A by a circle method Γ_λ. Thus in terms of the partial sums $s_n = \sum_{k=0}^n a_k$ and the functions u_n belonging to the method Γ_λ (Definition 5.2), $\sum_0^\infty s_n u_n(x)$ exists for x in the domain X of the functions u_n and*

$$F(x) = \sum_{n=0}^{\infty} s_n u_n(x) \to A \quad \text{as } x \to \infty \quad \text{in } X. \tag{12.1}$$

Suppose that Schmidt's Tauberian condition is satisfied:

$$\liminf\,(s_m - s_n) \ge 0 \quad \text{for } n \to \infty \text{ and } 0 \le \sqrt{m} - \sqrt{n} \to 0. \tag{12.2}$$

Then $s_n \to A$ as $n \to \infty$.

Proof. Let the hypotheses of the Theorem be satisfied. Then it follows from Theorem 8.1 that the sequence $\{s_n\}$ is bounded. As before it is convenient to introduce the functions $s(v) = s_n$ for $n \le v < n+1$, $n = 0, 1, 2, \dots$ and

$$S(y) = \begin{cases} s(y^2) & \text{for } y \ge 0, \\ 0 & \text{for } y < 0. \end{cases} \tag{12.3}$$

Then $S(\cdot)$ is bounded and hence by Theorems 9.1 and 9.2, the Γ_λ-limitability (12.1) is equivalent to the integral relation

$$\sqrt{\frac{4\lambda}{\pi}}\int_{\mathbb{R}} e^{-4\lambda(x-y)^2} S(y)dy \to A \quad \text{as } x \to \infty. \tag{12.4}$$

It is easy to see that the function

$$K(x) = K_\lambda(x) = \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda x^2} \tag{12.5}$$

is a Wiener kernel (Section II.8) and that $\int_{\mathbb{R}} K(x)dx = \hat{K}(0) = 1$. Indeed,

$$\hat{K}(t) = \int_{\mathbb{R}} \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda x^2} e^{-itx} dx = e^{-t^2/(16\lambda)} \neq 0;$$

cf. formula (I.25.7). We finally verify that the function S is slowly decreasing on $\mathbb{R}$ (Definition II.2.3): by (12.2),

$$\begin{gathered}\liminf \{S(y) - S(x)\} = \liminf \{s(y^2) - s(x^2)\} \geq 0 \\ \text{for } x \to \infty \text{ and } 0 \leq y - x \to 0.\end{gathered}$$

It thus follows from the Wiener–Pitt Theorem II.8.4 that $S(x) \to A$ as $x \to \infty$.

Remarks 12.2. For the equivalence of a limitation method Γ_λ to the limitation (12.4) one needs only the condition $s_n = o(\sqrt{n})$ or $S(y) = o(y)$; see Section 9. For *bounded* $\{s_n\}$ or $S(\cdot)$, it follows from Wiener's theory that the integral relations (12.4) for different values of λ are all equivalent (cf. Section II.8). Thus for bounded sequences $\{s_n\}$, any circle method Γ_λ is *equivalent* to any other circle method Γ_μ. For the case of *special* circle methods such results go back to Meyer-König [1949]; cf. Zeller and Beekmann [1958/70].

A predecessor of the general Theorem 12.1 can be found in Sitaraman and Swaminathan [1977]. Certain general Borel-type methods require adjustment of condition (12.2); cf. Kratz and Stadtmüller [1990b], Kiesel and Stadtmüller [1991].

13 Tauberian Theorem: Case of Positive s_n

Here we discuss results of Tenenbaum [1980] and subsequent authors, extending them to general circle methods.

Theorem 13.1. *Suppose that the sequence $\{s_n\}$ is Borel limitable to A, or more generally, limitable to A by a circle method Γ_λ (Definition 5.2). Then the Tauberian condition $s_n \geq -C$, $\forall n$, implies that the sum function $s^{(-1)}(v) = \sum_{n \leq v} s_n$ satisfies the relation*

$$\frac{s^{(-1)}(v + b\sqrt{v}) - s^{(-1)}(v)}{\sqrt{v}} \to Ab \quad \text{as } v \to \infty, \quad \forall b \in \mathbb{R}. \tag{13.1}$$

There are equivalent conclusions in terms of the function $s(v)$ *which is equal to* s_n *for* $v \le n < v+1$, $n = 0, 1, 2, \ldots$*:*

$$\int_x^{x+b} s(t^2)dt \to Ab, \quad \int_0^x s(t^2)e^t dt \sim Ae^x \quad \text{as } x \to \infty. \tag{13.2}$$

The final relation is equivalent to the Riesz-type summability $R(e^{\sqrt{n}}, 1)$ *of the series* $\sum_0^\infty a_n = \sum_0^\infty (s_n - s_{n-1})$ *to* A*; see formula* (13.11) *below.*

In the other direction, the relations (13.1) *(with no further condition on the numbers* s_n*) imply the limitability of* $\{s_n\}$ *to* A *by every circle method* Γ_λ.

Proof. Let $s_n \ge -C$. Adding C to the numbers s_n, so that $s^{(-1)}(v)$ is increased by $C[v+1]$, one may assume that the numbers s_n are ≥ 0. We will now use the notation of Theorem 10.3. In particular, the positive measure $d\Sigma(y)$ on $\mathbb{R}$ is defined in terms of $ds^{(-1)}$ by the formula

$$d\Sigma(y) = \begin{cases} (2y+1)^{-1} ds^{(-1)}(y^2) & \text{for } y \ge 0, \\ 0 & \text{for } y < 0. \end{cases} \tag{13.3}$$

The kernel involved in the integral (10.8) will be denoted by K as in (12.5):

$$K(x) = K_\lambda(x) = \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda x^2}. \tag{13.4}$$

By Theorem 10.3, the Γ_λ-limitability of the sequence $\{s_n\}$ to A implies that

$$\int_n^{n+1} d\Sigma(y) = \mathcal{O}(1) \quad \text{for } n \in \mathbb{Z}, \tag{13.5}$$

and the Γ_λ-limitability is equivalent to the relation

$$\int_{\mathbb{R}} K(x-y)d\Sigma(y) \to A = A\int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty. \tag{13.6}$$

We now apply Wiener's so-called second Tauberian theorem which involves measures; see Section II.13. Our kernel K is of the required class M: it is a continuous L^1 function such that

$$\|K\|_M = \sum_{n\in\mathbb{Z}} \sup_{n \le x \le n+1} |K(x)| < \infty.$$

It is also a Wiener kernel; cf. Section 12. Hence relation (13.6), which involves the positive measure $d\Sigma$ satisfying condition (13.5), implies the corresponding relation

$$\int_{\mathbb{R}} H(x-y)d\Sigma(y) \to A\int_{\mathbb{R}} H(y)dy \quad \text{as } x \to \infty \tag{13.7}$$

for every kernel $H \in M$. In particular the Γ_λ-limitability of $\{s_n\}$ to A implies its Γ_μ-limitability to A for every number $\mu > 0$.

We next take $H(z)$ equal to the characteristic function of an interval $[-b, 0]$ with $b > 0$. Strictly speaking such a function is not in M, but since $d\Sigma$ is positive, approximation from below and above by 'trapezoidal functions' will take care of the difficulty. In view of (13.3), it follows that

$$\Sigma(x+b) - \Sigma(x) = \int_x^{x+b} d\Sigma(y) = \int_x^{x+b} \frac{1}{2y+1} ds^{(-1)}(y^2)$$
$$= \sum_{x^2 < n \le (x+b)^2} \frac{1}{2\sqrt{n}+1} s_n \to Ab \quad \text{as } x \to \infty, \quad \forall b > 0. \tag{13.8}$$

Now $s_n \geq 0$ and $1/(2y+1)$ is decreasing, so that also

$$\frac{s^{(-1)}\{(x+b)^2\} - s^{(-1)}\{x^2\}}{2x+1} = \frac{1}{2x+1} \sum_{x^2 < n \le (x+b)^2} s_n \to Ab. \tag{13.9}$$

Taking b small one finds that $s_n = o(\sqrt{n})$, hence

$$s^{(-1)}(x^2 + 2bx) - s^{(-1)}(x^2) \sim 2Abx \quad \text{as } x \to \infty,$$

which implies (13.1).

In terms of the function $s(\cdot)$ one successively obtains the following equivalent forms of (13.9):

$$\int_{x^2}^{(x+b)^2} s(v)dv \sim 2Abx, \quad \int_x^{x+b} s(t^2)2tdt \sim 2Abx, \quad \int_x^{x+b} s(t^2)dt \to Ab,$$

$$\int_x^{x+b} s(t^2)g(t)dt \sim A \int_x^{x+b} g(t)dt \quad \text{as } x \to \infty \tag{13.10}$$

for every positive monotonic function g. Taking $g(t) = e^t$ one obtains the equivalent relations

$$\int_0^x s(t^2)e^t dt \sim Ae^x, \quad \int_1^z s(\log^2 y)dy \sim Az \quad \text{as } z = e^x \to \infty,$$

and likewise for $z \to \infty$,

$$\int_0^z \left(\sum_{e^{\sqrt{n}} \le y} a_n \right) dy \sim Az, \quad \sum_{e^{\sqrt{n}} \le z} \left(1 - \frac{e^{\sqrt{n}}}{z} \right) a_n \to A. \tag{13.11}$$

The final relation says that the series $\sum_0^\infty a_n$ is summable to A by the first order Riesz method corresponding to the sequence $\lambda_n = e^{\sqrt{n}}$; cf. the definition of the Riesz means of order k at the end of Section I.18. Cf. also Hardy and Riesz [1915], Chandrasekharan and Minakshisundaram [1952].

THE OTHER DIRECTION. Tenenbaum's method for the converse in the Borel case can be adapted to deal with the 'Abelian step' from (13.1) to Γ_λ-limitability. Replacing

s_n by $s_n - A$ in (13.1) one may assume that $A = 0$. One next shows that the validity of (13.1) for every real number b implies that

$$\frac{s^{(-1)}(v + b\sqrt{v}) - s^{(-1)}(v)}{\sqrt{v}} \to 0 \quad \text{uniformly for, say, } 0 \le b \le 1 \tag{13.12}$$

as $v \to \infty$. For this one may argue as in the case of slowly varying functions; see Section IV.3. It follows from (13.12) that

$$s^{(-1)}(v + w) - s^{(-1)}(v) = o(\sqrt{v} + w), \quad s^{(-1)}(v) = o(v) \tag{13.13}$$

for $v \to \infty$, $w \ge 0$. In particular $s_n = o(\sqrt{n})$. Now let $1/2 < \gamma < 2/3$. For the proof that the sequence $\{s_n\}$ is Γ_λ-limitable to 0, it is sufficient to show that

$$\sum_{|n-x| \le x^\gamma} s_n u_n(x) \to 0 \quad \text{as } x \to \infty;$$

cf. inequality (6.6), or by partial summation, that

$$\sum_{|n-x| \le x^\gamma} \{s^{(-1)}(n) - s^{(-1)}(x)\}\{u_n(x) - u_{n+1}(x)\} \to 0.$$

By (13.13) and (6.3), the last sum is of the form

$$\sum_{|n-x| \le x^\gamma} o(|n - x| + \sqrt{x}) \frac{|n - x| + 1}{x} u_n(x) = o(1) \quad \text{as } x \to \infty.$$

In the final step we have used (6.4) and comparison with an integral as in Section 6; cf. also (6.7).

Remarks 13.2. Although optimal as to order, the one-sided Tauberian condition $s_n \ge -C$ in Theorem 13.1 can be relaxed to boundedness of the numbers s_n from below in a certain average sense; see Section 18.

Theorem 13.1 had several predecessors involving Borel limitability. For that case relation (13.1) is due to Tenenbaum [1980], who was motivated by the application to number theory described below. For the Tauberian part he made use of Vitali's theorem; cf. the method of Hardy and Littlewood [1943] for the classical Borel Tauberian and Section 11. Special cases had been treated earlier by Moh [1972] and Diaconis and Stein [1978]. The latter discussed a result for Borel and Euler summability in the context of probability theory. Related results for a variety of circle methods and Riesz summability, which also include probabilistic elements, are in Bingham [1981], [1984a], [1984b], [1984c], [1985], [1988], [1989], Bingham and Goldie [1983], [1988], Bingham and Stadtmüller [1990], Kiesel [1993a], Kiesel and Stadtmüller [2000]. For their proofs, Moh, Bingham, and Bingham and Goldie used the extension of Wiener's theory due to Beurling which was discussed in Section IV.11.

Finally we remark that the equivalence of B-limitability and the relations (13.1) for positive s_n may also be derived from a general result of Wiener and Martin [1937].

14 An Application to Number Theory

Theorem 14.2 below is due to Tenenbaum [1980]. It is a far-reaching generalization of von Mangoldt's classical result to the effect that 'up to large x, there are roughly as many integers with an even number of prime factors as with an odd number'. Tenenbaum's proof makes use of a refined estimate by Halász [1971] which we state as a proposition.

Let E be any set of primes such that

$$\sum_{p\in E} \frac{1}{p} = \infty.$$

We let $g(n) = g_E(n)$ denote the number of prime factors of n which belong to E (consistently counting multiplicity, or not counting multiplicity). As usual, $g^{-1}(m)$ denotes the set of those positive integers n for which $g(n) = m$. The counting function for $g^{-1}(m)$ is called $N(m, x)$:

$$N(m, x) = \text{number of } n \le x \text{ such that } g_E(n) = m. \tag{14.1}$$

Proposition 14.1. *Let* $0 < \varepsilon < 1$ *and*

$$y = y_E(x) = \sum_{p\in E,\, p\le x} \frac{1}{p}. \tag{14.2}$$

Then for $|m - y| \le (1 - \varepsilon)y$ *and* $y \ge 2$, *the counting function* $N(m, x)$ *satisfies the uniform estimate*

$$N(m, x) = x\frac{y^m}{m!}e^{-y}\left\{1 + \mathcal{O}\left(\frac{|m-y|}{y}\right) + \mathcal{O}\left(\frac{1}{\sqrt{y}}\right)\right\}. \tag{14.3}$$

Observe that $(y^m/m!)e^{-y}$ is equal to the Borel function $u_m(y)$; cf. (5.2).

Now let $\mathcal{S}$ be any set of nonnegative integers. Its counting function is denoted by $\sigma = s^{(-1)}$:

$$\sigma(v) = s^{(-1)}(v) = \sum_{n\le v} s_n, \quad s_n = \begin{cases} 1 \text{ if } n \in \mathcal{S}, \\ 0 \text{ if } n \notin \mathcal{S}. \end{cases} \tag{14.4}$$

We write $\mathcal{T} = g^{-1}(\mathcal{S})$ for the set of those positive integers n for which the number $g_E(n)$ belongs to $\mathcal{S}$.

Theorem 14.2. *The set* $\mathcal{T}$ *has ordinary density* $\delta \in [0, 1]$ *if and only if the set* $\mathcal{S}$ *has 'Borel density'* δ. *That is, if and only if the sequence* $\{s_n\}$ *is Borel limitable to* δ *or equivalently, if and only if the counting function* $\sigma = s^{(-1)}$ *satisfies relation* (13.1) *with* $A = \delta$. *(In that case* $\mathcal{S}$ *will also have ordinary density* δ.*)*

Proof. Let τ be the counting function for $\mathcal{T}$:

$$\tau(x) = \text{number of } n \le x \text{ such that } m = g(n) \in \mathcal{S}.$$

We fix γ between 1/2 and 2/3. By (14.1)

$$\tau(x) = \sum_{m\in\mathcal{S}} N(m,x)$$
$$= \sum_{m\in\mathcal{S},\, |m-y|\le y^\gamma} N(m,x) + \theta \sum_{|m-y|>y^\gamma} N(m,x), \tag{14.5}$$

where the final summation is over *all* integers $m \ge 0$ with $|m-y| > y^\gamma$, and $0 \le \theta = \theta(x,\mathcal{S}) \le 1$. If $\mathcal{S}$ is the set of all nonnegative integers, then $\tau(x) = [x]$, while by (14.3) and (6.6),

$$\sum_{|m-y|\le y^\gamma} N(m,x) = x\{1+\mathcal{O}(y^{\gamma-1})\} \sum_{|m-y|\le y^\gamma} u_m(y)$$
$$= x\{1+\mathcal{O}(y^{\gamma-1})\}.$$

Hence for every set $\mathcal{S}$ the final term in (14.5) is majorized by

$$\sum_{|m-y|>y^\gamma} N(m,x) = [x] - x\{1+\mathcal{O}(y^{\gamma-1})\} = o(x) \quad \text{as } x\to\infty.$$

Substituting this estimate into (14.5) and using (14.3) and (6.6) once again, we obtain

$$\frac{\tau(x)}{x} = \{1+\mathcal{O}(y^{\gamma-1})\} \sum_{m\in\mathcal{S},\, |m-y|\le y^\gamma} u_m(y) + o(1)$$
$$= \{1+o(1)\}\sum_{m=0}^{\infty} s_m u_m(y) + o(1). \tag{14.6}$$

Thus $\tau(x)/x$ has a limit δ if and only if $\{s_n\}$ is Borel limitable to δ. By Theorem 13.1 such limitability is equivalent to (13.1) with $A=\delta$. □

For the case where E is the set of *all* primes and $\mathcal{S}$ is an arithmetic progression, Pillai and Delange had proved earlier that $\mathcal{T}$ has ordinary density (equal to that of $\mathcal{S}$). See Tenenbaum's paper for more complete references.

15 High-Indices Theorems

A Tauberian theorem for lacunary series in which there is no order condition on the terms is called a high-indices theorem. If there is still a weak order condition one may speak of a 'restricted' high-indices theorem. In the case of general circle methods (Section 5), it is appropriate to consider lacunary series with 'square-root gaps'.

Definition 15.1. We will say that the series $\sum_0^\infty a_n$ (is lacunary and) has *square-root gaps* if $a_n = 0$ for all n which do not belong to a sequence of positive integers $\{p_k\}$ with the property that

$$\sqrt{p_{k+1}} - \sqrt{p_k} \ge \delta \quad \text{for some number } \delta > 0 \text{ and all } k, \tag{15.1}$$

or equivalently, $p_{k+1} - p_k \ge \varepsilon\sqrt{p_k}$ for some number $\varepsilon > 0$.

We also say that the sequence $\{p_k\}$ has square-root gaps. Example: $p_k = k^2$.

In Section 16 we prove the following restricted high-indices theorem for general circle methods.

Theorem 15.2. *Let the series $\sum_0^\infty a_n$ have square-root gaps as in Definition 15.1, let the terms a_n satisfy an order condition*

$$a_n = \mathcal{O}\left(e^{b\sqrt{n}}\right) \quad \textit{for some constant } b, \tag{15.2}$$

and suppose that $\sum_0^\infty a_n$ is summable to A by a general circle method Γ_λ. In other words, the partial sums s_n are such that $\sum_0^\infty s_n u_n(x)$ is well-defined for x in the domain X of the functions u_n and

$$F(x) = \sum_0^\infty s_n u_n(x) \to A \quad \textit{as } x \to \infty \textit{ in } X. \tag{15.3}$$

Then $s_n \to A$ as $n \to \infty$.

Recall that by Definition 5.2, the domain X contains a sequence $x_k \nearrow \infty$ with bounded difference sequence $\{x_{k+1} - x_k\}$.

For which (special) circle methods is there a true high-indices theorem for series with square-root gaps – a theorem without any order condition on the terms? For Euler summability (Section 20) a positive answer was obtained by Erdős [1952] (who imposed the stronger condition $p_{k+1} - p_k \geq C\sqrt{p_k}$ for a sufficiently large constant C), and Meyer-König and Zeller [1956]. For the moment we limit ourselves to Borel summability.

Theorem 15.3. *Let the series $\sum_0^\infty a_n$ have square-root gaps and be* Borel *summable to A, that is, the series $\sum_0^\infty s_n x^n/n!$ converges for $x > 0$ and*

$$F(x) = \sum_0^\infty s_n \frac{x^n}{n!} e^{-x} \to A \quad \textit{as } x \to \infty. \tag{15.4}$$

Then $s_n \to A$ as $n \to \infty$.

Theorem 15.3 will be derived from Theorem 15.2 in Section 17. It will be shown in Section 18 that in these theorems, square-root gaps are optimal.

Remarks 15.4. In [1938a], Pitt stated a high-indices theorem for *Borel summability* under the order condition that a_n should be $\mathcal{O}(e^{\varepsilon n})$ for every positive number ε. However, he later found that he needed the stronger condition $a_n = \mathcal{O}(e^{b\sqrt{n}})$; see Pitt [1958] (p. 92). The order condition was relaxed to $a_n = \mathcal{O}(e^{cn})$ through the functional-analytic work of Meyer-König [1953], Zeller [1953c], and Meyer-König and Zeller [1956]. Finally Gaier [1965] and Mel'nik [1965] succeeded in removing the restrictions by reducing the general case to the result of Meyer-König and Zeller. Later Gaier found a simpler reduction to the case $a_n = \mathcal{O}(e^{b\sqrt{n}})$ [1966], [1967]; see Section 17. Ingham [1968a] has given an independent proof for Theorem 15.3 (also

in two steps) with the aid of his method of peak functions. Another reduction of the general case to the result of Meyer-König and Zeller may be found in Turán's book [1984] (section 22).

That the gap condition (15.1) was optimal in the Borel high-indices theorem could be derived from work of Lorentz [1948], [1949], [1951] and Erdős [1956].

For certain other special circle methods there also are high-indices theorems under weaker order conditions than (15.2); cf. Section 23. However, for most methods there is *no unrestricted* high-indices theorem. A striking example is provided by the discrete Borel method (domain $X = \mathbb{N}$), for which no gap condition can give such a theorem; see Meyer-König and Zeller [1960b].

Rajagopal [1969] proved a gap Tauberian theorem for Borel summability which connects the oscillation of $\{s_n\}$ and $F(x)$.

16 Restricted High-Indices Theorem for General Circle Methods

For the proof of Theorem 15.2 we adapt the method which Pitt used for the case of Borel summability. The first step is to change the condition of Γ_λ-summability (15.3) to integral form. Not knowing that $s_n = o(\sqrt{n})$, one cannot proceed as in Section 9.

By reduction of the domain X of the functions u_n, it may be assumed that X is a sequence $x_k \nearrow \infty$ such that $x_{k+1} - x_k \le \mu$ for some constant μ and that x_0 is suitably large. We now redefine $u_n(x)$ as $u_n(x_k)$ for $x_k \le x < x_{k+1}$, $k = 0, 1, 2, \ldots$ and set $u_n(x) = 0$ for $0 \le x < x_0$. One readily verifies that the new functions u_n on $\mathbb{R}$ satisfy condition (5.4) and the final estimate in (5.3). Thus they satisfy all the relations in Proposition 6.1, except perhaps the now dispensable formula (6.3) for the difference $u_n - u_{n+1}$. In terms of the new functions u_n, we *define*

$$J(x, y) = 2yu_n(x^2) \quad \text{if } \sqrt{n} \le y < \sqrt{n+1}, \quad n = 0, 1, 2, \ldots. \tag{16.1}$$

We set $J(x, y) = 0$ for $y < 0$ and also for $x < 0$. As usual, we define

$$S(y) = s_n \quad \text{for } \sqrt{n} \le y < \sqrt{n+1}, \qquad S(y) = 0 \ \text{ for } \ y < 0. \tag{16.2}$$

Then in terms of the function F of (15.3),

$$\begin{aligned} G(x) &\overset{\text{def}}{=} \int_{\mathbb{R}} J(x, y)S(y)dy = \sum_{n=0}^{\infty} \int_{\sqrt{n}}^{\sqrt{n+1}} 2yu_n(x^2)S(y)dy \\ &= \sum_0^\infty s_n u_n(x^2) = \begin{cases} F(x_k) \text{ if } \sqrt{x_k} \le x < \sqrt{x_{k+1}}, \\ 0 \quad\ \ \text{ if } x < \tilde{x}_0 = \sqrt{x_0}. \end{cases} \end{aligned} \tag{16.3}$$

Applied to the sequence $\{x_k\} \subset X$, condition (15.3) implies the limit relation

$$G(x) \to A \quad \text{as } x \to \infty \ \text{ along } \mathbb{R}^+. \tag{16.4}$$

By taking $\tilde{x}_0$ sufficiently large we may assume that G is bounded.

For the application of Pitt's method we show that the kernel $J(x, y)$ is *well-approximated* by the difference kernel

$$K(x-y) = \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda(x-y)^2} \tag{16.5}$$

in the sense described by the following Proposition; cf. Section V.10.

Proposition 16.1. *For any given number $d > 0$ and for $0 \le u \le d$, the products $J(x, x-z)e^{-uz}$ and $K(z)e^{-uz}$ are majorized by an integrable function $K^*(z)$ independent of $x \in \mathbb{R}$ and u, and the function*

$$\rho(x) = \sup_{0 \le u \le d} \int_{\mathbb{R}} |J(x, x-z) - K(z)| e^{-uz} dz \tag{16.6}$$

is bounded on $\mathbb{R}$ and tends to 0 as $x \to \infty$.

Proof. It is clear that the products $K(z)e^{-uz}$ are majorized by the L^1 function

$$K_1^*(z) = \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda z^2 + d|z|}. \tag{16.7}$$

We now turn to the majorization of $J(x, x-z)e^{-uz}$, where we may take $x - z = y > 0$ and $x \ge \tilde{x}_0 \ge 1$. It will be convenient to discuss the behavior of $J(x, x-z)$ in a number of cases separately. The starting point will be definition (16.1) for $J(x, y)$ as $2yu_n(x^2)$ when $n \le y^2 < n+1$. The cases depend on the distance between y and x, or between n and x^2. For each case we use an appropriate estimate from Sections 5, 6. Throughout, *let* $1/2 < \gamma < 2/3$ *and* $n \le y^2 < n+1$.

STEP 1. We begin with the case

$$|z| \le \frac{1}{3} x^{2\gamma - 1}. \tag{16.8}$$

Then $|y^2 - x^2| = |z(2x - z)| \le x^{2\gamma}$ and

$$\frac{(y^2 - x^2)^2}{x^2} = \frac{z^2(2x-z)^2}{x^2} = 4z^2 + \mathcal{O}(x^{6\gamma - 4}).$$

Hence by formula (6.10) with x^2 instead of x and $v = y^2$ (so that $n \le v < n+1$),

$$\begin{aligned} J(x, x-z) &= J(x, y) = 2yu_n(x^2) \\ &= 2y\sqrt{\frac{\lambda}{\pi x^2}} e^{-\lambda(y^2 - x^2)^2/x^2} \left\{ 1 + \mathcal{O}\left(\frac{|y^2 - x^2|^3}{x^4} + \frac{|y^2 - x^2| + 1}{x^2} \right) \right\} \\ &= \sqrt{\frac{4\lambda}{\pi}} e^{-4\lambda z^2} \left[1 + \mathcal{O}\{(x+1)^{6\gamma - 4}\} \right]. \end{aligned} \tag{16.9}$$

STEP 2. Next consider the case

$$|z| > \frac{1}{3} x^{2\gamma - 1} \quad \text{and} \quad |n - x^2| \le \delta x^2 \tag{16.10}$$

with small $\delta > 0$. We may then apply inequality (6.4), replacing x by x^2:

$$J(x, x-z) = 2yu_n(x^2) \le C\frac{y}{x}e^{-\lambda(n-x^2)^2/(2x^2)}. \tag{16.11}$$

Observe that

$$|n - x^2| \ge |y^2 - x^2| - 1 \ge \frac{1}{2}|y^2 - x^2| \tag{16.12}$$

provided $|y^2 - x^2| \ge 2$. In this case

$$\frac{(n-x^2)^2}{x^2} \ge \frac{(y^2-x^2)^2}{4x^2} = \frac{z^2(y+x)^2}{4x^2} \ge \frac{1}{4}z^2.$$

Then by (16.11) and (16.10), by which $y^2 < n + 1 \le (1+\delta)x^2 + 1$,

$$J(x, x-z) \le C'e^{-az^2} = (|z|+1)e^{-az^2}\mathcal{O}\{(x+1)^{1-2\gamma}\} \tag{16.13}$$

with $a = \lambda/8 > 0$. There is also such an inequality if $|y^2 - x^2| < 2$, which implies $|z| = |y - x| < 2/x$. Then on the one hand, $J(x, y) = \mathcal{O}\{(x+1)u_n(x^2)\} = \mathcal{O}(1)$ (cf. Proposition 6.1), while on the other hand, $e^{-az^2} \ge \eta > 0$.

Step 3. It remains to deal with the case

$$|z| > \frac{1}{3}x^{2\gamma-1} \quad \text{and} \quad |n - x^2| > \delta x^2. \tag{16.14}$$

We now appeal to estimate (5.4):

$$J(x, x-z) = 2yu_n(x^2) \le Cye^{-c|n-x^2|} \quad \text{with } c = c(\delta) > 0. \tag{16.15}$$

For $x \ge \tilde{x}_0$ with sufficiently large $\tilde{x}_0$ one has $|y^2 - x^2| \ge 2$; cf. (16.14), so that (16.12) is satisfied. Thus

$$|n - x^2| \ge \frac{1}{2}|y - x|(y + x) = \frac{1}{2}|z|(y + x) \ge \frac{1}{2}z^2. \tag{16.16}$$

The factor y in (16.15) can be estimated in terms of $|z|$. Indeed, there are positive numbers δ_j such that for $x \ge$ suitable $\tilde{x}_0$, $|y^2 - x^2| > \delta_1 x^2$, hence either

$$y > (1+\delta_2)x \quad \text{or} \quad y < (1-\delta_3)x.$$

It will be enough to deal with the first alternative. Then $x < y/(1+\delta_2)$, so that

$$-z = y - x > \delta_2 y/(1+\delta_2), \quad y < (1+\delta_2)|z|/\delta_2. \tag{16.17}$$

Combining (16.15)–(16.17), one obtains

$$J(x, x-z) \le C'|z|e^{-bz^2} = (|z|+1)^2 e^{-bz^2}\mathcal{O}\{(x+1)^{1-2\gamma}\} \tag{16.18}$$

with $b = c(\delta)/2 > 0$.

STEP 4. It follows from (16.9), (16.13) and (16.18) that for $u \in [0, d]$, the products $J(x, x-z)e^{-uz}$ (which vanish for $x < \tilde{x}_0$) are majorized by an integrable function of the form

$$K_2^*(z) = C(|z|+1)^2 e^{-\mu z^2 + d|z|}, \tag{16.19}$$

with $\mu = \min\{4\lambda, a, b\} > 0$. Thus, cf. (16.7), the function $\rho(x)$ of (16.6) is bounded on $\mathbb{R}$.

Furthermore, $\rho(x)$ will tend to 0 as $x \to \infty$. Indeed, by (16.9), (16.5), (16.7) and (16.19), taking $x \geq \tilde{x}_0 \geq 1$,

$$\begin{aligned}\rho(x) &\leq \int_{\mathbb{R}} |J(x, x-z) - K(z)| e^{d|z|} dz = \int_{|z| \leq x^{2\gamma-1}/3} + \int_{|z| > x^{2\gamma-1}/3} \\ &\leq \int_{|z| \leq x^{2\gamma-1}/3} e^{-4\lambda z^2 + d|z|} \cdot \mathcal{O}\{(x+1)^{6\gamma-4}\} dz \\ &\quad + \int_{|z| > x^{2\gamma-1}/3} \{K_2^*(z) + K_1^*(z)\} dz = o(1) \quad \text{as } x \to \infty. \end{aligned} \tag{16.20}$$

This completes the proof of Proposition 16.1, which shows that all requirements of Definition V.10.1 are satisfied for 'good approximation' of $J(x, y)$ by $K(x-y)$, relative to the weights $e^{-u(x-y)}$ with $0 \leq u \leq d$.

We are now ready to use the Pitt-type Boundedness Theorem V.10.2. For the convenience of the reader we state the relevant special case:

Theorem 16.2. *Let d be positive, let $J(x, y)$ be well-approximated by $K(x-y)$ relative to the weights $e^{-u(x-y)}$ with $0 \leq u \leq d$, and let $K(z)e^{-uz}$ be a Wiener kernel for $0 \leq u \leq d$. Let $S(y) = 0$ for $y < 0$ and*

$$S(y) = \mathcal{O}(e^{uy}) \quad \textit{for} \ \ y \geq 0 \ \ \textit{and some number} \ \ u < d. \tag{16.21}$$

In addition, let S be piecewise constant, with the intervals of constancy having length $\geq \delta > 0$. Finally suppose that

$$G(x) = \int_{\mathbb{R}} J(x, y) S(y) dy = \mathcal{O}\{(x+1)^\alpha\} \quad \textit{for} \ \ x \geq 0, \tag{16.22}$$

where $\alpha \geq 0$. Then $S(y) = \mathcal{O}\{(y+1)^\alpha\}$.

Proof of Theorem 15.2. We will verify that the functions J, K, S and G described by (16.1)–(16.5) satisfy the conditions of Theorem 16.2, with $\alpha = 0$:
— One has $S(y) = 0$ for $y < 0$ and it follows from hypothesis (15.2) that $s_n = \sum_{k=0}^n a_k = \mathcal{O}(e^{c\sqrt{n}})$ for some constant c, so that

$$S(y) = \mathcal{O}(e^{cy}) \quad \text{for} \ \ y \geq 0.$$

— It follows from Proposition 16.1 that for any number $d > c$, the kernel $J(x, y)$ is well-approximated by the kernel $K(x-y)$ in (16.5) relative to the weights $e^{-u(x-y)}$ with $0 \leq u \leq d$.

— Direct calculation shows that the Fourier transform of $K^u(z) = K(z)e^{-uz}$ is zero free for every value of u, so that the functions K^u are Wiener kernels:

$$\hat{K}^u(t) = \hat{K}(t - iu) = e^{-(t-iu)^2/(16\lambda)};$$

cf. formula (I.25.7).
— The function G in (16.3) is bounded.
— We finally show that S satisfies the condition of piecewise constancy. Indeed, the hypotheses of Theorem 15.2 imply that $s_n = s_{p_k}$ for $p_k \le n < p_{k+1}$, so that

$$S(y) = s_{p_k} \quad \text{for} \quad \sqrt{p_k} \le y < \sqrt{p_{k+1}}.$$

Moreover $\sqrt{p_{k+1}} - \sqrt{p_k} \ge \delta$ for some number $\delta > 0$. Thus S is piecewise constant, and the intervals of constancy have length $\ge \delta$.
Conclusion from Theorem 16.2: the function S of (16.2) is *bounded.*

By the boundedness of S and the fact that $\rho(x)$ in (16.6) tends to 0, one has

$$\left| \int_{\mathbb{R}} \{J(x, y) - K(x - y)\} S(y) dy \right| \le \sup |S| \cdot \rho(x) \to 0.$$

Hence by (16.4),

$$\int_{\mathbb{R}} K(x - y) S(y) dy \to A = A \int_{\mathbb{R}} K(z) dz \quad \text{as } x \to \infty.$$

The convergence of $S(y)$ (and hence of s_n) to A now follows from the Wiener–Pitt Theorem II.8.4. Indeed, the piecewise constant function S satisfies the 'step function condition' of Definition II.2.3.

17 The Borel High-Indices Theorem

In order to derive Theorem 15.3 from Theorem 15.2 we verify Gaier's result [1966] that the hypotheses of Theorem 15.3 imply the growth estimate (15.2). Let

$$f(z) = \sum_0^\infty a_n z^n / n! = \sum_0^\infty b_n z^n \quad \text{with } b_n = 0 \text{ for } n \notin \{p_k\}, \tag{17.1}$$

where we suppose as a minimum that

$$0 < p_1 < p_2 < \cdots \quad \text{and} \quad \sum \frac{1}{p_k} < \infty. \tag{17.2}$$

Proposition 17.1. *Let f be an entire function with lacunary power series as in* (17.1), (17.2) *and suppose that $|f(x)| \le e^x$ for $0 \le x < \infty$. Then there is a constant C such that for all k,*

$$|a_{p_k}| \le C p_k \Pi_k, \quad \textit{where} \quad \Pi_k = \prod_{j \ne k} \frac{p_j + p_k + 1}{|p_j - p_k|}. \tag{17.3}$$

If condition (17.2) *is replaced by the stronger condition that the sequence* $\{p_k\}$ *have square-root gaps as in* (15.1), *there is a constant b such that*

$$a_n = \mathcal{O}\left(e^{b\sqrt{n}}\right). \tag{17.4}$$

Proof. (i) Let Σ_k denote the linear span of the powers t^{p_j} with $j \ne k$. We will use the formula of Müntz [1914] and Szász [1915] for the distance d_2 in $L^2(0, 1)$ between t^{p_k} and Σ_k:

$$\begin{aligned} d_2(t^{p_k}, \Sigma_k) &= \inf_{\{c_j\}} \left(\int_0^1 \left| t^{p_k} - \sum_{j \ne k} c_j t^{p_j} \right|^2 dt \right)^{1/2} \\ &= \frac{1}{\sqrt{2p_k + 1}} \prod_{j \ne k} \frac{|p_j - p_k|}{p_j + p_k + 1} = \frac{1}{\sqrt{2p_k + 1}\,\Pi_k}. \end{aligned} \tag{17.5}$$

Note that d_2 is majorized by the distance d_∞ under the supremum norm:

$$d_2(t^{p_k}, \Sigma_k) \le \inf_{\{c_j\}} \sup_{0 \le t \le 1} \left| t^{p_k} - \sum_{j \ne k} c_j t^{p_j} \right| = d_\infty(t^{p_k}, \Sigma_k).$$

In order to find an upper bound for $|b_{p_k}| = |a_{p_k}|/p_k!$ one may assume that $b_{p_k} \ne 0$. Setting $x = Rt$ we now use the bound for $|f(x)|$ and the uniform convergence of the series for $f(x)$ on $[0, R]$:

$$\begin{aligned} e^R &\ge \sup_{0 \le x \le R} |f(x)| = |b_{p_k}| R^{p_k} \sup_{0 \le t \le 1} \left| t^{p_k} + \sum_{j \ne k} (b_{p_j}/b_{p_k}) R^{p_j - p_k} t^{p_j} \right| \\ &\ge |b_{p_k}| R^{p_k}\, d_\infty(t^{p_k}, \Sigma_k) \ge |b_{p_k}| R^{p_k}\, d_2(t^{p_k}, \Sigma_k). \end{aligned}$$

Hence by (17.5)

$$|b_{p_k}| \le \frac{e^R}{R^{p_k} d_2(t^{p_k}, \Sigma_k)} = \frac{e^R}{R^{p_k}} \sqrt{2p_k + 1}\, \Pi_k. \tag{17.6}$$

The optimal choice for R is $R = p_k$, so that by Stirling's formula for $p_k!$,

$$|b_{p_k}| \le (e/p_k)^{p_k} \sqrt{2p_k + 1}\, \Pi_k \le C p_k \Pi_k / p_k!, \tag{17.7}$$

which implies (17.3).

We remark that Gaier's estimate was a little more precise: using complex analysis, he obtained $\sqrt{p_k}$ in the final member of (17.7) where we have the factor p_k.

(ii) To derive (17.4) from (17.3) we will verify that for the case of square-root gaps (15.1),

$$\log \Pi_k = \mathcal{O}(\sqrt{p_k}). \tag{17.8}$$

Following Gaier we set $t_j = t_j^{(k)} = (p_j/p_k)^{1/2}$. Then

$$\log \Pi_k = \sum_{j \neq k} \log \left\{ \left| \frac{p_j + p_k}{p_j - p_k} \right| \left(1 + \frac{1}{p_j + p_k} \right) \right\} < \sum_{j \neq k} \log \left| \frac{t_j^2 + 1}{t_j^2 - 1} \right| + \log C_1,$$

where we may take $C_1 = \prod_j (1 + 1/p_j)$. Now by (15.1) with j instead of k,

$$t_{j+1} - t_j = (\sqrt{p_{j+1}} - \sqrt{p_j})/\sqrt{p_k} \geq \delta/\sqrt{p_k} \quad \text{with } \delta > 0.$$

Hence

$$\begin{aligned} \frac{\delta}{\sqrt{p_k}} \log \frac{\Pi_k}{C_1} &< \sum_{j<k} \left(\log \frac{1+t_j^2}{1-t_j^2} \right) (t_{j+1} - t_j) \\ &\quad + \sum_{j>k} \left(\log \frac{t_j^2 + 1}{t_j^2 - 1} \right) (t_j - t_{j-1}). \end{aligned} \tag{17.9}$$

Observe that the function $\phi(t) = \log(|t^2+1|/|t^2-1|)$ on $(0, \infty)$ is increasing for $t < 1 = t_k$ and decreasing for $t > 1$. Thus the right-hand side of (17.9) may be interpreted as a lower Riemann sum for the integral of ϕ over $(0, \infty)$, hence it is bounded by $C_2 = \int_0^\infty \phi(t)dt$. This proves (17.8) and by (17.3), also (17.4).

Proof of Theorem 15.3. Let the series $\sum_0^\infty a_n$ satisfy the conditions of the Theorem. Then by (15.4)

$$\begin{aligned} \sum_0^\infty a_n \frac{x^n}{n!} &= \sum_0^\infty s_n \frac{x^n}{n!} - \sum_0^\infty s_n \frac{x^{n+1}}{(n+1)!} = \sum_0^\infty s_n \frac{x^n}{n!} - \int_0^x \sum_0^\infty s_n \frac{t^n}{n!} dt \\ &= Ae^x - \int_0^x Ae^t dt + o(e^x) = o(e^x) \quad \text{as } x \to \infty. \end{aligned} \tag{17.10}$$

In particular there will be a constant C_3 such that

$$\left| \sum_0^\infty a_n x^n/n! \right| \leq C_3 e^x \quad \text{for } x \geq 0.$$

Thus it follows from Proposition 17.1 for the case of square-root gaps that $a_n = \mathcal{O}(e^{b\sqrt{n}})$ for some constant b. Under this condition the convergence of s_n to A follows from Theorem 15.2.

18 Discussion of the Tauberian Conditions

We recall the original Hardy–Littlewood condition

$$|a_n| \leq C/\sqrt{n} \tag{18.1}$$

in the 'Borel Tauberian', Theorem I.9.1; cf. Section 11. It follows from the work of Lorentz [1951] that condition (18.1) is optimal here as to order; cf. Peyerimhoff [1969] and Kwee [1983] (the latter also considered Tauberians involving Borel-type methods and Cesàro summability). We will show in Proposition 18.2 that (18.1) is an optimal order condition for all general circle methods Γ_λ. The proof is based on the following observation.

Lemma 18.1. *Let $s_n = \sum_{k=0}^n a_k \geq -C$ for all n. Then the condition*

$$s_n^{(-1)} = s_0 + s_1 + \ldots + s_n = An + o(\sqrt{n}) \tag{18.2}$$

implies the summability of $\sum_0^\infty a_n$ to A by every method Γ_λ.

For Borel summability this was shown by Hardy [1904b]; cf. Hardy [1949] (section 9.9). For the methods Γ_λ the result may be obtained from the final part of Theorem 13.1.

Proposition 18.2. *For any positive increasing function $\phi \nearrow \infty$, there is a* DIVERGENT *series $\sum_0^\infty a_n$ with $|a_n| \leq \phi(n)/\sqrt{n}$ which is summable by every method Γ_λ.*

Proof. (Cf. Section I.24) We will construct a generating function $F = \sum_0^\infty s_n u_n$ by adding nonoverlapping blocks $F_{p,q} = \sum s_n u_n$ with positive integers p and q. The block $F_{p,q}$ involves nonzero s_n only on the interval $\{p < n < p+4q\}$. The numbers s_n are defined by the following conditions:

s_n increases linearly from 0 to $q\phi(p)/\sqrt{p+4q}$
for $p \leq n \leq p+q$,
s_n decreases linearly from $q\phi(p)/\sqrt{p+4q}$ to $-q\phi(p)/\sqrt{p+4q}$
for $p+q \leq n \leq p+3q$,
s_n increases linearly from $-q\phi(p)/\sqrt{p+4q}$ to 0
for $p+3q \leq n \leq p+4q$.

Thus

$$|a_n| = |s_n - s_{n-1}| = \frac{\phi(p)}{\sqrt{p+4q}} \leq \frac{\phi(n)}{\sqrt{n}} \quad \text{for } p+1 \leq n \leq p+4q. \tag{18.3}$$

It is desired that

$$\max_{p<n<p+4q} s_n = -\min_{p<n<p+4q} s_n = \frac{q\phi(p)}{\sqrt{p+4q}} \approx 1 \quad \text{for large } p. \tag{18.4}$$

Replacing the original ϕ by a smaller positive function $\phi \nearrow \infty$ if necessary, one may assume that $1 \geq \phi(n)/\sqrt{n} \to 0$. We then ensure (18.4) by taking

$$q = [\sqrt{p}/\phi(p)]. \tag{18.5}$$

For the block $F_{p,q}$ the sums $s_n^{(-1)}$ are ≥ 0. They vanish for $n \leq p$ and for $n \geq p+4q$, while

$$\max_{p<n<p+4q} s_n^{(-1)} = s_{p+2q}^{(-1)} = \frac{q^2\phi(p)}{\sqrt{p+4q}} \approx q = o(\sqrt{p}) \quad \text{as } p \to \infty. \tag{18.6}$$

We now set

$$F(x) = \sum_{k=1}^{\infty} F_{p_k,q_k}(x),$$

where the terms are of the form $F_{p,q}$ with $p = p_k$, $q = q_k = [\sqrt{p_k}/\phi(p_k)]$ and $p_{k+1} \geq p_k + 4q_k$. Then the blocks $F_{p,q}$ in the sum do not overlap, so that for the calculation of a_n, s_n and $s_n^{(-1)}$, one never has to deal with more than one block. It follows that for the sum F, we have $|a_n| \leq \phi(n)/\sqrt{n}$ for all n and $s_n^{(-1)} = o(\sqrt{n})$ as $n \to \infty$. Since the sequence $\{s_n\}$ is bounded, Lemma 18.1 shows that the series $\sum_0^\infty a_n$ is Γ_λ-summable to 0. However, by (18.4) the partial sums s_n fail to converge. □

Essentially the same construction proves the optimality of the Schmidt condition (12.2) in Theorem 12.1. A small modification will show that the condition $s_n \geq -C$ in Theorem 13.1 cannot be replaced by a condition of the form $s_n \geq -\phi(n)$, where $0 < \phi \nearrow \infty$; cf. Tenenbaum [1980] for the Borel case.

However, the condition $s_n \geq -C$ in Theorem 13.1 can be relaxed to boundedness of the sequence $\{s_n\}$ from below in the following *average sense*:

$$\lim_{b \searrow 0} \liminf_{v \to \infty} \inf_{0<a\leq b} \frac{1}{b\sqrt{v}} \sum_{v<n\leq v+a\sqrt{v}} s_n > -\infty. \tag{18.7}$$

As in Section 13 we write $s^{(-1)}(v) = \sum_{n\leq v} s_n$.

Proposition 18.3. *Condition* (18.7) *is necessary and sufficient in order that the limitability of* $\{s_n\}$ *to* A *by a general circle method* Γ_λ *be equivalent to the condition that*

$$\frac{s^{(-1)}(v + b\sqrt{v}) - s^{(-1)}(v)}{\sqrt{v}} = \frac{1}{\sqrt{v}} \sum_{v<n\leq v+b\sqrt{v}} s_n \to Ab \quad as \quad v \to \infty \tag{18.8}$$

for every real number b.

Thus for a sequence $\{s_n\}$ satisfying condition (18.7), its limitability by one method Γ_λ implies its limitability by every method Γ_μ.

The important condition (18.7) was introduced by Bingham and Goldie [1983]. Bingham and coauthors proved Proposition 18.3 for a variety of special circle methods. Besides Bingham and Goldie, see Bingham [1984a], [1984b], [1985], [1988], and Bingham and Tenenbaum [1986]. In their proofs, Bingham et al. used the following auxiliary result. Condition (18.7) on a sequence $\{s_n\}$ implies (and is implied by) the existence of a representation

$$s_n = s_n' + s_n'',$$

where $\{s_n'\}$ satisfies condition (18.8) and $s_n'' \geq 0$. This decomposition will also establish Proposition 18.3 in its general form.

A related (fairly general) result was obtained by Stadtmüller [1995]; cf. also Motzer and Stadtmüller [1994].

Finally we show that in the high-indices theorems of Section 15, square-root gaps are optimal.

Proposition 18.4. *Let $\{p_k\}$ be any increasing sequence of positive integers such that*

$$\liminf_{k\to\infty} (\sqrt{p_{k+1}} - \sqrt{p_k}) = 0. \tag{18.9}$$

Then there exist DIVERGENT *series $\sum_0^\infty a_n$, with $a_n = 0$ for $n \notin \{p_k\}$ and $|a_n| \le 1$ for $n \in \{p_k\}$, which are summable to zero by every method Γ_λ.*

Proof. For the proof we choose a subsequence of $\{p_k\}$ from well-separated pairs (p_j, p_{j+1}) for which $\sqrt{p_{j+1}} - \sqrt{p_j} \to 0$. The pairs which we select are denoted by $(q_k, q_k + r_k)$. The precise requirements are

$$1 > \sqrt{q_k + r_k} - \sqrt{q_k} \to 0 \quad \text{and} \quad q_{k+1} \ge 4q_k. \tag{18.10}$$

Observe that as a result $q_k + r_k < 4q_k$ and $r_k = o(\sqrt{q_k})$. We now set $a_n = 1$ for $n = q_k$, $a_n = -1$ for $n = q_k + r_k$, $k = 1, 2, \ldots$ and $a_n = 0$ for all other n. Then $s_n = 1$ for $q_k \le n < q_k + r_k$, $k = 1, 2, \ldots$ and $s_n = 0$ otherwise.

It is clear that the series $\sum a_n$ is divergent. We describe how one can show that it is Γ_λ-summable to 0. Let the functions u_n satisfy the conditions of Definition 5.2 and form $F = \sum_{n=0}^\infty s_n u_n$. Then

$$F = \sum_{k=1}^\infty V_k, \quad \text{where } V_k = \sum_{q_k \le n < q_k + r_k} u_n. \tag{18.11}$$

Take k large and initially consider values of x such that $|x - q_k| \le x^\gamma/2$, where $1/2 < \gamma < 2/3$. Then $r_k = o(\sqrt{x})$ as $k \to \infty$ and $|n - x| \le x^\gamma$ for $q_k \le n < q_k + r_k$. Hence by (18.11) and (6.2),

$$\begin{aligned} V_k(x) &= \sum_{q_k \le n < q_k + r_k} u_n(x) \approx \sum_{q_k \le n < q_k + r_k} \sqrt{\frac{\lambda}{\pi x}} e^{-\lambda(n-x)^2/x} \\ &\approx \sqrt{\frac{\lambda}{\pi x}} \int_{q_k}^{q_k + r_k} e^{-\lambda(t-x)^2/x} dt = \sqrt{\frac{\lambda}{\pi}} \int_{(q_k - x)/\sqrt{x}}^{\{(q_k - x)/\sqrt{x}\} + r_k/\sqrt{x}} e^{-\lambda w^2} dw. \end{aligned} \tag{18.12}$$

The maximum value of $V_k(x)$ occurs for $x \approx q_k$ and by (18.12) it is $o(1)$ as $k \to \infty$. As the distance between x and q_k increases, the value of $V_k(x)$ falls off exponentially because of (5.4). Since the blocks V_k are well-separated, the value of $F = \sum V_k$ at a given point x is essentially determined by the one or two terms for which q_k is relatively close to x. It follows that $F(x) \to 0$ as $x \to \infty$, so that $\sum_0^\infty a_n$ is Γ_λ-summable to 0.

Remark 18.5. Rangachari [1968/69] and Stieglitz [1971] discussed Tauberian conditions which allow the presence of small nonzero terms in the gaps of Theorem 15.3 for Borel summability.

19 Growth of Power Series with Square-Root Gaps

Here we discuss another application of the method of Sections 16, 17. It involves the growth of entire functions with lacunary power series. Such functions grow at about the same rate in every direction, depending on the extent of the gaps. Let $\{p_k\}$ be a given increasing sequence of positive integers and let f be an entire function with a power series of the same form as before:

$$f(z) = \sum_0^\infty a_n z^n/n! = \sum_0^\infty b_n z^n \quad \text{with } a_n = 0 \text{ for } n \notin \{p_k\}. \tag{19.1}$$

For the case of Fabry gaps ($p_k/k \to \infty$), Pólya [1929] proved that f has the same order and type in every angle. Moreover, if the order is finite, it is the same on every ray.

For the case of 'Hadamard sequences' $\{p_k\}$, that is, $p_{k+1} - p_k \geq \alpha p_k$ with $\alpha > 0$, Gaier [1966] has shown that f has exactly the same growth on every ray $\arg z = \theta$. For example, if

$$|f(x)| \leq e^x \quad \text{for } x \geq 0, \tag{19.2}$$

then

$$|f(z)| \leq Ce^{|z|}, \quad \forall z \in \mathbb{C}. \tag{19.3}$$

Surprisingly, such a strong result holds also for everywhere convergent power series with square-root gaps; cf. Korevaar [2001b].

Definition 19.1. For $\alpha > 0$, let E_α denote the class of all entire functions f of the form (19.1) for which the sequence of exponents $\{p_k\}$ has square-root gaps which we describe as follows:

$$p_{k+1} - p_k \geq \alpha\sqrt{p_k} \quad \text{for } k = 1, 2, \ldots. \tag{19.4}$$

Theorem 19.2. *There are constants C_α and c_α depending only on α such that for any function $f \in E_\alpha$ which satisfies the growth condition* (19.2) *on the positive real axis, one has the (equivalent) inequalities*

$$|a_n| \leq c_\alpha\sqrt{n}, \quad \forall n, \qquad |f(z)| \leq C_\alpha e^{|z|}, \quad \forall z \in \mathbb{C}. \tag{19.5}$$

Square-root gaps are optimal for the step from (19.2) *to* (19.3)*: one cannot allow sequences $\{p_k\}$ as in* (18.9).

Outline of the proof. Let $f \in E_\alpha$ satisfy condition (19.2). Then by Proposition 17.1, $a_n = \mathcal{O}(e^{b\sqrt{n}})$ for some constant b.

We turn to the corresponding function $F(x) = \sum_0^\infty s_n x^n e^{-x}/n!$ as in (15.4) with $s_n = \sum_0^n a_k$, and the related function $G(x) = F(x^2)$. Writing $\phi(x) = \sum_0^\infty s_n x^n/n!$, the first line of (17.10) gives

$$f(x) = \sum_0^\infty a_n \frac{x^n}{n!} = \phi(x) - \int_0^x \phi(t)dt,$$

so that $f' = \phi' - \phi$. Notice also that $f(0) = \phi(0) = 0$. Hence for $x \geq 0$ condition (19.2) shows that

$$|F(x)| = |\phi(x)e^{-x}| = \left|\int_0^x f'(t)e^{-t}dt\right| = \left|f(x)e^{-x} + \int_0^x f(t)e^{-t}dt\right| \leq 1 + x.$$

Thus by (16.3), where we now keep the original Borel functions u_n,

$$|G(x)| = \left|\int_{\mathbb{R}} J(x, y)S(y)dy\right| = |F(x^2)| \leq x^2 + 1. \tag{19.6}$$

Since the present function $J(x, y)$ also is well-approximated by the difference kernel $K(x - y)$ of (16.5) we can apply Boundedness Theorem 16.2. Indeed, $S(y) = \mathcal{O}(e^{cy})$ for some constant c and $S(y)$ satisfies a suitable condition of piecewise constancy. Conclusion: $S(y) = \mathcal{O}(y^2)$ or $s_n = \mathcal{O}(n)$. This, however, is the best one could derive from (19.6).

To obtain the more precise estimate $a_n = \mathcal{O}(\sqrt{n})$ one has to refine the Tauberian method. One has to work directly with $f(x)$ instead of $\phi(x)$ and define a function $A(y)$ in terms of the numbers a_n analogously to the definition of $S(y)$ in (16.2). It is now necessary to manufacture a suitable Tauberian condition. For this one may introduce running averages of $A(y)e^{-cy}$ (with small c) over short intervals. Appropriate modification of the Tauberian argument in Section V.13 will then show that

$$A(y) = \mathcal{O}(y) \quad \text{or} \quad a_n = \mathcal{O}(\sqrt{n}).$$

This leads to the first inequality (19.5); for square-root gaps, the two inequalities in (19.5) are equivalent. For details see Korevaar (loc. cit.). The optimality of square-root gaps follows by a construction similar to the one used for Proposition 18.4.

Remark 19.3. Theorem 19.2 has an extension to the case of series

$$\sum_k a_{\lambda_k} z^{\lambda_k} / \Gamma(\lambda_k + 1)$$

in which the (positive) exponents λ_k need not be integers, but satisfy a square-root condition analogous to (19.4). The step from (19.2) to (19.3) now is a little more complicated; cf. Korevaar [2001c]. We mention the following application which is obtained by appropriate change of variables. If an entire function $f(z) = \sum_k b_{p_k} z^{p_k}$, involving positive exponents p_k as in (19.4), satisfies the growth condition

$$|f(x)| \leq e^{\tau x^\rho} \quad \text{for } x \geq 0, \quad \text{with } \rho,\ \tau > 0,$$

then

$$|f(z)| \leq C(\alpha, \rho)e^{\tau |z|^\rho}, \quad \forall z.$$

Open Problem. There should be a related growth result for lacunary power series with *finite* radius of convergence. What is the precise result?

20 Euler Summability

After Borel summability, the most important special circle method is Euler summability, which we discuss in some detail; cf. Knopp [1922–23], [1964], Hildebrand [1974].

It is perhaps of interest to begin with some heuristics. In the calculus of finite differences one introduces symbolic operators on semi-infinite sequences $a_0, a_1, \ldots$. We mention the forward shift E, which changes a_k to a_{k+1}, and the forward difference $\Delta = E - 1$. The classical *Euler transform* is used in numerical work to *accelerate convergence*. It replaces the series

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n E^n a_0 = \frac{1}{1+E}a_0 \tag{20.1}$$

by its formal equivalent,

$$\frac{1}{2+\Delta}a_0 = \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}(-\Delta)^k a_0 = \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\sum_{n=0}^{k}(-1)^n\binom{k}{n}a_n. \tag{20.2}$$

Examples 20.1. The convergence acceleration works especially well for alternating series $\sum(-1)^n a_n$ with regularly decreasing a_n such as $a_n = 1/(n+1)$. Here

$$\begin{aligned}(-\Delta)^k a_0 &= 1 - \binom{k}{1}\frac{1}{2} + \binom{k}{2}\frac{1}{3} - \cdots \\ &= \int_0^1 \left\{1 - \binom{k}{1}x + \binom{k}{2}x^2 - \cdots\right\} dx = \frac{1}{k+1},\end{aligned}$$

which gives

$$\log 2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{k=0}^{\infty}\frac{1}{(k+1)\cdot 2^{k+1}}.$$

Euler applied the method also to divergent series. Observe that $-\Delta z^0 = z^0 - z^1 = 1 - z, \cdots, (-\Delta)^k z^0 = (1-z)^k$; $\Delta^0 z^0$ should be read as 1. Thus the series

$$\sum_{n=0}^{\infty}(-1)^n z^n \quad \text{has Euler transform} \quad \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}(1-z)^k.$$

The new series converges for $|1-z| < 2$ to the sum $1/(1+z)$, which is the *analytic continuation* of the sum of the original series in the unit disc $\{|z| < 1\}$.

It is perhaps not so well known that the Euler transformation can be used to obtain the analytic continuation of the *Riemann zeta function*. Starting with the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1)^z} \quad \text{for} \quad \eta(z) = \left(1 - \frac{2}{2^z}\right)\sum_{n=1}^{\infty}\frac{1}{n^z},$$

Sondow [1994] proved that

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{n=0}^{k} (-1)^n \binom{k}{n} \frac{1}{(n+1)^z}, \quad \forall z \neq 1.$$

Returning to the general case, we prefer to write the original series as $\sum_{n=0}^{\infty} a_n$. If $(-1)^n a_n$ in (20.1) is replaced by a_n, the general term in the transformed series (20.2) becomes

$$b_k \stackrel{\text{def}}{=} \frac{1}{2^{k+1}} \sum_{n=0}^{k} \binom{k}{n} a_n = \frac{1}{2^{k+1}} (1+E)^k a_0. \tag{20.3}$$

Definition 20.2. The series $\sum_{n=0}^{\infty} a_n$ is *Euler summable* to A if the transformed series $\sum_{k=0}^{\infty} b_k$ converges to A. Equivalently (see below), the series $\sum_{n=0}^{\infty} a_n$ is Euler summable to A, and the sequence of the partial sums s_n is *Euler limitable* to A, if the sequence

$$t_k \stackrel{\text{def}}{=} \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} s_n \quad (k = 0, 1, \ldots) \text{ converges to } A. \tag{20.4}$$

The sequence $\{t_k\}$ is called the *Euler transform* of the sequence $\{s_n\}$.

The second part of the definition shows that the convergence of a series implies its Euler summability, to the same sum.

The two forms of the definition are indeed equivalent:

Proof. (i) We first show that the partial sums $w_k = \sum_{j=0}^{k} b_j$ are given by

$$w_k = \frac{1}{2^{k+1}} \sum_{n=0}^{k} \binom{k+1}{n+1} s_n. \tag{20.5}$$

For the verification one may replace b_j by the expression $(1/2^{j+1})(1+E)^j a_0$. Observe that one has the following identity for polynomials in the operator E:

$$\sum_{j=0}^{k} \frac{1}{2^{j+1}} (1+E)^j = \frac{1}{2} \frac{1 - \{(1+E)/2\}^{k+1}}{1 - (1+E)/2}$$

$$= \frac{1}{2^{k+1}} \frac{(1+1)^{k+1} - (1+E)^{k+1}}{1-E} = \frac{1}{2^{k+1}} \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{1^j - E^j}{1-E}$$

$$= \frac{1}{2^{k+1}} \sum_{j=1}^{k+1} \binom{k+1}{j} (1 + E + \cdots + E^{j-1}).$$

We apply the two sides of the identity to a_0. Then the left-hand side becomes $\sum_{j=0}^{k} b_j = w_k$. On the right-hand side, $(1 + E + \cdots + E^{j-1})a_0$ can be written as s_{j-1}. Replacing the dummy index j by $n+1$ one obtains (20.5).

(ii) By (20.4), (20.5) and (20.3),

$$2^k(t_k - w_{k-1}) = \sum_{n=0}^{k} \binom{k}{n} s_n - \sum_{n=0}^{k-1} \binom{k}{n+1} s_n$$
$$= \sum_{n=0}^{k} \binom{k}{n} s_n - \sum_{n=1}^{k} \binom{k}{n} s_{n-1} = \sum_{n=0}^{k} \binom{k}{n} a_n = 2^{k+1} b_k. \tag{20.6}$$

Suppose now that $\sum_0^\infty b_k = A$ or $w_k \to A$. Then $b_k \to 0$ and $w_{k-1} \to A$, hence by (20.6) also $t_k \to A$.

For the converse one observes that

$$2w_k - w_{k-1} = \frac{1}{2^k} \sum_{n=0}^{k} \left\{ \binom{k+1}{n+1} - \binom{k}{n+1} \right\} s_n = \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} s_n = t_k.$$

Thus

$$w_k = \frac{1}{2} t_k + \frac{1}{2^2} t_{k-1} + \cdots + \frac{1}{2^{k+1}} t_0,$$

hence if $t_k \to A$, then also $w_k \to A$. □

What can one say about the size of the terms in an Euler summable series?

Lemma 20.3. *For sequences $\{a_n\}$ and $\{c_n\}$, one has*

$$c_n = (1+E)^n a_0 = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \textit{if and only if} \quad a_n = \Delta^n c_0. \tag{20.7}$$

If $\sum_0^\infty a_n$ is Euler summable, then $a_n = o(3^n)$.

Proof. Defining c_j as $(1+E)^j a_0$, one finds that

$$\Delta^n c_0 = (E-1)^n c_0 = \sum_{k=0}^{n} \binom{n}{k} E^{n-k} (-1)^k c_0 = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} (-1)^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} (1+E)^{n-k} a_0 (-1)^k = (1+E-1)^n a_0 = a_n.$$

For the other direction one starts with $a_n = \Delta^n c_0$ and observes that then

$$\sum_{n=0}^{k} \binom{k}{n} a_n = \sum_{n=0}^{k} \binom{k}{n} (E-1)^n c_0 = (1+E-1)^k c_0 = c_k.$$

Suppose finally that $\sum_{n=0}^\infty a_n$ is Euler summable. Then by Definition 20.2 the series $\sum_{k=0}^\infty b_k$ is convergent, so that $b_k = o(1)$. Thus if we define c_n by the left-hand side of (20.7), then by (20.3), $c_n = 2^{n+1} b_n = o(2^n)$. The right-hand side of (20.7) now shows that

$$a_n = \Delta^n c_0 = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} (-1)^k = o\left\{ \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \right\} = o(3^n).$$

It follows from the Euler summability of the series $\sum_{n=0}^{\infty}(-1)^n z^n$ on the interval $\{-1 < z < 3\}$ that one cannot replace 3 by a smaller number. □

We can now prove an important 'inclusion' result.

Theorem 20.4. *If the series $\sum_{n=0}^{\infty} a_n$ is Euler summable to A, then it is also Borel summable to A.*

Proof. Euler summability of $\sum a_n$ implies that the numbers t_k in (20.4) remain bounded as $k \to \infty$ and by Lemma 20.3, that $a_n = o(3^n)$. Thus also $s_n = o(3^n)$ and by absolute convergence,

$$\sum_{k=0}^{\infty} 2^k t_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{k} \binom{k}{n} s_n = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} \sum_{k=n}^{\infty} \frac{x^{k-n}}{(k-n)!}$$
$$= \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} e^x.$$

Hence if $t_k \to A$, then for $x \to \infty$ also

$$e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} = e^{-2x} \sum_{k=0}^{\infty} t_k \frac{(2x)^k}{k!} \to A.$$

Euler Methods (E, q) and E_α. There is a whole family of Euler methods (E, q), $0 < q < \infty$. They are defined by the relations

$$t_k^{(q)} \stackrel{\text{def}}{=} \frac{1}{(q+1)^k} \sum_{n=0}^{k} \binom{k}{n} q^{k-n} s_n = \left(\frac{q+E}{q+1}\right)^k s_0 \to A \tag{20.8}$$

as $k \to \infty$. These methods become more powerful as q increases and formally tend to the Borel method as $q \to \infty$; cf. Hardy [1949] (sections 8.1–8.5). Every (E, q) summable series is also Borel summable; cf. the proof of Theorem 20.4 for the case $q = 1$. Most of the results in the present section are due to Knopp [1922–23]; cf. the expositions in Hardy (loc. cit.), Knopp [1964] and Baron [1966/77] (section 18).

With parameter $\alpha = 1/(q+1)$, the method (E, q) is known as the Euler–Knopp method E_α. In terms of α, the transform $t_k^{(q)}$ becomes

$$t_k = \sum_{n=0}^{k} \binom{k}{n} \alpha^n (1-\alpha)^{k-n} s_n. \tag{20.9}$$

21 The Taylor Method and Other Special Circle Methods

Under the classification used by Zeller and Beekmann [1958/70], the group of so-called circle methods includes the summation/limitation methods B (Section 2) and

E_α (Section 20), and the methods T_α, S_α and V_α described below. With different notation, the 'Taylor method' T_α and the 'Valiron method' V_α made their appearance in the work of Hardy and Littlewood [1916]. They used these methods (also in [1943]) as a tool in the study of Tauberians for Borel summability; cf. Hardy's book [1949]. Independently and in connection with analytic continuation, the method T_α was considered by Fekete; it was discussed in the Hungarian textbook Beke [1916] (pp 433–435); cf. Vermes [1951] (see his note added in proof), and Fekete [1958].

The TAYLOR METHOD T_α was studied in detail by Wais [1935], Meyer-König [1949] and Vermes [1949]. By way of motivation, suppose that the series $f(x) = \sum_{n=0}^\infty a_n x^n$ converges for $|x| < 1$. From the point of view of analytic continuation, it is natural to expand the sum function f around a point inside the unit disc, which for the theory one may take as $1 - \alpha \in (0, 1)$. [Some authors use the point α instead.] Setting $x = 1 - \alpha + \alpha y$ we obtain

$$f(1-\alpha+\alpha y) = \sum_{k=0}^\infty b_k y^k, \quad \text{with } b_k = \frac{f^{(k)}(1-\alpha)}{k!}\alpha^k. \tag{21.1}$$

It follows that

$$b_k = \alpha^k \sum_{n=k}^\infty \binom{n}{k} (1-\alpha)^{n-k} a_n. \tag{21.2}$$

If f has an analytic continuation (also called f) to a neighborhood of the point $x = 1$, complex analysis shows that the Taylor series for $f(x)$ around the point $x = 1 - \alpha$ has radius of convergence greater than α. In that case the series in (21.1) must have radius of convergence greater than 1, and then the series $\sum_{k=0}^\infty b_k$ will converge to the value $f(1)$.

Example 21.1. Taking $a_n = (-1)^n z^n$ and $\alpha = 1/2$, a short calculation shows that the series (21.2) for the terms b_k converge when $|z| < 2$:

$$\begin{aligned} b_k &= \sum_{n=k}^\infty \binom{n}{k}\left(\frac{-z}{2}\right)^n = \left(\frac{-z}{2}\right)^k \sum_{m=0}^\infty \binom{k+m}{m}\left(\frac{-z}{2}\right)^m \\ &= \left(\frac{-z}{2}\right)^k \sum_{m=0}^\infty \binom{-k-1}{m}\left(\frac{z}{2}\right)^m = \left(\frac{-z}{2}\right)^k \left(1+\frac{z}{2}\right)^{-k-1}. \end{aligned}$$

Keeping $|z| < 2$ so that the terms b_k are well-defined, the series $\sum_{k=0}^\infty b_k$ converges for $|z| < |z+2|$ or $\operatorname{Re} z > -1$ to the analytic continuation of the original sum $\sum_{n=0}^\infty (-1)^n z^n$:

$$\sum_{k=0}^\infty b_k = \frac{2}{2+z}\sum_{k=0}^\infty \left(\frac{-z}{2+z}\right)^k = \frac{1}{1+z}.$$

We return to the *General Case* of (21.1). Setting $\sum_{k=0}^n a_k = s_n$, the partial sum $t_k = \sum_{n=0}^k b_n$ is the coefficient of y^k in the expansion

$$\frac{1}{1-y}f(1-\alpha+\alpha y) = \frac{\alpha}{1-x}\sum_0^\infty a_n x^n = \alpha\sum_0^\infty s_n x^n$$
$$= \alpha\sum_{n=0}^\infty s_n(1-\alpha+\alpha y)^n = \alpha\sum_{n=0}^\infty s_n\sum_{k=0}^n\binom{n}{k}(1-\alpha)^{n-k}\alpha^k y^k$$
$$= \sum_{k=0}^\infty y^k\alpha^{k+1}\sum_{n=k}^\infty\binom{n}{k}(1-\alpha)^{n-k}s_n.$$

Thus

$$t_k = \sum_{n=0}^k b_n = \alpha^{k+1}\sum_{n=k}^\infty\binom{n}{k}(1-\alpha)^{n-k}s_n$$
$$= \alpha^{k+1}\sum_{n=0}^\infty\binom{k+n}{k}(1-\alpha)^n s_{k+n}. \tag{21.3}$$

We now forget about the earlier assumption of convergence for $\sum_0^\infty a_n x^n$ and define numbers b_k by the series in (21.2) with $\alpha \in (0, 1)$. One can use the asymptotic relation $\binom{k+n}{k} \sim n^k/k!$ as $n \to \infty$ to verify that these series all converge if and only if the sequence

$$\{n^k(1-\alpha)^n a_n\},\ n = 0, 1, 2, \ldots \text{ is bounded for every } k \in \mathbb{N}. \tag{21.4}$$

Under this condition one can derive formula (21.3) from (21.2) and conversely.

Definition 21.2. The series $\sum_{n=0}^\infty a_n$ is *T_α-summable* to A if the numbers b_k in (21.2) are well-defined and the series $\sum_{k=0}^\infty b_k$ converges to A. Or equivalently, if the numbers t_k in (21.3) are well-defined and $t_k \to A$. The sequence $\{s_n\}$ is then called *limitable* to A by the *Taylor method T_α*.

(Note that some authors use parameter $1-\alpha$ where we have α.)

Since

$$\sum_{n=0}^\infty\binom{k+n}{k}(1-\alpha)^n = \sum_{n=0}^\infty\binom{-k-1}{n}(\alpha-1)^n$$
$$= (1+\alpha-1)^{-k-1} = \alpha^{-k-1}, \tag{21.5}$$

it readily follows from (21.3) that a convergent sequence $\{s_n\}$ with limit A is also T_α-limitable to A.

By the first line in (21.3), the matrix for T_α-limitability is equal to α times the transpose of the matrix for E_α-limitability; cf. (20.9).

The METHOD S_α (Meyer-König [1949], Vermes [1949]) is obtained from T_α by a shift of indices: instead of (21.3) one uses the rule

$$t_k^* = \alpha^{k+1}\sum_{n=0}^\infty\binom{k+n}{k}(1-\alpha)^n s_n. \tag{21.6}$$

The VALIRON METHOD V_α (also called F_α) is given by

$$t_k = \sqrt{\frac{\alpha}{\pi k}} \sum_{n=0}^{\infty} e^{-\alpha(n-k)^2/k} s_n \qquad (\alpha > 0). \tag{21.7}$$

There are many relations between the above (and other) methods; see Zeller and Beekmann (loc. cit.) for formulas and references.

GENERALIZED BOREL METHODS. Several authors have investigated Borel-type methods; cf. Zeller and Beekmann. Here we mention an example from the extensive work of D. Borwein and coauthors. Slightly changing their notation, we will say that a series $\sum_{n=0}^{\infty} a_n$ with partial sums s_n is $B(\alpha, \beta)$-summable to A (with $\alpha > 0,\ \beta \geq 0$) if

$$f(t) = \sum_{n=0}^{\infty} s_n \frac{\alpha t^{\alpha n+\beta}}{\Gamma(\alpha n + \beta + 1)} e^{-t} \quad \text{exists and} \quad f(t) \to A \quad \text{as } t \to \infty. \tag{21.8}$$

See Borwein [1958], [1960] and subsequent papers, and Shawyer and Watson [1994]. For $\alpha = 1,\ \beta = 0$ one obtains the ordinary Borel method.

22 The Special Methods as Γ_λ-Methods

For appropriate choices of the index λ, the special limitability methods in Sections 20, 21 can all be considered as circle methods Γ_λ in the sense of Definition 5.2. In order to prove this for the matrix methods, one has to represent the corresponding sequence to sequence transformations

$$t_k = \sum_{n=0}^{\infty} u_{kn} s_n \quad \text{as} \quad \sum_{n=0}^{\infty} s_n u_n(x). \tag{22.1}$$

Example 22.1. (*The Taylor method T_α*) Comparing the defining relation (21.3) with (22.1), one sets $u_n(x) = u_{kn} = 0$ for $n < k$ and

$$u_n(x) = u_{kn} = \alpha^{k+1} \binom{n}{k} (1-\alpha)^{n-k} \quad \text{for } n \geq k. \tag{22.2}$$

Aiming for the Basic Properties 5.1 (cf. Proposition 6.1), we want the maximum of $u_n(x)$ to occur for $n \approx x$. Since u_{kn} is maximal for $n \approx k/\alpha$, the parameter $x = x_k$ is taken equal to k/α. Replacing k in (22.2) by αx, one obtains

$$u_n(x) = \alpha \cdot \alpha^{\alpha x} \binom{n}{\alpha x} (1-\alpha)^{n-\alpha x} \quad \text{for } n \geq \alpha x. \tag{22.3}$$

It follows from (21.5) that $\sum_{n=0}^{\infty} u_n(x) = 1$, hence the functions u_n are uniformly bounded.

Writing $n = x + h$, we now use Stirling's formula to obtain a representation of type (5.3). For $x = x_k = k/\alpha \geq 1$ the result has the form

$$\log u_n(x) = -\lambda\frac{h^2}{x} + \frac{1}{2}\log\left(\frac{\lambda}{\pi x}\right) + a_3\frac{h^3}{x^2} + a_4\frac{h^4}{x^3} + \cdots$$
$$+ b_1\frac{h}{x} + b_2\frac{h^2}{x^2} + \cdots + \mathcal{O}\left(\frac{1}{x}\right), \tag{22.4}$$

with

$$\lambda = \frac{1}{2}\frac{\alpha}{1-\alpha}. \tag{22.5}$$

The development will be uniformly convergent for $|h| \leq \delta x$ with $\delta < 1 - \alpha$.

A short calculation will also provide an estimate of the form (5.4); cf. the proof of the Basic Properties 5.1 for the case of the Borel functions. Thus T_α is a circle method of type Γ_λ, with index λ given by (22.5).

The proof in the other cases is similar. The result is

Theorem 22.2. *The special circle methods E_α, T_α, S_α (with $0 < \alpha < 1$), V_α (with $\alpha > 0$) and $B(\alpha, \beta)$ (with $\alpha > 0$, $\beta \geq 0$) can be considered as circle methods Γ_λ. The function $u_n(x)$ associated with s_n, the domain X of the variable x and the index λ can be obtained from the defining relations* (20.9), (21.3), (21.6), (21.7), (21.8) *and the following table:*

E_α	$x = \alpha k$	$\lambda = 1/\{2(1-\alpha)\}$
T_α	$x = k/\alpha$	$\lambda = \alpha/\{2(1-\alpha)\}$
S_α	$x = (1-\alpha)k/\alpha$	$\lambda = \alpha/2$
V_α	$x = k$	$\lambda = \alpha$
$B(\alpha, \beta)$	$x = (t-\beta)/\alpha$	$\lambda = \alpha/2$

Here k runs through the positive integers and t is a continuous parameter greater than β. To get an explicit expression for $u_n(x)$, one takes the coefficient of s_n in the defining relation and expresses the parameters k or t in terms of x.

Cf. also Meir [1963]. In conjunction with Theorems 12.1, 13.1 and Proposition 18.3 for the methods Γ_λ, Theorem 22.2 confirms the known fact that the various special circle methods have a great deal of common Tauberian theory:

Theorem 22.3. *Let $\sum_{n=0}^{\infty} a_n$ be summable to A by one of the methods E_α, T_α, S_α, V_α or $B(\alpha, \beta)$. Suppose that the partial sums $s_n = \sum_{k=0}^{n} a_k$ satisfy the Tauberian condition (Schmidt condition)*

$$\liminf\,(s_m - s_n) \geq 0 \quad \textit{for} \ \ m \to \infty \ \ \textit{and} \ \ 0 \leq \sqrt{m} - \sqrt{n} \to 0. \tag{22.6}$$

Then $s_n \to A$ as $n \to \infty$.

Theorem 22.4. *Let the sequence $\{s_n\}$ be limitable to A by one of the methods E_α, T_α, S_α, V_α or $B(\alpha, \beta)$. Set $s^{(-1)}(v) = \sum_{n \leq v} s_n$. Suppose that $s_n \geq 0$ or $s_n \geq -C$ for all n, or more generally, that the average-type condition*

$$\lim_{b\searrow 0}\liminf_{v\to\infty}\inf_{0<a\le b}\frac{1}{b\sqrt{v}}\sum_{v<n\le v+a\sqrt{v}} s_n > -\infty \tag{22.7}$$

is satisfied; cf. Section 18. Then

$$\frac{s^{(-1)}(v+b\sqrt{v})-s^{(-1)}(v)}{\sqrt{v}} \to Ab \quad as\ v\to\infty, \quad \forall b>0. \tag{22.8}$$

Remarks 22.5. The Γ_λ-limitability of a sequence $\{s_n\}$ and the Schmidt condition (22.6) imply that $\{s_n\}$ is bounded (Section 8). If $s_n = o(\sqrt{n})$, the limitability is equivalent to a limit relation for integrals (Section 9); if $s_n = \mathcal{O}(1)$, Γ_λ-limitability implies Γ_μ-limitability for every index μ (cf. Remarks 12.2). The latter is also true if the sequence $\{s_n\}$ is bounded from below or satisfies the average-type condition (22.7); cf. Sections 13, 18.

Extending early results in Hardy and Littlewood [1916], Meyer-König [1949] established the equivalence of a number of classical circle methods for bounded sequences $\{s_n\}$; cf. Tietz and Zeller [2000] for a recent discussion. Gaier [1952] proved corresponding equivalences if the function $f(z) = \sum_0^\infty a_n z^n$ is bounded on the unit disc.

Meyer-König obtained such results also under the weaker conditions $s_n = o(\sqrt{n})$ and $a_n = o(1)$. In this case a method with index α_1 is equivalent to another method with index α_2 whenever α_1 and α_2 correspond to the same value of λ in Theorem 22.2. Some of these equivalences hold for all sequences $\{s_n\}$ of finite order: $s_n = \mathcal{O}(n^\gamma)$; see Hyslop [1936a], [1936b], Faulhaber [1956], Zeller and Beekmann [1958/70], Shawyer and Watson [1994].

It may seem surprising that Möbius transformations of the unit disc need not preserve the convergence of a power series at points of the circumference (Turán [1958]). An analysis of this phenomenon involving circle methods was made by Ishiguro, Meyer-König and Zeller [1990].

Jurkat [1956b] proved that a B-summable series $\sum_0^\infty a_n$ for which one has $\sum_0^\infty s_n z^n/n! = \mathcal{O}(e^{|z|})$ is Euler summable. Other comparisons in this area were made by Fridy and Roberts [1980], and Boos and Tietz [1992].

Using various boundedness conditions, including (22.7), Bingham, and Bingham and coauthors, proved the equivalence of a large number of special methods to the limitation method given by relation (22.8). We mention Bingham [1981], [1984a], [1984b], [1984c], [1985], [1988], Bingham and Goldie [1983], and Bingham and Tenenbaum [1986].

Doetsch [1931] showed that the Borel summability of $\sum_0^\infty a_n$ implies its Abel summability if $\sum_0^\infty a_n x^n$ is convergent for $|x| < 1$; see Section I.25 and cf. Karamata [1938]. Several authors have investigated conditions under which Borel summability implies (C,k) summability; see Hardy and Littlewood [1913a], Lord [1935], Hardy [1949], Jakimovski [1954b], Zeller and Beekmann [1958/70], Parameswaran [1975], Kwee [1983], Borwein and Markovich [1988]. Related results involving Borel or Borel-type and Riesz methods were obtained by Kwee [1989] and D. Borwein [1992]. Borwein and Markovich [1992] compared circle methods and weighted means methods.

That the condition $|\sqrt{n}a_n| \leq C$ is a Tauberian condition for Euler summability was shown already by Knopp [1922–23]. The corresponding result for the method $B(\alpha, \beta)$ was proved by Robinson and Borwein [1975]; cf. also Tam [1992]. Sõrmus [2000] discussed o-conditions for triangular matrix methods in the setting of Banach spaces.

23 High-Indices Theorems for Special Methods

It follows from Section 22 that Theorem 15.2 and Proposition 18.4 for general circle methods Γ_λ apply also to special circle methods.

Theorem 23.1. *Let the series $\sum_0^\infty a_n$ have square-root gaps as in Definition 15.1, let the terms a_n satisfy an order condition*

$$a_n = \mathcal{O}\left(e^{b\sqrt{n}}\right) \quad \textit{for some constant } b, \tag{23.1}$$

and suppose that $\sum_0^\infty a_n$ is summable to A by one of the methods E_α, T_α, S_α, V_α or $B(\alpha, \beta)$. Then $\sum_0^\infty a_n$ converges to A. One cannot allow smaller gaps than square-root gaps for any of the methods.

Since Euler summability implies Borel summability (Theorem 20.4), the former method also allows an unrestricted high-indices theorem; cf. Section 15. For the other special methods the situation is more complicated; cf. Remarks 15.4. Here the functional-analytic method of Meyer-König and Zeller, discussed in Chapter V, makes it possible to obtain sharper results for a number of methods. See Meyer-König and Zeller [1956], [1958], [1962], [1981], Schieber [1962], Zeller and Beekmann [1958/70].

For some special methods it is possible to obtain the desired sharper results directly from Theorem 23.1. We use the Taylor method T_α to illustrate this potential.

Theorem 23.2. *Let the series $\sum_0^\infty a_n$ have square-root gaps and be T_α-summable to A. Suppose that not only $a_n = \mathcal{O}\{n^{-k}(1-\alpha)^{-n}\}$ for every positive integer k, as is implied by the T_α-summability, but that $a_n = \mathcal{O}(c^{-n})$ for some number $c > 1-\alpha$, or that at least the point $x = 1-\alpha$ is a regular point for the function $f(x) = \sum_0^\infty a_n x^n$. Then $\sum_0^\infty a_n$ converges to A.*

The regularity condition is necessary and sufficient for the high-indices theorem; see Meyer-König and Zeller [1958]. Here we verify the sufficiency.

Proof. The T_α-summability of $\sum_0^\infty a_n$ to A means that the numbers b_k in (21.2) are well-defined (so that (21.4) holds) and that the series $\sum_0^\infty b_k$ converges to A. The hypothesis that $f(x)$ is analytic at the point $x = 1-\alpha$ implies that the numbers b_k are the expansion coefficients of $f(1-\alpha+\alpha y)$ as in (21.1). Since $\sum b_k$ converges, the power series $\sum_0^\infty b_k y^k$ for $f(1-\alpha+\alpha y)$ converges for $|y| < 1$. The sum function $g(y)$, when expressed in terms of x, provides an analytic continuation of $f(x)$ over the interval $(1-2\alpha, 1)$.

We now use the fact that the series $\sum_0^\infty a_n$ has square-root gaps. Then by Fabry's theorem, the circle of convergence for $\sum_0^\infty a_n x^n$ is natural boundary for the sum function $f(x)$; cf. Remarks 7.4. It follows that the power series has radius of convergence ≥ 1. Moreover, since $\sum_0^\infty b_k y^k \to \sum_0^\infty b_k = A$ as $y \nearrow 1$, the function $f(x)$ is bounded on the interval $\{0 \leq x < 1\}$. The lemma below will then show that the numbers a_n satisfy an inequality (23.1), so that Theorem 23.2 becomes a corollary to Theorem 23.1.

Lemma 23.3. *Let $\sum_0^\infty a_n$ have square-root gaps and suppose that the power series $\sum_0^\infty a_n x^n$ with $0 \leq x < 1$ converges to a bounded sum function $f(x)$. Then the numbers a_n satisfy an inequality* (23.1).

Proof. Setting $\sup_{0 \leq x < 1} |f(x)| = M$, one has

$$|f(\theta x)| = \left|\sum_0^\infty \theta^n a_n x^n\right| \leq M \quad \text{for } 0 \leq x \leq 1 \text{ whenever } 0 < \theta < 1. \tag{23.2}$$

For fixed θ, the series in (23.2) is uniformly convergent for $0 \leq x \leq 1$. Hence by the theorem of Müntz [1914] and Szász [1915], cf. Section 17 for the notation,

$$\begin{aligned} M \geq \sup_{0 \leq x \leq 1} |f(\theta x)| &= |\theta^{p_k} a_{p_k}| \sup_{0 \leq x \leq 1} \left| x^{p_k} + \sum_{j \neq k} (a_{p_j}/a_{p_k}) \theta^{p_j - p_k} x^{p_j} \right| \\ &\geq |\theta^{p_k} a_{p_k}| \, d_\infty(x^{p_k}, \Sigma_k) \geq |\theta^{p_k} a_{p_k}| \, d_2(x^{p_k}, \Sigma_k). \end{aligned}$$

Thus by (17.5),

$$|\theta^{p_k} a_{p_k}| \leq \frac{M}{d_2(x^{p_k}, \Sigma_k)} = M\sqrt{2p_k + 1} \prod_{j \neq k} \frac{p_j + p_k + 1}{|p_j - p_k|}.$$

The hypothesis of square-root gaps implies that the right-hand side has a bound $MCe^{b\sqrt{p_k}}$ independent of θ; cf. Section 17. Hence

$$|\theta^{p_k} a_{p_k}| \leq MCe^{b\sqrt{p_k}}. \tag{23.3}$$

Since this holds for every number $\theta \in (0, 1)$, inequality (23.1) follows.

24 Power Series Methods

It is possible to treat Abel and Borel summability together as special cases of so-called *power series methods*, for which we will use the notation J_μ instead of (J, p_n) or J_p. Such methods for the limitation of sequences $\{s_n = a_0 + \cdots + a_n\}$ occur in Hardy [1949] (section 4.12) and have received increasing attention from the Tauberian point of view since 1980. We limit ourselves to a sketch of the developments. Papers by Ishiguro [1964], [1965], Štěpánek [1966] and Kwee [1972] were followed

by work of Jakimovski and Tietz [1980], Mihalin [1980], D. Borwein [1981], Borwein and Meir [1987], Tietz and Trautner [1988], Borwein and Kratz [1989], and Kratz and Stadtmüller [1989a], [1990b]. This led to Theorem 24.4 below and various refinements. See the comments and references at the end of the section and cf. Boos [2000].

To define the method J_μ, one starts with a *comparison function*

$$m(x) = \sum_{n=0}^{\infty} \mu_n x^n, \quad \text{where } \mu_n \geq 0 \quad (\text{with } \mu_0 > 0), \tag{24.1}$$

and the series has positive radius of convergence R (which may be infinite).

Definition 24.1. We say that the sequence $\mathbf{s} = \{s_n\}$ is limitable to A by the power series method J_μ if

$$\text{the series } g(x) = \sum_{n=0}^{\infty} s_n \mu_n x^n \text{ converges for } |x| < R, \tag{24.2}$$

$$\text{and} \quad f(x) = \frac{g(x)}{m(x)} = \frac{\sum s_n \mu_n x^n}{\sum \mu_n x^n} \to A \quad \text{as } x \nearrow R. \tag{24.3}$$

Examples 24.2. For $\mu_n \equiv 1$ one obtains the Abel method. Indeed, in this case $R = 1$, $m(x) = 1/(1-x)$, and the sequence $\{s_n\}$ is J_μ-limitable to A if

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} a_n x^n \to A \text{ as } x \nearrow 1.$$

For $\mu_n = 1/n!$ one obtains the Borel method: $R = \infty$, $m(x) = e^x$, and $\{s_n\}$ is J_μ-limitable to A if

$$f(x) = e^{-x}\sum_{n=0}^{\infty} s_n x^n/n! \to A \quad \text{as } x \to \infty.$$

The choice $\mu_n = 1/(n+1)$ leads to *logarithmic* summability or limitability: $R = 1$, $m(x) = -(1/x)\log(1-x)$, and $\{s_n\}$ is J_μ-limitable to A if

$$f(x) = \frac{\sum s_n x^n/(n+1)}{\sum x^n/(n+1)} \to A \quad \text{as } x \nearrow 1.$$

If $0 < R < \infty$, we can and will normalize the method J_μ through replacement of the numbers μ_n by $\mu_n R^n$. This gives an equivalent method for which the radius of convergence is equal to 1. In this case one speaks of an *Abel-type method* J_μ. If $R = \infty$ one speaks of a *Borel-type method.* One now has the following criterion for regularity.

Proposition 24.3. *An Abel-type method J_μ is regular if and only if $m(x)$ tends to infinity as $x \nearrow 1$, or equivalently, $\sum_{k=0}^{n} \mu_k \to \infty$ as $n \to \infty$. A Borel-type method J_μ is regular if and only if $m(x)/x^n \to \infty$ as $x \to \infty$ for every $n \in \mathbb{N}_0$.*

Proof. *The case* $R = 1$. If J_μ is regular, it must limit the convergent sequence $\mathbf{e}_0 = \{1, 0, 0 \cdots\}$ to its limit 0, hence $\mu_0/m(x) \to 0$ as $x \nearrow 1$, or $m(x) \to \infty$. Conversely, if $m(x) \to \infty$ and $s_n \to A$, it readily follows from (24.3) that $g(x)/m(x) \to A$ as $x \nearrow 1$.

The case $R = \infty$. If $m(x)$ is a polynomial with highest nonzero coefficient μ_k, then $\mu_k x^k/m(x) \to 1$ as $x \to \infty$. This means that the sequence $\mathbf{e}_k = \{\delta_{kn}\}$ is limited to 1 instead of to its limit 0, so that the method J_μ fails to be regular. Thus for a regular method, infinitely many numbers μ_n are $\neq 0$. Conversely, if this is so and $s_n \to A$, (24.3) shows that $g(x)/m(x) \to A$.

In terms of the numbers

$$\delta_n \stackrel{\text{def}}{=} \sup_{0<x<R} x^n/m(x), \qquad n = 0, 1, 2, \ldots, \tag{24.4}$$

one has the following general 'little o'-Tauberian theorem; cf. Kratz and Stadtmüller (loc. cit.).

Theorem 24.4. *Let the sequence* $\{s_n\}$ *be limitable to* A *by the regular power series method* J_μ. *Suppose that* $a_{n+1} = s_{n+1} - s_n = o(\mu_n \delta_n)$, *or more precisely, that there is a positive sequence* $\{\varepsilon_n\}$ *with limit* 0 *such that*

$$|a_{n+1}| = |s_{n+1} - s_n| \leq \varepsilon_n \mu_n \delta_n, \qquad \forall n. \tag{24.5}$$

Then $s_n \to A$ *as* $n \to \infty$.

The proof will be given in Section 25. Although this theorem is not a 'big $\mathcal{O}$'-result, it has the advantage that the Tauberian condition (24.5) works for all methods J_μ. Moreover the condition is essentially optimal as to order.

Examples 24.5. In the case of Abel summability one has

$$\delta_n = \sup_{0<x<1} x^n(1-x) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \sim \frac{1}{en},$$

so that the Tauberian condition (24.5) reduces to Tauber's original condition $a_n = o(1/n)$. In the case of Borel summability one has

$$\delta_n = \sup_{0<x<\infty} x^n e^{-x} = n^n/e^n,$$

so that by Stirling's formula, $\delta_n \mu_n \sim 1/\sqrt{2\pi n}$. Here the Tauberian condition becomes $a_n = o(1/\sqrt{n})$. In the logarithmic case one finds $\delta_n \sim 1/\log n$, so that the Tauberian condition becomes $a_n = o\{1/(n \log n)\}$.

The proof of Theorem 24.4 will imply a *boundedness result* under a big $\mathcal{O}$-condition; cf. the situation in the case of Tauber's Theorem I.5.1.

Corollary 24.6. *Let J_μ be regular and let the sequence $\{s_n\}$ be J_μ-bounded, that is, $g(x)$ in* (24.2) *exists for $|x| < R$ and $f(x) = g(x)/m(x) = \mathcal{O}(1)$ as $x \nearrow R$. Then the Tauberian condition*

$$a_{n+1} = s_{n+1} - s_n = \mathcal{O}(\mu_n \delta_n) \tag{24.6}$$

implies that $\{s_n\}$ is bounded.

BIG $\mathcal{O}$-RESULTS. Are there Tauberian results which extend the $\mathcal{O}$-theorems of Hardy and Littlewood for Abel and Borel summability to the methods J_μ? One may in particular ask

Question 24.7. *Is condition* (24.6) *a Tauberian condition for regular power series methods J_μ?*

It turns out that this is so provided the weights μ_n satisfy suitable regularity conditions. Jakimovski and Tietz [1980] (among other results) gave a Tauberian condition related to (24.6). They required regular variation of the sequence $\{\mu_n\}$, that is, $\mu_n = \mu(n)$ with a regularly varying function $\mu(v)$; cf. Section IV.2. Kratz and Stadtmüller [1989a] proved that (24.6) is a Tauberian condition whenever $M(v) = \sum_{k \le v} \mu_k$ is regularly varying, or just $\mathcal{O}$-regularly varying with $M^*(1+) = 1$; cf. Section IV.6. In [1990a], Kratz and Stadtmüller treated weights

$$\mu_n \sim e^{\rho(n)},$$

where $\rho(v)$ is a smooth positive function with $\rho'(v) \searrow 0$ and $v\rho'(v) \to \infty$; such weights give Abel-type methods. In [1990b] they treated weights

$$\mu_n \sim e^{-\rho(n)},$$

with smooth positive functions $\rho(v)$ such that $\rho'(v) \nearrow \infty$ and $\rho''(v) \to 0$; such weights give Borel-type mehods. For these cases they proved that (24.6) is a Tauberian condition, with

$$\mu_n \delta_n \sim \sqrt{|\rho''(n)|/2\pi}.$$

More generally, Kratz, Stadtmüller and others have considered Schmidt-type oscillation conditions for methods J_μ and Dirichlet-series methods of summability. Besides the references for this section given before, we mention Tietz [1989], [1990], Kratz and Stadtmüller [1989b], Kiesel and Stadtmüller [1991], [1994], Ishiguro and Tietz [1992], Kiesel [1993b], [1995], Stadtmüller [1993], [1995], Borwein and Kiesel [1994], Borwein and Kratz [1994], Motzer and Stadtmüller [1994], Beurer, Borwein and Kratz [1999], Borwein, Kratz and Stadtmüller [2001].

Some of these authors also discussed gap series, as did Jakimovski, Meyer-König and Zeller [1981], [1987].

25 Proof of Theorem 24.4

Using the notation of Section 24 and proceeding roughly as in Kratz and Stadtmüller [1989a], [1990b], we begin with an auxiliary result.

Lemma 25.1. *Let J_μ be regular and $R = 1$ or $R = \infty$. Then there are numbers $x_n \in [0, R)$ such that*

$$\delta_n = \sup_{0<x<R} x^n/m(x) = x_n^n/m(x_n), \quad n = 0, 1, 2, \dots. \tag{25.1}$$

For any sequence $\{x_n\}$ as in (25.1) and $k = 0, 1, 2, \dots$,

$$x_n \le x_{n+1} \nearrow R, \quad x_k^{n-k} \le \delta_n/\delta_k \le x_n^{n-k}. \tag{25.2}$$

Furthermore

$$\frac{1}{\delta_n} \ge \begin{cases} \sum_{k=0}^{n} \mu_k \text{ if } R = 1, \\ \sum_{k=n}^{\infty} \mu_k \text{ if } R = \infty \text{ and } x_n \ge 1. \end{cases} \tag{25.3}$$

Proof. By Proposition 24.3, $m(x) \nearrow \infty$ as $x \nearrow R$, and $m(x)/x^n \to \infty$ for every $n \ge 1$ as $x \searrow 0$ and $x \nearrow R$. Hence $\delta_0 = 1/m(0)$, while for $n \ge 1$, the supremum in (25.1) is assumed for some $x_n \in (0, R)$.

By (25.1) and the inequality $\delta_k \ge x_n^k/m(x_n)$,

$$\frac{\delta_n}{\delta_k} = \frac{x_n^n/m(x_n)}{\delta_k} \le \frac{x_n^n/m(x_n)}{x_n^k/m(x_n)} = x_n^{n-k};$$

the inequality for δ_n/δ_k from below in (25.2) follows by symmetry. In particular $x_{n-1} \le x_n$. To show $x_n \nearrow R$, observe that for $n \ge 1$ the quotient $x^n/m(x)$ has derivative 0 at the maximum point $x = x_n \in (0, R)$:

$$nx_n^{n-1}m(x_n) - x_n^n m'(x_n) = 0, \quad \text{or} \quad n = x_n m'(x_n)/m(x_n).$$

It follows that x_n cannot remain below some number $r < R$, for that would imply $n < rm'(r)/m(0)$.

To verify (25.3) one observes that

$$\frac{1}{\delta_n} = \frac{m(x_n)}{x_n^n} = \sum_{k=0}^{\infty} \mu_k x_n^{k-n} \ge \begin{cases} \sum_{k=0}^{n} \mu_k \text{ if } x_n \le 1, \\ \sum_{k=n}^{\infty} \mu_k \text{ if } x_n \ge 1. \end{cases}$$

Proof of Theorem 24.4. We compare the partial sum $s_n = \sum_{k=0}^n a_k$ with the value of the function f in (24.3) at a point x_n given by (25.1). Writing

$$f(x) = \sum_{k=0}^{\infty} s_k \mu_k x^k/m(x), \quad m(x) = \sum_{k=0}^{\infty} \mu_k x^k,$$

one has

$$
\begin{aligned}
m(x_n)\{s_n - f(x_n)\} &= \sum_{k=0}^{\infty}(s_n - s_k)\mu_k x_n^k \\
&= \sum_{k=0}^{n-1}\sum_{j=k}^{n-1} a_{j+1}\mu_k x_n^k - \sum_{k=n+1}^{\infty}\sum_{j=n}^{k-1} a_{j+1}\mu_k x_n^k = S_1 - S_2,
\end{aligned}
\tag{25.4}
$$

say. Hence by hypothesis (24.5)

$$
\begin{aligned}
|S_1| &\le \sum_{k=0}^{n-1}\sum_{j=k}^{n-1} \varepsilon_j\mu_j\delta_j\mu_k x_n^k = \sum_{j=0}^{n-1}\varepsilon_j\mu_j\delta_j\sum_{k=0}^{j}\mu_k x_n^k \\
&= \sum_{j=0}^{n-1}\varepsilon_j\mu_j\delta_j(x_n/x_j)^j\sum_{k=0}^{j}\mu_k x_j^k(x_j/x_n)^{j-k}.
\end{aligned}
$$

To estimate the final sum over k we apply Lemma 25.1. Since $j < n$ one has $x_j \le x_n$. Also $j - k \ge 0$, so that the final sum is bounded by $\sum_{k=0}^{j}\mu_k x_j^k$, which does not exceed $m(x_j) = x_j^j/\delta_j$. As a result

$$
|S_1| \le \sum_{j=0}^{n-1}\varepsilon_j\mu_j x_n^j. \tag{25.5}
$$

Similarly

$$
\begin{aligned}
|S_2| &\le \sum_{k=n+1}^{\infty}\sum_{j=n}^{k-1}\varepsilon_j\mu_j\delta_j\mu_k x_n^k = \sum_{j=n}^{\infty}\varepsilon_j\mu_j\delta_j\sum_{k=j+1}^{\infty}\mu_k x_n^k \\
&= \sum_{j=n}^{\infty}\varepsilon_j\mu_j\delta_j(x_n/x_j)^j\sum_{k=j+1}^{\infty}\mu_k x_j^k(x_n/x_j)^{k-j} \\
&\le \sum_{j=n}^{\infty}\varepsilon_j\mu_j\delta_j(x_n/x_j)^j m(x_j) = \sum_{j=n}^{\infty}\varepsilon_j\mu_j x_n^j.
\end{aligned}
\tag{25.6}
$$

Combining (25.4)–(25.6) one obtains the inequality

$$
|s_n - f(x_n)| = \frac{|S_1 - S_2|}{m(x_n)} \le \frac{\sum_{j=0}^{\infty}\varepsilon_j\mu_j x_n^j}{\sum_{j=0}^{\infty}\mu_j x_n^j}. \tag{25.7}
$$

Here the final quotient will tend to 0 as $n \to \infty$ since $\varepsilon_j \to 0$, while the denominator goes to ∞. Indeed, the method J_μ is regular, hence $x_n \nearrow R$ by Lemma 25.1, and thus $m(x_n) \to \infty$ by Proposition 24.3. Conclusion: $\lim s_n = \lim f(x_n) = A$.

Proof of Corollary 24.6. One need only replace the numbers ε_j in the preceding proof by a constant C such that $|a_{j+1}| \le C\mu_j\delta_j$ for all j.

Remarks 25.2. In proofs of Tauberian theorems involving the $\mathcal{O}$-condition (24.6), the Corollary helps to go from (regular) J_μ-limitability of the sequence $\{s_n\}$ to *weighted-mean* limitability,

$$\sum_{k=0}^{n} \mu_k s_k \Big/ \sum_{k=0}^{n} \mu_k \to A \quad \text{as } n \to \infty. \tag{25.8}$$

If one sets $\sum_{k=0}^{n} \mu_k = \lambda_n$, the weighted mean (25.8) becomes the (shifted) Riesz mean

$$a_0 + \sum_{k=0}^{n} (1 - \lambda_k/\lambda_n) a_{k+1};$$

cf. Section I.18. A simple Tauberian condition for the step from (25.8) to convergence would now be $a_{n+1} = \mathcal{O}(\mu_n/\lambda_n)$; cf. the discussion in Hardy's book [1949] (section 6.1). One may also mention Reimers [1984] and Boos [2000]. More refined results for this step are in Móricz and Rhoades [1995], Tietz and Zeller [1997], and Móricz and Stadtmüller [2001]; cf. also Karamata [1937b] (theorem V) for an early integral form.

VII

Tauberian Remainder Theory

1 Introduction

This chapter deals with real Tauberian remainder theory, for both series and integrals, in the case where one has information on their transforms only in the real domain. There are three main topics.

The first part of the chapter, comprising Sections 1–12, deals with remainder theory for the case of Abel summability, so that the relevant transforms are power series or Laplace transforms. The remainders are obtained with the aid of polynomial approximation and, in some cases, complex analysis. The principal contributors to this theory were Freud, the author and Ganelius; see Sections 2–4 for the method and a detailed account of its development.

At first sight, the remainders provided by the real theory would seem rather weak. For example, in the case

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{A}{1-x} + \mathcal{O}(1) \quad \text{as } x \nearrow 1, \quad a_n \geq -C,$$

Freud obtained the estimate

$$s_N = \sum_{n=0}^{N} a_n = N\left\{A + \mathcal{O}\left(\frac{1}{\log N}\right)\right\} \quad \text{as } N \to \infty.$$

From the point of view of possible applications (cf. Section 22), the result was disappointing, and Freud's first article was held up for several years. When it became clear from Erdős-inspired examples of the author that the estimate was optimal, Turán quickly gave the green light for publication!

Freud obtained his results by an important improvement of Karamata's method for Hardy–Littlewood theorems (Section I.11): he introduced the technique of optimal *one-sided L^1 approximation* by polynomials. Independently using L^1 approximation and helped by Freud's work, the author obtained a variety of more general remainder theorems; cf. Section 2 and Section 3 on Laplace transforms. Shortly thereafter,

Ganelius formulated the very general Theorem 2.5 below for Laplace–Stieltjes transforms.

The theory of one-sided L^1 approximation by polynomials takes up Sections 5–8. Examples to show the optimality of various remainder estimates are given in Section 10. Some interesting results of real remainder theory are derived by a complex method, which involves harmonic-measure arguments. We mention the proof of Theorem 3.3 on vanishing remainders in Section 9 and the remainder estimates for general Dirichlet series in Sections 11, 12. The latter extend work of Freud and Halász.

Our second topic, taking up Sections 13–20, is the powerful Fourier integral method developed mostly by Ganelius, which applies to large classes of Wiener kernels K. Ganelius took his cue from Beurling, who proposed to use complex-analytic properties of the reciprocal of the Fourier transform, $1/\hat{K}$, for remainder estimates. Beurling treated a special case, which was later worked out by Lyttkens. Starting at about the same time (1954), Ganelius greatly enlarged the scope of the analytic theory; cf. Section 13. Sections 14–18 contain a thorough discussion of selected remainder theorems of Ganelius; additional results can be found in his Springer Lecture Notes [1971]. Section 19 is devoted to Frennemo's application of Ganelius's method to Laplace transforms.

Later, Lyttkens replaced Ganelius's analyticity and growth conditions on $1/\hat{K}$ in a strip or half-plane by growth conditions on the derivatives of $1/\hat{K}$ along the real axis. This made her results more general, but also more complicated; cf. Section 20.

The third part of the chapter is devoted to some special nonlinear problems of Erdős involving convolution of sequences; see Section 21 for an introduction. Sections 25–28 deal with Erdős's striking Tauberian remainder theorem related to the elementary proof of the prime number theorem. The present version of the proof makes use of a 'Fundamental Relation' due to Siegel. In Sections 22–24 a related 'square-root problem' is treated by a complex method developed by the author.

It is sometimes possible to obtain stronger remainder estimates, say for series $\sum_0^\infty a_n$, if one has information on the transform, in this case the power series $\sum_0^\infty a_n x^n$, also in the complex domain. Some examples of 'complex remainder theory' were treated at the end of Chapter III.

Many results of early remainder theory can be found in the short monograph by Postnikov [1980]. Here one should also mention the more elaborate volume by Subhankulov [1976] (Russian).

The reader will notice that the notation in this chapter occasionally differs from that in earlier chapters, because certain letters are needed for other variables.

2 Power Series and Laplace Transforms: How the Theory Developed

We consider Abel summability of series and integrals. Let $f(x)$ denote the sum of a convergent power series with real coefficients a_n,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad 0 \le x < 1. \tag{2.1}$$

Setting $x = e^{-1/u}$ and $s(t) = \sum_{n \le t} a_n$, one can write $f(x) = f(e^{-1/u})$ as an absolutely convergent Laplace–Stieltjes transform,

$$F(u) = \mathcal{L}ds(1/u) = \int_{0-}^{\infty} e^{-t/u} ds(t), \quad 0 < u < \infty. \tag{2.2}$$

In the case of general Laplace–Stieltjes transforms $F = \mathcal{L}ds$ we always assume that $s(\cdot)$ satisfies the following *standard conditions*: the function $s(t)$ vanishes for $t < 0$, is real-valued and of bounded variation on every finite interval, continuous from the right and such that the (possibly improper) integral for $F(u)$ exists for $0 < u < \infty$; cf. Section I.13. If $s(t)$ is absolutely continuous, so that $s(t) = \int_0^t a(v)dv$ with locally integrable $a(\cdot)$, one has

$$F(u) = \mathcal{L}a(1/u) = \int_{0}^{\infty-} a(t) e^{-t/u} dt. \tag{2.3}$$

Under the usual Tauberian conditions, this integral will be absolutely convergent; cf. Section 3.

In the following, the letters $C,\ c,\ C',\ C_1,\ \cdots$ denote positive constants, whose value may change from one formula to the next.

First Model Result. In [1951], [1952/53] Freud published remainder estimates for the Hardy–Littlewood Theorem I.7.4. The following result is typical. *Suppose that for a constant A, a* positive *number α and some number $\beta > 0$,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = [A + \mathcal{O}\{(1-x)^{\beta}\}] \frac{1}{(1-x)^{\alpha}} \quad \text{as } x \nearrow 1, \tag{2.4}$$

while $a_n \ge 0$, or $a_n \ge -Cn^{\alpha-1}$, for all n. Then

$$s_N = \sum_{n=0}^{N} a_n = \left\{ A + \mathcal{O}\left(\frac{1}{\log N}\right) \right\} \frac{N^{\alpha}}{\Gamma(\alpha+1)} \quad \text{as } N \to \infty. \tag{2.5}$$

Here the case $a_n \ge -Cn^{\alpha-1}$ is an easy consequence of the case $a_n \ge 0$. Indeed, one may replace a_n by $a'_n = a_n + Cn^{\alpha-1}$ for $n \ge 1$ (and a_0 by 0); comparisons with integrals show that

$$\sum_{n=1}^{N} n^{\alpha-1} = N^{\alpha}/\alpha + \mathcal{O}(N^{\alpha-1} + 1) \quad \text{as } N \to \infty,$$

$$\sum_{n=1}^{\infty} n^{\alpha-1} e^{-n/u} = \Gamma(\alpha) u^{\alpha} + \mathcal{O}(u^{\alpha-1} + 1) \quad \text{as } u = \frac{1}{|\log x|} \to \infty.$$

The model result follows from the analog for Laplace–Stieltjes transforms: *Suppose that $s(\cdot)$ satisfies the standard conditions, is nondecreasing and such that*

$$F(u) = \int_{0-}^{\infty} e^{-t/u} ds(t) = \{A + \mathcal{O}(u^{-\beta})\}u^{\alpha} \quad \text{as } u \to \infty, \tag{2.6}$$

where α, $\beta > 0$. Then

$$s(u) = \left\{A + \mathcal{O}\left(\frac{1}{\log u}\right)\right\} \frac{u^{\alpha}}{\Gamma(\alpha + 1)} \quad \text{as } u \to \infty. \tag{2.7}$$

For his proof, Freud refined Karamata's polynomial approximation method (Section I.11) by introducing precise one-sided L^1 approximation; cf. Section 5. At the same time, Postnikov [1951] and Korevaar [1951] published somewhat weaker results. The author's work implied that the order of Freud's remainder estimates was optimal. Cf. also the historical remarks in Ganelius [1986] and Nevai [1986].

Second Model Result. There is a corresponding remainder estimate for Littlewood's Theorem I.7.1 and for the Hardy–Littlewood Theorem I.7.2. *Suppose that for a constant A and some number $\beta > 0$,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = A + \mathcal{O}\{(1 - x)^{\beta}\} \quad \text{as } x \nearrow 1, \tag{2.8}$$

while

$$a_n = \mathcal{O}(1/n), \quad \text{or } a_n \geq -C/n. \tag{2.9}$$

Then

$$s_N = \sum_{n=0}^{N} a_n = A + \mathcal{O}\left(\frac{1}{\log N}\right) \quad \text{as } N \to \infty. \tag{2.10}$$

See Freud [1954] and Korevaar [1953], [1954a]. Formally, the second model result is the case $\alpha = 0$ of the first. However, the proof is more difficult, because condition (2.9) cannot be reduced to the condition $a_n \geq 0$. Also, the detour via Cesàro summability, which Karamata used for Littlewood's theorem (Section I.11), does not give the optimal remainder shown in (2.10). The present author used the basic idea of Wielandt's more direct method for Littlewood's theorem; cf. Section I.12. In order to obtain the optimal order of L^1 approximation, Korevaar [1953] used Jackson's kernel (Jackson [1930]), essentially the square of the Fejér kernel. This approach is simpler than Freud's, but made it necessary to impose the two-sided Tauberian condition $a_n = \mathcal{O}(1/n)$. In [1954a] the author therefore switched to Freud's technique of one-sided approximation.

The proofs by Freud and Korevaar were based on corresponding results for Laplace transforms, but the two authors used different analogs; see Theorems 3.5 and 3.2, respectively.

Remark 2.1. Both authors showed that under the conditions (2.8), (2.9), one has

$$\frac{s_0 + s_1 + \cdots + s_N}{N+1} = A + \mathcal{O}\left\{\frac{1}{(\log N)^2}\right\}, \tag{2.11}$$

with progressively smaller remainders for higher-order Cesàro means. There are corresponding results in connection with the subsequent remainder estimates; see Theorems 3.2, 3.5 and cf. formula (2.29).

A GENERAL RESULT. Korevaar [1954a], [1954b] and Ganelius [1954/55], [1956a] treated the 'general case' where $f(x)$ may approach its limit A quite rapidly. In the following $\varepsilon(u)$ stands for an arbitrary positive, continuous, nonincreasing function on $\mathbb{R}^+$ with limit 0 as $u \to \infty$. Suppose that

$$|f(e^{-1/u}) - A| = \left|\sum_0^\infty a_n e^{-n/u} - A\right| \le \varepsilon(u) \qquad (0 < u < \infty). \tag{2.12}$$

For efficient transition to a Laplace integral we set

$$a(t) = \begin{cases} a_0 - A & \text{for } 0 \le t < 1, \\ a_n & \text{for } n \le t < n+1, \quad n = 1, 2, \ldots. \end{cases} \tag{2.13}$$

Then

$$\begin{aligned} F(u) = \mathcal{L}a(1/u) &= \int_0^\infty a(t) e^{-t/u} dt \\ &= (a_0 - A) \int_0^1 e^{-t/u} dt + \sum_{n=1}^\infty a_n \int_n^{n+1} e^{-t/u} dt \\ &= \left(\sum_{n=0}^\infty a_n e^{-n/u} - A\right) \int_0^1 e^{-t/u} dt \\ &= \{f(e^{-1/u}) - A\} \frac{1 - e^{-1/u}}{1/u}. \end{aligned} \tag{2.14}$$

Hence in the case of (2.12)

$$|F(u)| \le \varepsilon(u) \qquad (u > 0). \tag{2.15}$$

We impose the Tauberian condition

$$a_n \ge -C/n, \quad \text{or more generally,} \quad a(t) \ge -C'/t \qquad (t > 0). \tag{2.16}$$

Under these conditions the author's Theorem 3.2 for Laplace transforms will give the 'remainder estimate'

$$\left|\int_0^u a(t) dt\right| \le \rho(u) \stackrel{\text{def}}{=} \min_{k \ge k_0} \left\{\frac{C_1}{k} + C_2^k \varepsilon\left(\frac{u}{k}\right)\right\} \qquad (u \ge 1), \tag{2.17}$$

where $k \in \mathbb{N}$ and $C_2 > 1$. In the power series case, an estimate for $s_N - A = (\sum_{n \le N} a_n) - A$ is obtained by setting $u = N + 1$. Since $\varepsilon(\cdot)$ is nonincreasing, we may replace $u = N + 1$ on the right-hand side of (2.17) by N, so that

$$|s_N - A| \le \rho(N).$$

ALTERNATIVE FORM OF THE REMAINDER ESTIMATE. In order to compare the estimate (2.17) with the general Theorem 2.5 below, we write

$$\varepsilon(u) = e^{-\omega(u)}, \tag{2.18}$$

where ω is a positive, continuous, *nondecreasing* function on $\mathbb{R}^+$ which is (usually) supposed to be *unbounded*. For the desired transformation of (2.17), set $C_2 = e^c$ with $c > 0$. To ensure that the term $e^{ck-\omega(u/k)}$ becomes $\mathcal{O}(1/k)$ one must take $\omega(u/k)$ somewhat larger than ck. It may be verified afterwards that the order of the estimate is not affected if one takes u such that for given $k \ge 1$, $\omega(u/k) = 2ck$. Then

$$(u/k)\omega(u/k) = 2cu. \tag{2.19}$$

Let $\theta(\cdot)$ be the *inverse function* of $v\omega(v)$, so that the relation $v\omega(v) = t$ is equivalent to $v = \theta(t)$. We need the following properties of the function $\theta(\cdot)$:

$$\begin{aligned} &\theta(t) \text{ is increasing}, \quad \theta(t)/t \text{ is nonincreasing}, \\ &\theta(t) = o(t) \text{ as } t \to \infty. \end{aligned} \tag{2.20}$$

Indeed, the inverse of a continuous increasing function is increasing. If $\theta(t) = v$ then $\theta(t)/t = v/\{v\omega(v)\} = 1/\omega(v)$. Finally, $v = o\{v\omega(v)\}$ if $\omega(v) \to \infty$.

Setting $\max\{2c, 1\} = c'$, the relations above give

$$\begin{aligned} &u/k = \theta(2cu) \le c'\theta(u), \quad C_1/k \le c'C_1\theta(u)/u, \\ &C_2^k\varepsilon(u/k) = e^{ck-\omega(u/k)} = e^{-ck} < 1/(ck) \le C\theta(u)/u, \end{aligned}$$

so that $\rho(u) \le C'\theta(u)/u$.

Theorem 2.2. *Let ω and θ be continuous monotonic functions on $\mathbb{R}^+$ as described above. Suppose that $F(u) = \mathcal{L}a(1/u)$ as in* (2.3) *satisfies the inequality*

$$|F(u) - A| \le e^{-\omega(u)} \qquad (0 < u < \infty), \tag{2.21}$$

and that $a(t) \ge -C'/t$. Then

$$|s(u) - A| = \left| \int_0^u a(t)dt - A \right| \le C^* \frac{\theta(u)}{u} \qquad (u \ge 1). \tag{2.22}$$

Examples 2.3. For $\beta, \gamma > 0$, simple calculations give the following relations for $u \ge u_0$:

$$\begin{aligned} \varepsilon(u) &= Cu^{-\beta} &&\Rightarrow \quad \rho(u) \le C^*(\beta)/\log u, \\ \varepsilon(u) &= Ce^{-\beta u} &&\Rightarrow \quad \rho(u) \le C^*(\beta)/\sqrt{u}, \\ \varepsilon(u) &= Ce^{-\beta u^\gamma} &&\Rightarrow \quad \rho(u) \le C^*(\beta, \gamma)/u^{\gamma/(\gamma+1)}. \end{aligned}$$

Generalizing the first example, Freud [1954] (cf. his article [1951]) discussed the case where $\varepsilon(u/k) \le e^{ck}\varepsilon(u)$. An equivalent growth condition on $\omega(\cdot)$ would be

$$\omega(eu) \le \omega(u) + b. \tag{2.23}$$

In this chapter we will occasionally refer to these conditions as 'Freud's condition'. In this case

$$\rho(u) \le \frac{C^*}{|\log \varepsilon(u)|} = \frac{C^*}{\omega(u)}, \tag{2.24}$$

but this is not true for general functions ε or ω. Results corresponding to the second and third example were obtained earlier by Avakumović [1950a], [1950b], who used complex methods; cf. also Vučković [1953].

Other Tauberian Conditions. The Tauberian condition may involve a *slowly varying function* L. We use comparison functions of the form $\phi(t) = \max\{t^{\alpha_1}, t^{\alpha_2}\}L(t)$, where $\alpha_1 \le \alpha_2$, $\alpha_1 \le 0$ and L is a positive continuous function on $[0, \infty)$ such that $L(tu)/L(u) \to 1$ as $u \to \infty$ for every number $t > 0$ (cf. Section IV.2). The condition

$$a_n \ge -\phi(n) \quad \text{implies} \quad a(t) \ge -C\phi(t) \tag{2.25}$$

for the function $a(t)$ of (2.13). In this case the constant C_1 in formula (2.17) has to be replaced by $C_1 u\phi(u)$; see the special case $\alpha = 0$ of Theorem 3.2.

Example 2.4. Under condition (2.25) the estimate

$$|f(e^{-1/u}) - A| \le \varepsilon(u) = Ce^{-\beta u}$$

(with $\beta > 0$) implies

$$|s_N - A| \le C'\sqrt{N}\phi(N) \qquad (N \ge 1).$$

In particular the series $\sum a_n$ will *converge* if $\phi(u) = o(1/\sqrt{u})$ as $u \to \infty$, or equivalently, if $a_n \ge -o(1/\sqrt{n})$. Information on the *rate of convergence* $f(e^{-1/u}) \to A$ enables one to *relax* the usual Tauberian conditions in the Hardy–Littlewood theorems involving power series (Section I.7).

More on Laplace–Stieltjes Transforms. The preceding results deal mostly with power series and ordinary Laplace transforms. However, for applications to general Dirichlet series one needs an extension of Theorem 2.2 to Laplace–Stieltjes transforms. Freud [1954] obtained a result for the case where $\varepsilon(u/k) \le e^{ck}\varepsilon(u)$; cf. Theorem 3.5 below. In a Comptes Rendus note [1956a], Ganelius formulated a very general extension in which the Tauberian condition is of *Schmidt type* and depends on the asymptotic behavior of the Laplace–Stieltjes transform:

Theorem 2.5. *Let $s(\cdot)$ satisfy the standard conditions listed after* (2.2)*, but with* $s(0) = 0$. *Let $\omega(\cdot)$ be a positive continuous nondecreasing function on $\mathbb{R}^+$ and let $\theta(\cdot)$ be the inverse function of $v\omega(v)$, so that $\theta(t) = o(t)$ if $\omega(v) \to \infty$. Let $\phi(t) = t^\alpha L(t)$ where α is real and L is slowly varying. Suppose that*

$$F(u) = \int_0^{\infty-} e^{-t/u} ds(t) = \mathcal{O}\{\phi(u)e^{-\omega(u)}\} \quad as \ u \to \infty, \tag{2.26}$$

and that $s(\cdot)$ satisfies the Tauberian condition

$$\inf_{u \le v \le u+\theta(u)} \{s(v) - s(u)\} \ge -C\phi(u)\frac{\theta(u)}{u} \quad as \ u \to \infty. \tag{2.27}$$

Then

$$s(u) = \mathcal{O}\left\{\phi(u)\frac{\theta(u)}{u}\right\} \quad as \ u \to \infty. \tag{2.28}$$

Ganelius sketched a proof based on polynomial approximation. He did not publish the details, but, putting the polynomial method aside, subsequently developed a powerful Fourier integral method with wider applicability in [1958], [1962], [1964], [1971]; see Sections 13–18. For the case $\phi(t) = 1$ and functions ω satisfying Freud's condition (2.23), the estimate (2.28) may be derived from Ganelius's Model Theorem 14.1. For the case $\phi(t) = t^\alpha$ with $\alpha > -1$ and unrestricted functions ω, the estimate follows from the application of the Fourier method by Ganelius's student Frennemo [1966–67], [1967]; see Section 19. [The case $\phi(t) = t^\alpha$ with $\alpha \le -1$ may be more difficult.] The special case $\phi(t) = 1$, $\omega(u) = \beta u^\gamma$ of (2.28), with the corresponding Tauberian condition (2.27), was treated by complex analysis in the work of Avakumović and Vučković (loc. cit.).

Half-Open Problem. It would be worth having a complete proof of Theorem 2.5 by polynomial approximation. Such a proof might be simpler than a proof by the Fourier integral method!

Under the conditions of Theorem 2.5, Ganelius's article [1956a] also contains the following estimate for the Cesàro means of $s(\cdot)$ of integral order m:

$$\sigma_m(u) = \int_{0-}^{u} (1 - t/u)^m ds(t) = \mathcal{O}\left\{\phi(u)\left(\frac{\theta(u)}{u}\right)^{m+1}\right\} \quad \text{as } u \to \infty. \tag{2.29}$$

Examples in Section 10 show that the orders of the remainder estimates in this section are *optimal* provided $\varepsilon(u) = e^{-\omega(u)}$ does not tend to zero too rapidly as $u \to \infty$. The *critical rate* corresponds to the case $\gamma = 1$ in the final Example 2.3; cf. Vučković (loc. cit.). More precisely, the results in Section 9 imply the following *vanishing theorem*:

Theorem 2.6. (Korevaar [1954a]) *Suppose that*

$$\limsup_{u \to \infty} \frac{\omega(u)}{u} = \infty. \tag{2.30}$$

Then the true remainders in Theorems 2.2 and 2.5 are identically equal to zero, so that $s(u) \equiv A$ or $s(u) \equiv 0$ for $u \ge 0$.

3 Theorems for Laplace Transforms

Let the function $s(\cdot)$ satisfy the *standard conditions* mentioned in Section 2: $s(t)$ vanishes for $t < 0$, is real-valued and of bounded variation on every finite interval, continuous from the right and such that the Laplace–Stieltjes integral

$$F(u) = \mathcal{L}ds(1/u) = \int_{0-}^{\infty} e^{-t/u} ds(t) \tag{3.1}$$

converges for $0 < u < \infty$. In the theorems of this section, the Tauberian conditions will imply that the integral in (3.1) is absolutely convergent, so that we do not have to write $F(u)$ as an improper integral. As before, $\varepsilon(u) = e^{-\omega(u)}$ will denote a positive, continuous, nonincreasing function on $(0, \infty)$ with limit zero.

Theorem 3.1 below can be applied immediately to power series and Dirichlet series with nonnegative coefficients. If the coefficents change sign infinitely often, Theorem 3.2 gives good results for the case of power series, as indicated in Section 2. To treat the remainder in Littlewood's theorem and an extension to Dirichlet series, Freud [1954] devised the special Theorem 3.5. We will derive a more general result for Dirichlet series in Section 11.

Theorem 3.1. *Let $s(t)$ be nondecreasing and satisfy the standard conditions, so that in particular the Laplace–Stieltjes transform $F(u) = \mathcal{L}ds(1/u)$ exists for $0 < u < \infty$. Suppose that for some number $\alpha \geq 0$ and some constant A (necessarily ≥ 0)*

$$F(u) = \{A + R(u)\}u^{\alpha}, \quad \textit{with} \quad |R(u)| \leq \varepsilon(u) = e^{-\omega(u)} \quad (0 < u < \infty). \tag{3.2}$$

Then there are numbers $C_1 \geq 0$, $C_2 > 1$ and $k_0 \in \mathbb{N}$ (which depend only on α) such that

$$\left| s(u) - \frac{A}{\Gamma(\alpha+1)} u^{\alpha} \right| \leq \rho_1(u) = \min_{k \geq k_0} \left\{ C_1 \frac{A}{k} + C_2^k \, \varepsilon\left(\frac{u}{k}\right) \right\} u^{\alpha} \tag{3.3}$$

for all $u > 0$. If $\alpha = 0$ one may take $C_1 = 0$, but in that case one has the trivial bound $\rho_1(u) = e\varepsilon(u)$.

In [1965] Ingham used peak functions to treat the special case where ω satisfies Freud's condition (2.23).

If $\alpha = 0$ the monotonicity condition on $s(\cdot)$ is too restrictive. In the following theorem we take $s(\cdot)$ absolutely continuous, $s(u) = \int_0^u a(t)dt$, and require that $a(\cdot)$ satisfy a suitable one-sided Tauberian condition. Our condition involves *arbitrary* real numbers $\alpha_1 \leq \alpha_2$ and a slowly varying function L:

$$a(t) \geq -\phi(t), \quad \phi(t) = \max\{t^{\alpha_1}, t^{\alpha_2}\}L(t) \quad (t > 0). \tag{3.4}$$

We assume that $L(t)$ is positive and continuous for $t \geq 0$, so that the quotient $L(tu)/L(u)$ is $\mathcal{O}(\max\{t^{\eta}, t^{-\eta}\})$ for every number $\eta > 0$, and this uniformly for $u \geq 0$; cf. Section IV.2.

Theorem 3.2. *Let $a(\cdot)$ be real, integrable over every finite interval $(0, B)$ and such that the Laplace transform*

$$F(u) = \mathcal{L}a(1/u) = \int_0^\infty a(t)e^{-t/u}dt \tag{3.5}$$

exists for $0 < u < \infty$. Suppose that $a(t) \geq -\phi(t)$ as in (3.4) and that $F(u)$ satisfies condition (3.2) with $\alpha \geq 0$. Then there are nonnegative numbers C_j (with $C_2 > 1$) and $k_0 \in \mathbb{N}$ (which depend only on α and ϕ) such that $s(u) = \int_0^u a(t)dt$ satisfies the estimate

$$\begin{aligned} &\left| s(u) - \frac{A}{\Gamma(\alpha+1)}u^\alpha \right| \\ &\leq \rho_2(u) = \min_{k \geq k_0} \left\{ \frac{C_0 A}{k}u^\alpha + \frac{C_1}{k}u\phi(u) + C_2^k\,\varepsilon\left(\frac{u}{k}\right)u^\alpha \right\} \end{aligned} \tag{3.6}$$

for $u \geq 1$. More generally, the Cesàro mean

$$\sigma_m(u) = \int_0^u (1 - t/u)^m a(t)dt \tag{3.7}$$

of integral order $m \geq 0$ satisfies an estimate

$$\begin{aligned} &\left| \sigma_m(u) - \frac{\Gamma(m+1)A}{\Gamma(m+\alpha+1)}u^\alpha \right| \\ &\leq \min_{k \geq k_0} \left\{ \frac{C_0 A}{k^{m+1}}u^\alpha + \frac{C_1}{k^{m+1}}u\phi(u) + C_2^k\,\varepsilon\left(\frac{u}{k}\right)u^\alpha \right\} \end{aligned} \tag{3.8}$$

for $u \geq 1$. Here the numbers C_j and k_0 may depend also on m. If $\alpha = 0$ one can take $C_0 = 0$. There is a corresponding result for $\alpha < 0$ if condition (3.2) holds with $A = 0$.

Examples in Section 10 will show that the remainders in (3.3) and (3.6) are essentially optimal when $\varepsilon(u)$ does not go to zero faster than e^{-Cu} as $u \to \infty$. More rapid decrease leads to a much stronger result. More precisely:

Theorem 3.3. (Vanishing Theorem) *Suppose that in Theorem 3.1 or 3.2*

$$\liminf_{u\to\infty} \frac{\log \varepsilon(u)}{u} = -\infty, \quad \textit{or equivalently,} \quad \limsup_{u\to\infty} \frac{\omega(u)}{u} = \infty. \tag{3.9}$$

Then the remainders $\rho_j(u)$ are identically zero, so that

$$s(u) \equiv Au^\alpha/\Gamma(\alpha+1) \quad \textit{for} \quad u \geq 0. \tag{3.10}$$

See Korevaar [1954a]. Vanishing theorems for other transforms can be found in Ganelius [1971]; cf. also Johansson [1996]. The proof of Theorem 3.3 uses complex analysis and will be given in Section 9.

In Section 4 we use the Karamata–Wielandt method of Sections I.11, I.12 to derive Theorems 3.1 and 3.2 from the following quantitative result on one-sided approximation; cf. Korevaar [1954a], [1954b] for the case $m = 0$.

Theorem 3.4. *Let $m \geq 0$ be an integer,*

$$G_m(t) = \begin{cases} (1-t)^m & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } 1 < t < \infty, \end{cases} \tag{3.11}$$

and let β and γ be arbitrary real numbers. Then there are constants $B_1 > 0$, $B_2 > 1$ and $k_0 \in \mathbb{N}$ depending only on m, β and γ such that the following holds. For every integer $k \geq k_0$, there are polynomials in the variable e^{-t} *of degree $\leq k$:*

$$p_k(t) = \sum_{j=1}^{k} \alpha_{kj} e^{-jt}, \quad P_k(t) = \sum_{j=1}^{k} \beta_{kj} e^{-jt}, \tag{3.12}$$

which satisfy the relations

$$p_k(t) \leq G_m(t) \leq P_k(t) \quad \text{for } 0 < t < \infty, \quad p_k(0) = P_k(0) = 1, \tag{3.13}$$

$$\int_0^\infty t^{-\beta} e^{\gamma t} \{P_k(t) - p_k(t)\} dt \leq \frac{B_1}{k^{m+1}}, \tag{3.14}$$

$$\sum_{j=1}^{k} |\alpha_{kj}| \leq B_2^k, \quad \sum_{j=1}^{k} |\beta_{kj}| \leq B_2^k. \tag{3.15}$$

The optimal order of L^1 approximation for arbitrary m was determined earlier by S.M. Nikol'skiĭ [1947], [1950]. One-sided approximation was studied by Freud in papers starting in 1951; cf. Freud [1955]. Our proof of Theorem 3.4 will take up Sections 5–8.

Finally we state a result of Freud [1954] for Laplace–Stieltjes integrals. We restrict ourselves to the principal case, which corresponds to the special case of Theorem 3.2 given by $\alpha = 0$, $\phi(t) = C/t$ and functions $\varepsilon(u)$ such as $Cu^{-\beta}$.

Theorem 3.5. *Let $s(t)$ satisfy the standard conditions, so that in particular the Laplace–Stieltjes transform $F(u) = \mathcal{L}ds(1/u)$ exists for $0 < u < \infty$. Suppose that for some constant A and a function $\varepsilon(u) \searrow 0$ which satisfies the condition $\varepsilon(u/k) \leq e^{ck}\varepsilon(u)$,*

$$F(u) = \int_{0-}^\infty e^{-t/u} ds(t) = A + \mathcal{O}\{\varepsilon(u)\} \quad \text{as } u \to \infty. \tag{3.16}$$

Suppose furthermore that there is a nondecreasing function $s_1(t)$ satisfying the standard conditions and a positive constant A_1 such that

$$F_1(u) = \int_{0-}^\infty e^{-t/u} ds_1(t) \text{ exists and equals } [1 + \mathcal{O}\{\varepsilon(u)\}]u \tag{3.17}$$

as $u \to \infty$, while

$$s_2(t) = A_1 s_1(t) + \int_0^t v ds(v) \quad \text{is nondecreasing.} \tag{3.18}$$

Then the Cesàro mean (2.29) *of s of integral order $m \geq 0$ satisfies the estimate*

$$\sigma_m(u) = A + \mathcal{O}\left\{\frac{1}{\log^{m+1}\{1/\varepsilon(u)\}}\right\} \quad as \ u \to \infty. \tag{3.19}$$

A Tauberian condition related to (3.17), (3.18) was used by Hardy in his book [1949] (theorem 102) to obtain convergence $s(u) \to A$ from convergence $F(u) \to A$. We will not pursue this approach because Theorem 3.5 was superseded by Ganelius's general Theorem 2.5 for Laplace–Stieltjes transforms; cf. the treatment in Sections 14 and 19.

In closing we mention that there are still other kinds of remainder estimates for Laplace transforms; see for example Mel'nik [1982].

4 Proof of Theorems 3.1 and 3.2

Here we derive Theorems 3.1 and 3.2 from Approximation Theorem 3.4.

Proof of Theorem 3.1. Let u be a point of continuity for $s(\cdot)$. Then the function $G_0(t)$ of formula (3.11) (which jumps at $t = 1$) and $s(tu)$ are never discontinuous at the same point t, so that we may write

$$s(u) = \int_{0-}^{u} ds(t) = \int_{0-}^{1} d_t s(tu) = \int_{0}^{\infty} G_0(t) d_t s(tu). \tag{4.1}$$

Hence if $P_k \geq G_0$ with P_k as in (3.12) for the case $m = 0$, then by (3.1) and (3.2)

$$\begin{aligned} s(u) &\leq \int_{0-}^{\infty} P_k(t) d_t s(tu) = \sum_{j=1}^{k} \beta_{kj} \int_{0-}^{\infty} e^{-jt} d_t s(tu) \\ &= \sum_{j=1}^{k} \beta_{kj} F\left(\frac{u}{j}\right) = \sum_{j=1}^{k} \beta_{kj} \left\{A + R\left(\frac{u}{j}\right)\right\}\left(\frac{u}{j}\right)^{\alpha}. \end{aligned} \tag{4.2}$$

We first take $\alpha > 0$. Then if $p_k \leq G_0 \leq P_k$,

$$\begin{aligned} &\sum_{j=1}^{k} \beta_{kj} \frac{1}{j^{\alpha}} = \int_{0}^{\infty} P_k(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \\ &= \int_{0}^{\infty} G_0(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{0}^{\infty} \{P_k(t) - G_0(t)\} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \\ &\leq \frac{1}{\alpha\Gamma(\alpha)} + \int_{0}^{\infty} \{P_k(t) - p_k(t)\} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt. \end{aligned} \tag{4.3}$$

Now let p_k and P_k be as in Theorem 3.4 with $m = 0$, $-\beta = \alpha - 1$ and $\gamma = 0$. For $k \geq k_0$ we then obtain from (3.14) that

$$\sum_{j=1}^{k} \beta_{kj} \frac{1}{j^{\alpha}} \leq \frac{1}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha+1)} \frac{B_1}{k}. \tag{4.4}$$

This result is also valid for $\alpha = 0$, because in that case (3.12) and (3.13) give $\sum_{j=1}^{k} \beta_{kj} = P_k(0) = 1$. Finally, by (3.2) and (3.15)

$$\sum_{j=1}^{k} \beta_{kj} R\left(\frac{u}{j}\right) \frac{1}{j^{\alpha}} \leq \sum_{j=1}^{k} |\beta_{kj}| \varepsilon \left(\frac{u}{j}\right) \leq B_2^k \varepsilon \left(\frac{u}{k}\right). \tag{4.5}$$

Using (4.3)–(4.5) together with (4.2), one finds that

$$s(u) - \frac{A}{\Gamma(\alpha+1)} u^{\alpha} \leq \left\{ \frac{\alpha B_1}{\Gamma(\alpha+1)} \frac{A}{k} + B_2^k \varepsilon \left(\frac{u}{k}\right) \right\} u^{\alpha}, \quad \forall k \geq k_0.$$

The inequality will hold on a dense set of values u, hence by the monotonicity of $s(\cdot)$ and $\varepsilon(\cdot)$, it will hold for all $u > 0$. For a corresponding inequality in the other direction one would use the minorants p_k of G_0 in (4.2) instead of the majorants P_k. Thus one obtains an estimate (3.3). Analysis shows that the numbers C_j and k_0 depend only on α; for $\alpha = 0$ one has $C_1 = 0$. □

Proof of Theorem 3.2. We take $u \geq 1$ and write

$$\sigma_m(u) = u \int_0^1 (1-t)^m a(tu) dt = u \int_0^{\infty} a(tu) G_m(t) dt. \tag{4.6}$$

By (3.4) one has $a(t) \geq -\phi(t)$, and since $L(tu)/L(u) = \mathcal{O}(\max\{t, 1/t\})$ and $\alpha_1 \leq \alpha_2$, we find that for any number $\beta \geq 1 - \alpha_1$ (which will be specified later),

$$\begin{aligned} \frac{\phi(tu)}{\phi(u)} &= \frac{\max\{t^{\alpha_1} u^{\alpha_1}, t^{\alpha_2} u^{\alpha_2}\} L(tu)}{u^{\alpha_2} L(u)} \leq \frac{\max\{t^{\alpha_1}, t^{\alpha_2}\} L(tu)}{L(u)} \\ &\leq C \max\{t^{\alpha_1 - 1}, t^{\alpha_2 + 1}\} \leq C' t^{-\beta} e^t. \end{aligned}$$

Thus for functions $p_k \leq G_m \leq P_k$,

$$\begin{aligned} &u \int_0^{\infty} \{-a(tu)\}\{P_k(t) - G_m(t)\} dt \leq u \int_0^{\infty} \phi(tu)\{P_k(t) - G_m(t)\} dt \\ &\leq C' u \phi(u) \int_0^{\infty} t^{-\beta} e^t \{P_k(t) - p_k(t)\} dt. \end{aligned}$$

We now let P_k and p_k be polynomials in e^{-t} as in Theorem 3.4, where we take $\gamma = 1$. By (3.14) we then obtain the inequality

$$u \int_0^{\infty} a(tu)\{G_m(t) - P_k(t)\} dt \leq C' u \phi(u) B_1 / k^{m+1}, \tag{4.7}$$

provided $k \geq k_0$. For the estimation of

$$u\int_0^\infty a(tu)P_k(t)dt = \sum_{j=1}^{k} \beta_{kj} F(u/j) = \sum_{j=1}^{k} \beta_{kj}\left\{A + R\left(\frac{u}{j}\right)\right\}\left(\frac{u}{j}\right)^\alpha \tag{4.8}$$

we proceed as in the preceding proof, first taking $\alpha > 0$. This time

$$\begin{aligned}
\sum_{j=1}^{k} \beta_{kj}\frac{1}{j^\alpha} &= \int_0^\infty [G_m(t) + \{P_k(t) - G_m(t)\}]\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt \\
&\le \int_0^1 (1-t)^m \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt + \int_0^\infty \{P_k(t) - p_k(t)\}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt \\
&\le \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} + \frac{\alpha}{\Gamma(\alpha+1)}\frac{C''B_1}{k^{m+1}}.
\end{aligned} \tag{4.9}$$

To estimate the final integral we have chosen $\beta = \max\{1-\alpha, 1-\alpha_1\}$ and used an inequality $t^{\alpha-1} \le C''t^{-\beta}e^t$ before applying (3.14) with $\gamma = 1$. Inequality (4.9) is also valid for $\alpha = 0$ and one can again use (4.5). Combining the results, we find that for $u \ge 1$ and $k \ge k_0$,

$$\sigma_m(u) - \frac{\Gamma(m+1)A}{\Gamma(m+\alpha+1)}u^\alpha \le \frac{C'B_1}{k^{m+1}}u\phi(u) + \left\{\frac{\alpha C''B_1}{\Gamma(\alpha+1)}\frac{A}{k^{m+1}} + B_2^k\,\varepsilon\left(\frac{u}{k}\right)\right\}u^\alpha.$$

There is a corresponding inequality in the other direction. These inequalities imply an estimate (3.6), with constants C_j depending only on α, ϕ and m. If $\alpha = 0$ one finds that $C_0 = 0$.

In the case $\alpha < 0$ and $A = 0$, the inequalities (4.7) and (4.8) remain valid, while (4.5) may be replaced by the inequality

$$\sum_{j=1}^{k} \beta_{kj} R\left(\frac{u}{j}\right)\frac{1}{j^\alpha} \le B_2^k \varepsilon\left(\frac{u}{k}\right)k^{|\alpha|}.$$

The factor $k^{|\alpha|}$ can be combined with B_2^k to give (3.6) with a larger number C_2. □

5 One-Sided L^1 Approximation

For the proof of Theorem 3.4 we have to determine the optimal order of one-sided L^1 approximation to certain well-behaved functions by polynomials of degree $\le k$. It is also necessary to control the size of the coefficients in the approximating polynomials $P(x) = \sum_{j=0}^{k} b_j x^j$. To that end we introduce the following *norm*:

$$\nu(P) = \nu\left(\sum_{j=0}^{k} b_j x^j\right) = \sum_{j=0}^{k} |b_j|. \tag{5.1}$$

In the constructions it will be convenient to work with the interval $(-1, 1)$ and the standard Chebyshev polynomials

$$T_r(x) = \cos rt, \quad \text{where } x = \cos t; \tag{5.2}$$

cf. Korevaar [1954a], [1954b], Freud [1955]. In his earlier papers Freud used orthogonal polynomials for weight functions which depend on the Tauberian problem under consideration. This might give better constants, but made the calculations less explicit. Ganelius [1956b] determined optimal L^1 approximations by trigonometric polynomials. Cf. Bojanic and DeVore [1966] for additional results on L^1 approximation.

We start by approximating functions of bounded variation.

Theorem 5.1. *Let f on $(-1, 1)$ be real and of bounded variation. Then there are constants A_j such that for every $k \in \mathbb{N}$, there are polynomials p and P of degree $\leq k$ with the following properties:*

$$p \leq f \leq P \quad \textit{on } (-1, 1), \tag{5.3}$$

$$\int_{-1}^{1} \{P(x) - p(x)\} \frac{dx}{\sqrt{1 - x^2}} \leq A_1 \frac{V}{k}, \tag{5.4}$$

$$\nu(p),\ \nu(P) \leq A_2 A_3^k, \tag{5.5}$$

where V is the total variation of f.

The proof will be derived from the special case where f is the step function

$$g(x, a) = \begin{cases} 0 \text{ for } x < a, \\ 1 \text{ for } x \geq a. \end{cases} \tag{5.6}$$

Proposition 5.2. *There are constants C_j independent of $a \in (-1, 1)$ such that for every $k \in \mathbb{N}$, there are polynomials $q = q(\cdot, a)$ and $Q = Q(\cdot, a)$ of degree $\leq k$ with the following properties:*

$$\begin{gathered} q \leq g(\cdot, a) \leq Q \quad \textit{on } (-1, 1), \\ \int_{-1}^{1} \{Q(x) - q(x)\} \frac{dx}{\sqrt{1 - x^2}} \leq \frac{C_1}{k}, \\ \nu(q),\ \nu(Q) \leq C_2 C_3^k. \end{gathered} \tag{5.7}$$

Derivation of Theorem 5.1. Denote the total variation of f by V. It may be assumed that f is defined on $[-1, 1)$ and continuous at -1. For given $k \in \mathbb{N}$ and $h = 0, 1, \ldots, k$, set

$$a_h = -\cos(h\pi/k), \quad \text{so that} \quad \int_{a_{h-1}}^{a_h} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{k} \quad \text{when } h \geq 1.$$

We first approximate f by piecewise constant functions with possible jumps at the points a_h, $h = 1, \ldots, k - 1$. Denoting the interval $[a_{h-1}, a_h)$ by I_h, $1 \leq h \leq k$, we let m_h and M_h stand for, respectively, the infimum and the supremum of f on I_h. Now set

$$s(x) = m_h \text{ and } S(x) = M_h \text{ for } x \in I_h, \quad h = 1, \dots, k.$$

Then $s \le f \le S$ on $[-1, 1)$ and

$$\int_{-1}^{1} \{S(x) - s(x)\} \frac{dx}{\sqrt{1-x^2}} = \sum_{h=1}^{k} (M_h - m_h) \frac{\pi}{k} \le \frac{\pi V}{k}. \tag{5.8}$$

The functions s and S can be represented in terms of step functions $g(\cdot, a_h)$ given by (5.6). We focus on S:

$$S(x) = M_1 + (M_2 - M_1) g(x, a_1) + \cdots + (M_k - M_{k-1}) g(x, a_{k-1}). \tag{5.9}$$

We now use majorants as in Proposition 5.2 to majorize the terms in S. The constant M_1 is majorized by itself. Denoting the polynomial majorant of degree $\le k$ for $g(x, a_h)$ by $Q(x, a_h)$, the following polynomial P of degree $\le k$ majorizes S and hence f on $[-1, 1)$:

$$P(x) = M_1 + \sum_{h=1}^{k-1} (M_{h+1} - M_h) Q(x, a_h). \tag{5.10}$$

By Proposition 5.2 and (5.9),

$$\begin{aligned} &\int_{-1}^{1} \{P(x) - S(x)\} \frac{dx}{\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \sum_{h=1}^{k-1} (M_{h+1} - M_h) \{Q(x, a_h) - g(x, a_h)\} \frac{dx}{\sqrt{1-x^2}} \\ &\le \sum_{h=1}^{k-1} (M_{h+1} - M_h) \frac{C_1}{k} \le C_1 \frac{V}{k}. \end{aligned} \tag{5.11}$$

There is a corresponding minorant p for s and f, for which the integral involving $s - p$ is bounded by $C_1 V/k$. Combining the results one obtains (5.4), with $A_1 = 2C_1 + \pi$.

The proof of (5.5) follows from (5.10), its analog for p and (5.7). The result is

$$\nu(p), \ \nu(P) \le \sup |f| + C_2 C_3^k V. \tag{5.12}$$

6 Proof of Proposition 5.2

Let $T_r(x)$ be the Chebyshev polynomial of degree $r \ge 2$ given by (5.2). Arranged in decreasing order, its zeros are given by

$$x_j = \cos t_j, \quad t_j = (j - 1/2)\pi/r, \quad j = 1, 2, \dots, r. \tag{6.1}$$

For given $a \in (-1, 1)$, let $g = g(\cdot, a)$ be the step function of (5.6). We will determine good approximating polynomials $q = q(\cdot, a)$ and $Q = Q(\cdot, a)$ which satisfy the

inequalities $q \leq g = g(\cdot, a) \leq Q$. Here we may restrict ourselves to polynomials of even degree $k = 2r - 2 \geq 2$.

If T_r has a zero $\geq a$, let x_s (with $s = s(r, a) \leq r$) be the smallest such zero, otherwise denote by x_{s+1} (with $s = s(r, a) \geq 0$) the largest zero of T_r less than a. In the 'generic case' one has $x_{s+1} < a \leq x_s$, where both x_s and x_{s+1} are zeros of T_r.

STEP I. In a construction which goes back to Markov and Stieltjes, cf. Szegő [1939/75] (section 3.411), a majorant $Q = Q(\cdot, a)$ of $g = g(\cdot, a)$ is obtained as the unique polynomial of degree $\leq k = 2r - 2$ which satisfies the following $2r - 1$ conditions (or $2r$ conditions if $s = r$):

$$Q(x_j) = \begin{cases} 1 \text{ for } 1 \leq j \leq s + 1, \\ 0 \text{ for } s + 2 \leq j \leq r, \end{cases} \quad Q'(x_j) = 0 \text{ for } j \neq s + 1. \tag{6.2}$$

If $s = r$ or $r - 1$ our majorant is $Q \equiv 1$. Suppose now that $s + 1 < r$ so that $Q \not\equiv 1$. Then by Rolle's theorem, the polynomial Q' of degree $\leq k - 1$ has a (smallest) zero on each of the $r - 2$ intervals (x_{j+1}, x_j) with $j \neq s + 1$. Together with the $r - 1$ zeros of Q' prescribed in (6.2) this adds up to $k - 1$ zeros. It follows that deg $Q' = k - 1$ and that Q' has no other zeros than those that we have indicated, and that all these zeros are simple. In particular Q must be monotonic on the interval $[x_{s+2}, x_{s+1}]$, hence increasing because $Q(x_{s+1}) = 1$ and $Q(x_{s+2}) = 0$. Conclusion: the polynomial Q must have a local minimum at each point x_j with $j \neq s + 1$ and a local maximum on every interval (x_{j+1}, x_j) with $j \neq s + 1$. It follows in particular that $Q \geq g$; cf. Figure VII.6, where $r = 6$ and $a \approx -0.13$.

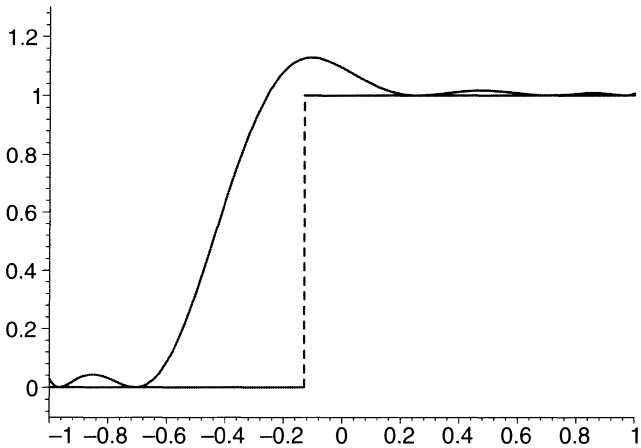

Fig. VII.6. The functions $g(\cdot, a)$ and Q

One similarly obtains a polynomial $q \leq g$ of degree $\leq k$ with the aid of the conditions

$$q(x_j) = \begin{cases} 1 \text{ for } 1 \leq j \leq s - 1, \\ 0 \text{ for } s \leq j \leq r, \end{cases} \quad q'(x_j) = 0 \text{ for } j \neq s. \tag{6.3}$$

When $s = 0$ or 1 the minorant q is identically zero. Observe that $q(x_j) = Q(x_j)$ at the zeros x_j of T_r with $j \le s-1$ or $j \ge s+2$.

STEP 2. We proceed to the inequality for the integral involving $Q-q$. It is convenient to introduce the auxiliary polynomials

$$U_j(x) = \left(\frac{T_r(x)}{T_r'(x_j)(x-x_j)}\right)^2, \quad j = 1, 2, \ldots, r. \tag{6.4}$$

The nonnegative polynomial U_j of degree $k = 2r-2$ is equal to 1 at x_j and has double zeros at the points $x_i \ne x_j$. Taking $1 \le s \le r-1$ for the time being, it follows from (6.2) and (6.3) that the polynomial

$$V = Q - q - U_s - U_{s+1} \tag{6.5}$$

of degree $\le k$ vanishes at all points x_i, hence $V = T_r W_{r-2}$ where W_{r-2} is a polynomial of degree $\le r-2$. Thus by the well-known orthogonality relations for trigonometric polynomials,

$$\int_{-1}^{1} V(x)\frac{dx}{\sqrt{1-x^2}} = \int_0^\pi (\cos rt)W_{r-2}(\cos t)dt = 0.$$

Hence by (6.5)

$$\begin{aligned}&\int_{-1}^{1}\{Q(x)-q(x)\}\frac{dx}{\sqrt{1-x^2}}\\ &= \int_0^\pi U_s(\cos t)dt + \int_0^\pi U_{s+1}(\cos t)dt = \lambda_s + \lambda_{s+1},\end{aligned} \tag{6.6}$$

say. (If $s = 0$ the integral reduces to λ_1 and if $s = r$, it reduces to λ_r.)

Allowing $1 \le s \le r$ we will estimate the integral for λ_s. Writing τ for $t_s = (s-1/2)\pi/r$, one has

$$T_r'(x_s) = \frac{r\sin r\tau}{\sin\tau} = \pm\frac{r}{\sin\tau}, \qquad U_s(\cos t) = \frac{\sin^2\tau}{r^2}\,\frac{\cos^2 rt}{(\cos t - \cos\tau)^2}.$$

Now for τ and t ranging over $(0,\pi)$, there is a constant $c > 0$ such that

$$\frac{|\cos t - \cos\tau|}{\sin\tau} = 2\left|\sin\frac{1}{2}(t-\tau)\right|\frac{\sin\{(t+\tau)/2\}}{\sin\tau} \ge c|t-\tau|.$$

It follows that

$$\lambda_s = \int_0^\pi U_s(\cos t)dt \le \frac{1}{r^2}\int_{\mathbb{R}}\frac{\sin^2 r(t-\tau)}{c^2(t-\tau)^2}dt = \frac{c'}{r} = \frac{2c'}{k+2}.$$

The upper bound applies also to λ_{s+1}. By (6.6) this completes the proof of the second inequality (5.7).

Incidentally, by the Gauss-Jacobi theory of quadrature based on the zeros of orthogonal polynomials, the numbers λ_s and λ_{s+1} are so-called Christoffel numbers. They belong to the weight function $(1-x^2)^{-1/2}$ on $(-1, 1)$ and the Chebyshev polynomials. For the polynomial T_r of degree r all the Christoffel numbers are equal to π/r; cf. Szegő (loc. cit., sections 3.4, 15.3), which confirms the second inequality (5.7).

STEP 3. We still have to estimate the norms of the approximating polynomials q and Q. The definition in (5.1) implies the following simple properties:

$$\nu(P_1 + P_2) \leq \nu(P_1) + \nu(P_2), \qquad \nu(P_1 P_2) \leq \nu(P_1)\nu(P_2). \tag{6.7}$$

Thus if all zeros of P have absolute value ≤ 1 and

$$P(x) = c_N \prod_{l=1}^{N} (x - z_l), \quad \text{then } \nu(P) \leq |c_N| 2^N. \tag{6.8}$$

For later use we observe that

$$\begin{aligned} T_r(x) &= \cos rt = \operatorname{Re}(\cos t + i \sin t)^r = \sum_{h \leq r/2} (-1)^h \binom{r}{2h} x^{r-2h} (1-x^2)^h \\ &= 2^{r-1} \prod_{l=1}^{r} (x - x_l) \qquad (x = \cos t, \;\; x_l = \cos\{(l - 1/2)\pi/r\}). \end{aligned} \tag{6.9}$$

For the estimation of $\nu(Q)$ it is convenient to introduce the polynomials V_j of degree $k = 2r - 2$, determined by the conditions

$$V_j(x_j) = 1, \quad V_j(x_i) = 0 \;\text{ for } i \neq j, \quad V_j'(x_i) = 0 \;\text{ for } i \neq s+1. \tag{6.10}$$

If $Q \equiv 1$ one has $\nu(Q) = 1$, hence we may take $s + 1 < r$. Then it follows from (6.2) that

$$Q(x) = \sum_{j=1}^{s+1} V_j(x) = 1 - \sum_{j=s+2}^{r} V_j(x), \tag{6.11}$$

so that it is sufficient to estimate $\nu(V_j)$. First take $j \neq s+1$. Then the product $(x - x_{s+1})V_j(x)$ has double zeros at the points $x_i \neq x_j$. Comparison with $U_j(x)$ in (6.4) shows that

$$(x - x_{s+1})V_j(x) = (a_j x + b_j) \left(\frac{T_r(x)}{x - x_j} \right)^2. \tag{6.12}$$

Here the coefficients a_j and b_j may be determined from the conditions

$$V_j(x_j) = 1, \quad V_j'(x_j) = 0.$$

Using the definition $T_r(\cos t) = \cos rt$, they give the equations

$$x_j - x_{s+1} = (a_j x_j + b_j)\frac{r^2}{1-x_j^2},$$

$$1 = a_j \frac{r^2}{1-x_j^2} + (a_j x_j + b_j)\frac{x_j}{1-x_j^2}\frac{r^2}{1-x_j^2}.$$

It follows that

$$a_j = (1 - 2x_j^2 + x_j x_{s+1})/r^2, \quad b_j = (x_j^3 - x_{s+1})/r^2. \tag{6.13}$$

Substituting these values into (6.12) and using (6.8), (6.9), one finds that for $j \neq s+1$

$$V_j(x) = (a_j x + b_j) 2^{2r-2} \prod_{i \neq j, s+1} (x - x_i) \prod_{i \neq j} (x - x_i), \tag{6.14}$$

with $|a_j|$, $|b_j| \leq 2/r^2$. For $j = s+1$ one has $V_j = U_j$; in this case one has to replace the factor $a_j x + b_j$ in (6.14) by a_{s+1}. In view of (6.8) and (6.11) the end result is

$$\nu(V_j) \leq \frac{1}{r^2} 2^{4r-3}, \quad \nu(Q) \leq 4^k. \tag{6.15}$$

A similar argument works for the norm of q. This completes the proof of (5.7).

7 Approximation of Smooth Functions

For the case $m \geq 1$ of Theorem 3.4 we also have to consider one-sided polynomial approximation to m-times differentiable functions f, whose mth derivative is of bounded variation. Here we use Freud's approach [1955]. Alternatively one could start from the optimal results of Ganelius [1956b] for corresponding trigonometric approximation. Cf. also Freud and Ganelius [1957], and Nevai [1972].

Theorem 7.1. *Let f on $[-1, 1]$ be real and an indefinite integral of order $m \geq 1$ of a function of bounded variation. Then there are constants A_j such that for every $k \in \mathbb{N}$, there are polynomials p and P of degree $\leq k$ with the following properties:*

$$p \leq f \leq P \quad \textit{on } (-1, 1), \tag{7.1}$$

$$\int_{-1}^{1} \frac{P(x) - p(x)}{\sqrt{1-x^2}} dx \leq \frac{A_1}{k^{m+1}}, \tag{7.2}$$

$$\nu(p), \ \nu(P) \leq A_2 A_3^k. \tag{7.3}$$

Proof. The pattern of the proof is the same as that of Theorem 5.1 and Proposition 5.2. For $a \in (-1, 1)$ one approximates an indefinite integral g_m of $g(\cdot, a)$ of order m:

$$g_m(x, a) = \begin{cases} 0 & \text{for } x < a, \\ (x-a)^m/m! & \text{for } x \geq a. \end{cases} \tag{7.4}$$

The approximation for f is derived from the one for g_m with the aid of Taylor's formula with base point -1, in a form which may be obtained by repeated integration by parts. It may be assumed here that the derivative $f^{(m)}$ is continuous from the right and continuous at the point 1. Substituting the canonical representation for $f^{(m)}$ as the difference of two nondecreasing functions, f_{m1} and f_{m2}, the Taylor formula becomes

$$\begin{aligned} f(x) &= \Pi_m(x) + \int_{-1}^{x} \frac{(x-a)^m}{m!} df^{(m)}(a) \\ &= \Pi_m(x) + \int_{-1}^{1} g_m(x,a)\{df_{m1}(a) - df_{m2}(a)\}, \end{aligned} \tag{7.5}$$

where Π_m is a polynomial of degree $\leq m$.

A majorant $Q_m = Q_m(\cdot, a)$ and minorant q_m for g_m may be obtained by multiplying approximants Q and q for g by $(x-a)^m/m!$. Suitable Q and q are easier to find for even than for odd m. We discuss the case $m = 2$ here; cf. the notation of Step 1 in Section 6. Suppose for simplicity that we are in the 'generic situation' where $x_{s+1} < a \leq x_s$ and there are additional zeros of T_r on both sides of a. Then one forms the majorant Q of $g(\cdot, a)$ of degree $\leq k-2 = 2r-4$, determined by the $2r-3$ conditions

$$Q(x_j) = \begin{cases} 1 \text{ for } 1 \leq j \leq s+1, \\ 0 \text{ for } s+3 \leq j \leq r, \end{cases} \quad Q'(x_j) = 0 \text{ for } j \neq s+1,\ s+2. \tag{7.6}$$

For a minorant q of degree $\leq 2r-4$ one imposes the conditions

$$q(x_j) = \begin{cases} 1 \text{ for } 1 \leq j \leq s-2, \\ 0 \text{ for } s \leq j \leq r, \end{cases} \quad q'(x_j) = 0 \text{ for } j \neq s-1,\ s. \tag{7.7}$$

Q increases from 0 to 1 for $x_{s+3} \leq x \leq x_{s+1}$ and q does so for $x_s \leq x \leq x_{s-2}$. One will have $Q = q$ at all points x_j with $j \neq s-1, s, s+1, s+2$ and $0 \leq Q - q \leq 1$ at the exceptional x_j. Observe also that $|x_j - a| \leq 2\pi/r$ for $s-1 \leq j \leq s+2$.

We now apply quadrature theory to $Q_2 - q_2 = (1/2)(x-a)^2(Q-q)$:

$$\begin{aligned} &\int_{-1}^{1} \{Q_2(x) - q_2(x)\} \frac{dx}{\sqrt{1-x^2}} = \sum_{j=s-1}^{s+2} \{Q_2(x_j) - q_2(x_j)\}\lambda_j \\ &\leq \sum_{j=s-1}^{s+2} (1/2)(x_j - a)^2 \frac{\pi}{r} \leq \left(\frac{2\pi}{r}\right)^3 = \left(\frac{4\pi}{k+2}\right)^3; \end{aligned} \tag{7.8}$$

cf. Section 6. Estimates for the norms of Q_2 and q_2 may also be obtained by adaptation of the earlier method.

How to proceed when m is odd? For $m = 1$ one may start with a polynomial $Q_1(x) = (x-a)Q(x)$, where Q of degree $\leq 2r-3$ is $\geq g(\cdot, a)$ for $x \geq x_s$ and $Q \leq g$ for $x \leq x_{s+1}$. The product $Q_1(x)$ will majorize $g_1(x,a) = (x-a)g(x,a)$ for $x \geq a$ and for $x \leq x_{s+1}$. To take care of the interval (x_{s+1}, a) one has to add a small nonnegative polynomial; cf. Freud (loc. cit.). Nevai (loc. cit.) treated the odd and even case together.

8 Proof of Approximation Theorem 3.4

Which functions f on $[-1, 1]$ do we have to approximate in Theorem 7.1 in order to obtain Theorem 3.4? We will map the interval $0 < t < \infty$ onto $1 > x > -1$ by setting $x = 2e^{-t} - 1$, so that $-t = \log\{(x+1)/2\}$. Taking $b = 2e^{-1} - 1$, so that $-1 = \log\{(b+1)/2\}$, we have $1 - t = \log\{(x+1)/(b+1)\}$. Thus the function $G_m(t)$ of (3.11) will correspond to

$$F_m(x) = \begin{cases} 0 & \text{for } -1 \le x < b, \\ \log^m\{(x+1)/(b+1)\} & \text{for } b \le x \le 1. \end{cases} \tag{8.1}$$

This function satisfies the conditions of Theorem 7.1, but inequality (7.2) is not strong enough to give inequality (3.14) if $\beta > 1/2$ or $\gamma > -1/2$. The following theorem will take care of the problem.

Theorem 8.1. *For an integer $m \ge 0$ and fixed $b \in (-1, 1)$, let F_m be the function given by* (8.1), *and let μ be a nonnegative integer. Then there are constants D_j such that for every integer $k \ge 2\mu + 1$, there are polynomials q and Q of degree $\le k$ with the following properties:*

$$\begin{aligned} &q \le F_m \le Q \quad \text{on } (-1, 1), \\ &q(1) = Q(1) = 1, \quad q(-1) = Q(-1) = 0, \\ &\int_{-1}^{1} \{Q(x) - q(x)\} \frac{dx}{(1-x^2)^{\mu+1/2}} \le \frac{D_1}{k^{m+1}}, \\ &\nu(q),\ \nu(Q) \le D_2 D_3^k. \end{aligned} \tag{8.2}$$

For $m = 0$ the result is in Korevaar [1954b], where it was shown in addition that the approximating polynomials can be chosen such that for some constant D_4,

$$|q^{(j)}(1)|,\ |Q^{(j)}(1)| \le D_4^j k^{j-1} \quad \text{for } j = 1, \dots, k.$$

Proof of the Theorem. It may be assumed that $\mu \ge 1$; cf. Theorem 7.1. We begin by determining a (the) polynomial R of degree $2\mu - 1$ such that the function

$$H(x) = \frac{F_m(x) - R(x)}{(1-x^2)^\mu} \tag{8.3}$$

is an indefinite integral of order m of a function of bounded variation on $(-1, 1)$. This requirement means that $R(x)$ must be divisible by $(x+1)^\mu$ and that $R(x) - F_m(x)$ must be divisible by $(x-1)^\mu$ near the point $x = 1$. More precisely, the expansion of $R(x)$ around the point $x = 1$ must have the form

$$\begin{aligned} R(x) &= \{1 + (x-1)/2\}^\mu \sum_{i=0}^{\mu-1} c_i (x-1)^i \\ &= \sum_{j=0}^{\mu-1} F_m^{(j)}(1) \frac{(x-1)^j}{j!} + d_\mu (x-1)^\mu + \cdots \end{aligned}$$

From this condition the coefficients c_i can be determined recursively. If $m = 0$ the polynomial $\sum_0^{\mu-1} c_i(x-1)^i$ will consist of the first μ terms in the binomial series for $\{1+(x-1)/2\}^{-\mu}$.

With the resulting polynomial R, Theorem 7.1 may be applied to the function $f = H$ in (8.3). Thus there are constants A_j such that for every integer $k \geq 2\mu + 1$, there are polynomials p^* and P^* of degree $\leq k - 2\mu$ with the following properties:

$$
\begin{aligned}
&p^* \leq H \leq P^* \quad \text{on } (-1,1),\\
&\int_{-1}^{1} \{P^*(x) - p^*(x)\} \frac{dx}{\sqrt{1-x^2}} \leq \frac{A_1}{(k-2\mu)^{m+1}},\\
&\nu(p^*),\ \nu(P^*) \leq A_2 A_3^{k-2\mu}.
\end{aligned}
\tag{8.4}
$$

Now set

$$q = (1-x^2)^\mu p^* + R, \qquad Q = (1-x^2)^\mu P^* + R.$$

Then by (8.3), (8.4) and since $\nu\{(1-x^2)^\mu\} = 2^\mu$ by (5.1),

$$
\begin{aligned}
&q(x) \leq (1-x^2)^\mu H(x) + R(x) = F_m(x) \leq Q(x) \quad \text{on } (-1,1),\\
&q(1) = Q(1) = 1, \quad q(-1) = Q(-1) = 0,\\
&\int_{-1}^{1} \{Q(x) - q(x)\} \frac{dx}{(1-x^2)^{\mu+1/2}} \leq \frac{A_1}{(k-2\mu)^{m+1}},\\
&\nu(q),\ \nu(Q) \leq 2^\mu A_2 A_3^{k-2\mu} + \nu(R).
\end{aligned}
$$

These relations imply the desired result (8.2).

Derivation of Theorem 3.4. Consider the map from $0 < t < \infty$ onto $1 > x > -1$ given by $x = 2e^{-t} - 1$. For $b = 2e^{-1} - 1$, the function $G_m(t)$ of (3.11) corresponds to the function $F_m(x)$ of (8.1). Taking μ equal to the least nonnegative integer $\geq \beta - 1/2$ and $> \gamma + 1/2$, one has in particular

$$t^{-\beta} e^{\gamma t} \leq C(\beta,\gamma)(1-e^{-t})^{-\mu-1/2} e^{(\mu-1/2)t}.$$

We now define polynomials $p_k(t)$ and $P_k(t)$ of degree $\leq k$ in e^{-t} by the polynomials $q(x)$ and $Q(x)$ provided by Theorem 8.1. Then by (8.2)

$$p_k(\infty-) = q(-1) = 0, \qquad P_k(\infty-) = Q(-1) = 0,$$

so that p_k and P_k have constant term zero as required by (3.12). Observe that also $p_k \leq G_m \leq P_k$ on $(0,\infty)$ and $p_k(0) = P_k(0) = 1$, which is (3.13). Furthermore

$$
\begin{aligned}
&\int_0^\infty (1-e^{-t})^{-\mu-1/2} e^{(\mu-1/2)t} \{P_k(t) - p_k(t)\} dt\\
&= 2^{2\mu} \int_{-1}^{1} \{Q(x) - q(x)\} \frac{dx}{(1-x^2)^{\mu+1/2}} \leq \frac{2^{2\mu} D_1}{k^{m+1}},
\end{aligned}
\tag{8.5}
$$

which implies (3.14).

It remains to consider sums of absolute values of coefficients. We may write

$$Q(x) = \sum_{h=0}^{k} b_h x^h = \sum_{h=0}^{k} b_h (2e^{-t} - 1)^h = P_k(t) = \sum_{j=0}^{k} \beta_j e^{-jt},$$

so that

$$\beta_j = \sum_{h=j}^{k} b_h \binom{h}{j} 2^j (-1)^{h-j} \qquad (\beta_0 = 0).$$

It follows that

$$\sum_{j=0}^{k} |\beta_j| \leq \sum_{j=0}^{k} \sum_{h=j}^{k} |b_h| \binom{h}{j} 2^j$$
$$= \sum_{h=0}^{k} |b_h| \sum_{j=0}^{h} \binom{h}{j} 2^j = \sum_{h=0}^{k} |b_h| 3^h \leq 3^k \nu(Q). \tag{8.6}$$

There is a corresponding result involving the coefficients α_j of p_k. In conjunction with the norm inequality in (8.2) one thus obtains an inequality (3.15).

9 Vanishing Remainders: Theorem 3.3

The proof of the Vanishing Theorem 3.3 will be derived from the following

Proposition 9.1. *Let $G(w) = G(u+iv)$ be a bounded analytic function on the (open) right half-plane $\{u > 0\}$ such that*

$$|G(u)| \leq \varepsilon(u) = e^{-\omega(u)} \quad \textit{for } 0 < u < \infty, \tag{9.1}$$

where ω is a nondecreasing function on $\mathbb{R}^+$ for which

$$\limsup_{u\to\infty} \frac{\omega(u)}{u} = \infty. \tag{9.2}$$

Then $G \equiv 0$.

There is a related but somewhat weaker result for Laplace and other transforms in Hirschman and Widder [1955] (section 10.3). The present Proposition may be deduced from a precise result of Ahlfors and Heins [1949] on the size of the set, where a bounded analytic function on a half-plane can be small without being identically zero; cf. Boas [1954] (section 7.2). We will give a simple proof by a *harmonic-measure argument*, a technique developed by Nevanlinna [1936/70].

Proof of the Proposition. It may be assumed that $|G(w)| \leq 1$ and, supposing $G \not\equiv 0$, that $G(1) \neq 0$; otherwise, one could first divide out a power of $w - 1$. Then $\log |G|$

is a subharmonic function which is bounded above by 0. Below we will obtain a contradiction by showing that condition (9.2) implies $G(1) = 0$.

For $B > 0$, let D_B be the domain obtained from the open right half-plane by deleting the half-line $L_B : \{v = 0,\ u \geq B\}$. Thus L_B becomes part of the boundary of D_B. (More precisely, the upper and lower side of L_B become part of the boundary.) Observe that by (9.1) and the monotonicity of ω,

$$\sup_{u\in[B,\infty)} \log|G(u)| \leq -\omega(B). \tag{9.3}$$

Let $h_B(\cdot)$ denote the so-called harmonic measure of the two-sided half-line L_B relative to D_B. That is, h_B is the harmonic function on D_B, with values between 0 and 1, which has boundary values 1 on L_B and 0 on the imaginary axis. Then $\log|G| + \omega(B)h_B$ is subharmonic on D_B and its boundary values are ≤ 0 (the lim sup on approach to the boundary is ≤ 0). Taking $B > 1$, the maximum principle for subharmonic functions (that are bounded from above) now shows that

$$\log|G(1)| + \omega(B)h_B(1) \leq 0. \tag{9.4}$$

Using the fact that conformal mappings preserve harmonic measure, one could compute the harmonic function h_B explicitly by using a suitable sequence of such mappings; cf. Heins [1962] for a discussion of the equivalent 'Milloux problem'. However, we need only a good lower bound for $h_B(1)$, and such a bound can be obtained from simpler mappings.

By an initial change of scale we replace D_B by D_1 and the point 1 by $b = 1/B < 1$; one has $h_B(1) = h_1(b)$. Next it is convenient to carry out a reflection in the imaginary axis and a translation to the right over a distance 1. Thus D_1 goes over into the domain $\tilde{D}$, consisting of the half-plane $\{u < 1\}$ minus the negative u-axis. One has $h_1(b) = \tilde{h}(1-b)$, where $\tilde{h}(\cdot)$ stands for the harmonic measure of the negative real axis relative to $\tilde{D}$. Finally we apply the square-root transformation $z = x + iy = \sqrt{w}$, which maps $\tilde{D}$ onto $D^* = (\tilde{D})^{1/2}$. This domain is bounded on the left by the imaginary axis $\{x = 1\}$ and on the right by the branch of the hyperbola $\{x^2 = y^2 + 1\}$ on which $x \geq 1$; see Figure VII.9. Let $h^*(\cdot)$ denote the harmonic measure of the imaginary axis relative to D^*, so that $\tilde{h}(1-b) = h^*(c)$, where $c = \sqrt{1-b}$.

Observe now that D^* contains the infinite strip $\Sigma : \{0 < x < 1\}$. By the maximum principle for (bounded) harmonic functions, $h^*(z) \geq h(z)$ in Σ, where h denotes the harmonic measure of the imaginary axis relative to Σ. Indeed, one has $h^*(iy) = h(iy) = 1$ and $h^*(1+iy) \geq 0 = h(1+iy)$. The harmonic measure $h(x+iy)$ is of course equal to $1-x$. Putting it all together, we find that

$$\begin{aligned} &h_B(1) = h_1(b) = \tilde{h}(1-b) = h^*(c) \geq h(c) = 1-c; \\ &1-c = 1-\sqrt{1-b} > b/2 = 1/(2B). \end{aligned} \tag{9.5}$$

Thus, combining (9.4) and (9.5),

$$\log|G(1)| \leq -\omega(B)h_B(1) \leq -\omega(B)/(2B), \quad \forall\, B > 1. \tag{9.6}$$

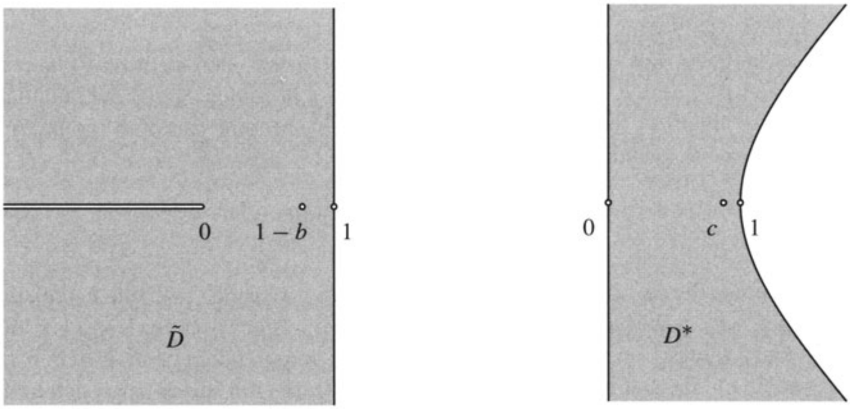

Fig. VII.9. The domains $\tilde{D}$ and D^*

By (9.2) the right-hand side has lim inf equal to $-\infty$ as $B \to \infty$, so that $G(1)$ must be zero, the promised contradiction. Conclusion: $G \equiv 0$.

Proof of Theorem 3.3. It will be enough to deal with the case where $s(\cdot)$ and F satisfy the hypotheses of Theorem 3.1, now with a function ε or ω as in (3.9). We begin by using the conclusion provided by (3.3) when we take $k = k_0$ and $u \geq 1$. Setting $s_1(t) = 0$ for $t < 0$ and

$$s_1(t) = s(t) - At^\alpha/\Gamma(\alpha+1) \quad \text{for } t \geq 0, \tag{9.7}$$

and observing that $\varepsilon(\cdot)$ is nonincreasing, one finds that

$$|s_1(t)| \leq \{C_1(A/k_0) + C_2^{k_0}\varepsilon(1/k_0)\}t^\alpha = \mathcal{O}(t^\alpha + 1) \quad \text{for } t \geq 1. \tag{9.8}$$

By the boundedness of $s_1(t)$ for $0 \leq t < 1$, the final estimate will hold for $0 \leq t < \infty$.

As a function of the complex variable $w = u + iv$, the Laplace–Stieltjes transform

$$\begin{aligned} F_1(w) &= \mathcal{L}ds_1(1/w) = F(w) - Aw^\alpha \\ &= \int_{0-}^{\infty} e^{-t/w} ds_1(t) = (1/w)\int_0^\infty s_1(t)e^{-t/w}dt \end{aligned} \tag{9.9}$$

is an analytic function on the right half-plane $\{u > 0\}$. By (9.8) it satisfies an inequality

$$|F_1(w)| \leq (C/|w|)\int_0^\infty (t^\alpha + 1)e^{-tu/|w|^2}dt = \mathcal{O}(|w|^{2\alpha+1}) \quad \text{for } u > 1. \tag{9.10}$$

Thus the analytic function

$$G(w) \stackrel{\text{def}}{=} \frac{F_1(w+1)}{(w+1)^{2\alpha+1}} \tag{9.11}$$

is bounded for $u = \operatorname{Re} w > 0$.

We now use hypothesis (3.2), which implies that

$$|G(u)| \leq \frac{R(u+1)(u+1)^{\alpha}}{(u+1)^{2\alpha+1}} \leq \frac{e^{-\omega(u+1)}}{(u+1)^{\alpha+1}} \leq e^{-\omega(u)} \quad \text{for } u > 0. \tag{9.12}$$

Here $\omega(\cdot)$ satisfies condition (9.2) by hypothesis (3.9) in Theorem 3.3. Hence by Proposition 9.1,

$$G(w) \equiv 0, \quad \text{so that } F_1(w) \equiv 0.$$

The uniqueness theorem for Laplace transforms finally implies that $s_1(t) = 0$ almost everywhere, so that $s(t) = At^{\alpha}/\Gamma(\alpha+1)$ almost everywhere for $t > 0$; see (9.7). Since $s(\cdot)$ is continuous from the right, the final equality holds for all $t \geq 0$. □

10 Optimality of the Remainder Estimates

The first two examples involve *power series*. They will show that the orders of the remainders in Theorem 2.2/Examples 2.3 are optimal when

$$\varepsilon(u) = Cu^{-\beta} \quad \text{or} \quad \varepsilon(u) = Ce^{-\beta u} \qquad (\beta > 0); \tag{10.1}$$

additional examples are in Korevaar [1951], [1954a]. It will be convenient to use the variable $x = e^{-1/u}$ instead of u; observe that

$$u = -\frac{1}{\log x} = \frac{1}{1-x} - \frac{1}{2} + o(1) \quad \text{as } x \nearrow 1.$$

We will write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{so that } \sum_{n=0}^{\infty} a_n e^{-n/u} = f(e^{-1/u}). \tag{10.2}$$

By formula (2.14), the Laplace transform $F(u)$ in Theorem 2.2 and the difference $f(e^{-1/u}) - A$ behave in exactly the same way as $u \to \infty$.

Example 10.1. The condition $f(e^{-1/u}) - A = \mathcal{O}(u^{-\beta})$ as $u \to \infty$ corresponds to

$$f(x) - A = \mathcal{O}\{(1-x)^{\beta}\} \quad \text{as } x \nearrow 1. \tag{10.3}$$

Taking $\beta = 2$ for convenience, it will be shown that even under Littlewood's two-sided Tauberian condition $a_n = \mathcal{O}(1/n)$, the remainder estimate $s_N - A = \mathcal{O}(1/\log N)$ is optimal.

The following construction is a refinement of one used in Section I.24 in connection with Littlewood's theorem. The desired function f is obtained by adding well-separated blocks of terms, each block $f_{p,q,r}$ involving positive integers $p,\ q,\ r$:

$$f_{p,q,r}(x) = \sum_{p+1\le n\le p+(2r+3)q} a_n x^n$$
$$= \frac{1}{p+(2r+3)q} x^{p+1}(1+x+\cdots+x^{q-1}) \frac{(1-x^q)^{2r+2}}{\binom{2r+2}{r+1}}. \tag{10.4}$$

The final quotient represents a polynomial whose coefficients have absolute value ≤ 1, hence

$$|a_n| \le \frac{1}{p+(2r+3)q} \le \frac{1}{n}.$$

For the partial sums s_N of $\sum a_n$ one has

$$|s_{p+(r+2)q} - s_{p+(r+1)q}| = \frac{q}{p+(2r+3)q}. \tag{10.5}$$

Taking r *large* we choose $p = 2rq \approx e^r$. Then the right-hand side of (10.5) is about $q/(2p) \approx 1/(4\log p)$, hence at least one of the partial sums in (10.5) has absolute value $\ge 1/(9\log p)$. It follows that for $N = p+(r+1)q$ or for $N = p+(r+2)q$ one has $|s_N| \ge 1/(10\log N)$ when r is sufficiently large.

A crude estimate will show that

$$0 \le f_{p,q,r}(x) \le \frac{1}{r}(1-x)^2. \tag{10.6}$$

Indeed,

$$0 \le \frac{rf_{p,q,r}(x)}{(1-x)^2} \le \frac{r}{p}x^p(1+x+\cdots+x^{q-1})^3(1-x^q)^{2r}$$
$$\le \frac{r}{p}q^3\{x^q(1-x^q)\}^{2r} < \frac{q^2}{4^{2r}} < \frac{e^{2r}}{4^{2r}} < 1.$$

To complete the example one adds up an infinite number of nonoverlapping blocks $f_{p,q,r}$ with rapidly increasing numbers $r = r_k$. Then $f(x) = \mathcal{O}\{(1-x)^2\}$ and $|a_n| \le 1/n$, while $|s_N| \ge 1/(10\log N)$ for a sequence of N tending to ∞.

□

Remarks 10.2. The construction above may be adjusted to treat the more general Tauberian condition (2.25) and functions $\varepsilon(u) = Cu^{-\beta}$ with other values of β. It can also be used to show the optimality of the remainder for the arithmetic means in (2.11) and for the higher-order Cesàro means. Cf. Korevaar [1951].

What is the best order estimate for the sequence $\{a_n\}$ *itself* if $a_n \ge -C/n$, while f satisfies condition (10.3)? It turns out that the numbers a_n can occasionally be about as large as the remainder estimate $s_n - A = \mathcal{O}(1/\log n)$ allows; cf. Vaughan [1983]. He constructed a series $\sum b_n$ with $b_n \ge 0$, for which infinitely many b_N are nearly of order $N/\log N$, while $\sum(b_n - 1)x^n = \mathcal{O}\{(1-x)^{\beta-1}\}$. Integration gives a series $\sum a_n x^n$ with $a_n > -1/n$ which satisfies (10.3), while infinitely many a_N are nearly of order $1/\log N$.

Continuing with the variable $x = e^{-1/u}$, we now turn to the case of

$$\varepsilon(u) = Ce^{-\beta u} \sim C' \exp\{-\beta/(1-x)\} \quad \text{as } x \nearrow 1. \tag{10.7}$$

Here one may begin with an estimate based on one of Jacobi's formulas for theta functions:

$$\begin{aligned} \vartheta_4(x) &= 1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \cdots \\ &= \prod_{k=1}^{\infty} \frac{1-x^k}{1+x^k} \le \exp\left(-\frac{2x}{1-x}\right) \\ &= e^2 \exp\left(-\frac{2}{1-x}\right) \qquad (0 \le x < 1); \end{aligned} \tag{10.8}$$

cf. Whittaker and Watson [1927/96], Hardy and Wright [1979] or Andrews [1976].

Example 10.3. Starting with ϑ_4 we form the following functions:

$$\begin{aligned} f_1(x) &= \frac{\vartheta_4(x)}{1-x} = 1 - x - x^2 - x^3 + x^4 + \cdots + x^8 - x^9 - \cdots, \\ f_2(x) &= \int_x^1 f_1 = \left(\int_0^1 f_1\right) - x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 \\ &\quad - \frac{1}{5}x^5 - \cdots - \frac{1}{9}x^9 + \frac{1}{10}x^{10} + \cdots, \\ f_3(x) &= \frac{f_1(x)}{1-x} = 1 - x^2 - 2x^3 - x^4 \\ &\quad + x^6 + 2x^7 + 3x^8 + 2x^9 + x^{10} - x^{12} - \cdots, \\ f_4(x) &= \int_x^1 f_3 = \left(\int_0^1 f_3\right) - x + \frac{1}{3}x^3 + \frac{2}{4}x^4 + \frac{1}{5}x^5 \\ &\quad - \frac{1}{7}x^7 - \frac{2}{8}x^8 - \frac{3}{9}x^9 - \frac{2}{10}x^{10} - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} + \cdots. \end{aligned}$$

One has $f_j(x) \to A = 0$ as $x \nearrow 1$ and in fact, $f_j(x) = \mathcal{O}\{\exp[-1/(1-x)]\}$ for each j. In the case of f_1 one has $|a_n| = 1$, while $|s_{k^2-1}| = k$ (cf. f_3), so that $|s_N| > \sqrt{N}$ for infinitely many N. More generally, if we set

$$\phi_1(v) = 1, \quad \phi_2(v) = 1/v, \quad \phi_3(v) = 2\sqrt{v}, \quad \phi_4(v) = 1/\sqrt{v},$$

then in every case $|a_n| \le \phi_j(n)$, while infinitely many remainders $|s_N - 0|$ are of order $\ge c\sqrt{N}\phi_j(N)$ with $c > 0$. □

In examples for the case of *Laplace transforms* it is easier to vary the parameters. The focus here will be on Theorem 3.1.

Example 10.4. For $\lambda > 0$, the nondecreasing function

$$s(u) = \int_0^u (1 + \sin 2\lambda\sqrt{v})dv \qquad (u \ge 0) \tag{10.9}$$

will satisfy the relation

$$F(u) = \mathcal{L}ds(1/u) = u + \mathcal{O}(e^{-\beta u}) \quad \text{as } u \to \infty \tag{10.10}$$

for any positive number $\beta < \lambda^2$. Hence by Theorem 3.1, $s(u) - u = \mathcal{O}(\sqrt{u})$ as $u \to \infty$. This order is precise:

$$\limsup_{u\to\infty} \frac{|s(u) - u|}{\sqrt{u}} = \frac{1}{\lambda}. \tag{10.11}$$

We first verify relation (10.11):

$$s(u) - u = \int_0^u (\sin 2\lambda\sqrt{v})dv = \int_0^{\sqrt{u}} (\sin 2\lambda t)2t dt$$
$$= -\frac{d}{d\lambda}\int_0^{\sqrt{u}} (\cos 2\lambda t)dt = -\frac{d}{d\lambda}\frac{\sin 2\lambda\sqrt{u}}{2\lambda} = -\frac{\cos 2\lambda\sqrt{u}}{\lambda}\sqrt{u} + \mathcal{O}(1).$$

For (10.10) observe that by formula (I.25.7),

$$F(u) - u = \int_0^\infty e^{-v/u}(\sin 2\lambda\sqrt{v})dv = \int_0^\infty e^{-t^2/u}(\sin 2\lambda t)2t\, dt$$
$$= -\frac{d}{d\lambda}\int_0^\infty e^{-t^2/u}(\cos 2\lambda t)dt = -\frac{d}{d\lambda}(\sqrt{\pi u}/2)e^{-\lambda^2 u} = \lambda\sqrt{\pi}u^{3/2}e^{-\lambda^2 u}.$$

The function $\sqrt{v}$ in formula (10.9) for $s(u)$ may be replaced by $v^{\gamma/(\gamma+1)}$ to treat the case where $F(u) - u = \mathcal{O}(e^{-\beta u^\gamma})$ with $0 < \gamma < 1$; cf. Examples 2.3 and see Korevaar [1954a]. The next example goes back to a review by Karamata [1952]; cf. also Tenenbaum [1995] (section 7.4).

Example 10.5. The nondecreasing function

$$s(u) = \int_0^u \{1 + \cos(\log^2 v)\}dv \qquad (u \geq 0) \tag{10.12}$$

will satisfy the relation

$$F(u) = \mathcal{L}ds(1/u) = u + \mathcal{O}(1) \quad \text{as } u \to \infty. \tag{10.13}$$

Hence by Theorem 3.1, $s(u) - u = \mathcal{O}(u/\log u)$ as $u \to \infty$. This order is optimal:

$$\limsup_{u\to\infty} \frac{|s(u) - u|}{u/\log u} = \frac{1}{2}. \tag{10.14}$$

We first verify (10.14):

$$s(u) - u - \int_0^1 \cos(\log^2 v)dv = \int_1^u \cos(\log^2 v)dv = \int_0^{\log u} (\cos t^2)e^t dt$$
$$= \int_0^{\log u} \frac{e^t}{2t}d(\sin t^2) = \frac{u\sin(\log^2 u)}{2\log u} - \int_0^{\log u} \frac{(t-1)e^t}{4t^3}d(1 - \cos t^2)$$
$$= \frac{u\sin(\log^2 u)}{2\log u} + \mathcal{O}\left(\frac{u}{\log^2 u}\right) \quad \text{as } u \to \infty.$$

The final estimate may be obtained through integration by parts, followed by the observation that $\int_1^{\log u}(e^t/t^2)dt = \mathcal{O}(u/\log^2 u)$.

For (10.13) we use complex integration, replacing the original real half-line where $1 \le v < \infty$ by the arc $\Gamma_1 : \{v = e^{i\theta}, 0 \le \theta \le 1\}$, followed by the half-line Γ_2 given by $\{v = e^i r, 1 \le r < \infty\}$. By Cauchy's theorem and an appropriate estimate,

$$F(u) - u - \int_0^1 e^{-v/u}\cos(\log^2 v)dv = \int_1^\infty e^{-v/u}\cos(\log^2 v)dv$$
$$= \int_1^\infty \mathrm{Re}\{e^{-(v/u)+i\log^2 v}\}dv = \int_{\Gamma_1}\cdots + \int_{\Gamma_2}\cdots = I_1 + I_2,$$

say. Taking $u > 0$ we estimate the absolute value of the final integrand by the absolute value of the exponential. On Γ_1 the real part of the exponent is ≤ 0 and on Γ_2 it is equal to $\mathrm{Re}\{-(e^i r/u) + i(i + \log r)^2\} = -(r\cos 1)/u - 2\log r$. It follows that I_1 and I_2, and hence $F(u) - u$, represent bounded functions of u.

11 Dirichlet Series and High Indices

General Dirichlet Series. A remainder theorem for Dirichlet series with nonnegative coefficients can be obtained from Theorem 3.1. To deal with other coefficients we appeal to Ganelius's Theorem 2.5. Accordingly, let ω be a positive continuous nondecreasing function on $\mathbb{R}^+$ and let $\theta(t)$ be the inverse function of $v\omega(v)$, so that $\theta(t)$ is increasing and $t/\theta(t)$ is nondecreasing; cf. Section 2.

Theorem 11.1. *Let $0 = \lambda_0 < \lambda_1 < \cdots$, $\lambda_n \to \infty$, and let the Dirichlet series $\sum_{n=0}^\infty a_n e^{-\lambda_n/u}$ converge for $0 < u < \infty$. Suppose that*

$$\lambda_{n+1} - \lambda_n = \mathcal{O}\{\theta(\lambda_n)\}, \quad a_n \ge -C\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \quad (n \ge 1), \tag{11.1}$$

and

$$F(u) \overset{\text{def}}{=} \sum_{n=0}^\infty a_n e^{-\lambda_n/u} = A + \mathcal{O}\{e^{-\omega(u)}\} \quad \text{as } u \to \infty. \tag{11.2}$$

Then

$$s(u) = \sum_{\lambda_n \le u} a_n = A + \mathcal{O}\left\{\frac{\theta(u)}{u}\right\} \quad \text{as } u \to \infty. \tag{11.3}$$

Proof. We proceed as in Section I.22. By the definition of $s(\cdot)$ one has

$$F(u) = \int_{0-}^\infty e^{-t/u}ds(t) = A + \mathcal{O}\{e^{-\omega(u)}\}$$

as $u \to \infty$. Adjusting a_0 one may assume that $A = 0$. Taking $u \le v \le u + \theta(u)$ with large u, we define p and q by requiring $\lambda_{p-1} \le u < \lambda_p$ and $\lambda_q \le v < \lambda_{q+1}$. Then

$$s(v) - s(u) = \sum_{u<\lambda_n \le v} a_n \ge -C \sum_{n=p}^{q} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}$$

$$\ge -\frac{C}{u}(\lambda_p - \lambda_{p-1} + \lambda_q - \lambda_p) \ge -\frac{C}{u}[\mathcal{O}\{\theta(\lambda_{p-1})\} + v - u] \ge -C' \frac{\theta(u)}{u}.$$

Conclusion (11.3) now follows from the case $\phi = 1$ of Theorem 2.5.

In [1954] Freud used Theorem 3.5 to treat the case where $\omega(\cdot)$ satisfies his growth condition (2.23). A typical example is $\omega(u) = \beta \log u$ (with $\beta > 0$), for which $\theta(u) \sim (1/\beta)u/\log u$. Instead of the first condition (11.1) he had to require that $\lambda_{n+1} - \lambda_n = \mathcal{O}(\lambda_n^{1-\delta})$ with $\delta > 0$. In [1965] Ingham used a special case of Theorem 3.1 to obtain another result for Dirichlet series under Freud's condition on ω.

THE CASE OF HIGH INDICES. Using the complex method which Halász [1967a] devised for the High-Indices Theorem I.23.1, we will extend some of the latter's remainder estimates.

Theorem 11.2. *Suppose that*

$$\lambda_{n+1}/\lambda_n \ge \rho > 1 \qquad (n \ge 1), \tag{11.4}$$

and that $F(u) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n/u}$ *exists for* $u > 0$ *and satisfies condition* (11.2). *Then there is a constant* $c > 0$ [which may be taken equal to $1/(2\sqrt{\rho})$] *such that*

$$s(u) = A + \mathcal{O}\{e^{-cu/\theta(u)}\} \quad as \quad u \to \infty. \tag{11.5}$$

In special cases the estimate can be made more precise:

(*i*) *If* $F(u) = A + \mathcal{O}(u^{-\beta})$ *as* $u \to \infty$ *for some number* $\beta > 0$, *then also* $s(u) = A + \mathcal{O}(u^{-\beta})$;

(*ii*) *If* $\limsup_{u\to\infty} \omega(u)/u > 0$, *or more generally,*

$$\int_1^{\infty} \frac{\sqrt{\omega(u)}}{u\sqrt{u}} du = \infty, \tag{11.6}$$

the remainder $s(u) - A$ *is identically equal to zero for* $u \ge 0$.

Proof. We explain first how to obtain formula (11.5); the discussion will be continued in Section 12. It may again be assumed that $A = 0$.

STEP I. Since F is bounded, the Laplace transform

$$G(z) \stackrel{\text{def}}{=} \int_0^{\infty} F(1/t) e^{zt} dt \tag{11.7}$$

exists and is holomorphic for $\operatorname{Re} z < 0$. For the following it is convenient and no essential restriction to assume that the sequence $\{a_n\}$ is bounded. Indeed, by the high-indices theorem, the series $\sum a_n$ is convergent!

Remark. If one does not want to assume that theorem, one may first consider the special case of Theorem 11.2 where $|F(u)| \le M$ for $u > 0$, so that one can take

$\omega(u) = 1$ and $\theta(u) = u$. Now replace the original sequence $\{a_n\}$ by the sequence $\{a_n e^{-\lambda_n \delta}\}$ with $\delta > 0$, which surely is bounded. The function $F(u)$ is then replaced by $F_\delta(u) = F\{u/(1+\delta u)\}$, which is also bounded by M. A simpler form of the proof below will then give an inequality

$$|s_\delta(u)| = \left| \sum_{\lambda_n \le u} a_n e^{-\lambda_n \delta} \right| \le K_\rho M,$$

where K_ρ depends only on ρ. Letting $\delta \searrow 0$ it follows that $|s(u)| \le K_\rho M$, which implies the boundedness of the sequence $\{a_n\}$. In fact, this is how Halász obtained his proof for the standard high-indices theorem. The inequality $|s(u)| \le K_\rho M$ provides the principal step in the proof of that theorem, cf. Section I.23.

We resume our argument. For $\operatorname{Re} z < 0$, the boundedness of $\{a_n\}$ allows one to write

$$G(z) = \int_0^\infty \sum_{n=0}^\infty a_n e^{-\lambda_n t} e^{zt} dt = \sum_{n=0}^\infty \frac{-a_n}{z - \lambda_n}. \tag{11.8}$$

This formula gives an analytic continuation of G to a meromorphic function (also called G) on the whole complex plane $\mathbb{C}$. It has simple poles at the points λ_n with residues $-a_n$. Taking z outside the union U of the discs $\{|z - \lambda_n| \le 1\}$, one may split the sum in (11.8) into the three parts, given by $\lambda_n \le |z|/2$, $|z|/2 < \lambda_n < 2|z|$ and $\lambda_n \ge 2|z|$, to show that $G(z) = \mathcal{O}(|z|)$ on $\mathbb{C} \setminus U$.

STEP 2. We proceed to estimate the transform G on the negative real axis. Since $\sup_{u>0} |F(u)| = M < \infty$, it follows from (11.7) that

$$|G(-x)| \le M/x \quad \text{for } x > 0, \tag{11.9}$$

but in our case there also is another bound. By (11.2) with $A = 0$,

$$|G(-x)| \le C \int_0^\infty e^{-\omega(u) - x/u} (du)/u^2 \qquad (x > 0). \tag{11.10}$$

Recalling that $v\omega(v) = x \Leftrightarrow v = \theta(x)$ one finds that for $u \ge \theta(x)$ one has $\omega(u) \ge \omega\{\theta(x)\} = x/\theta(x)$. Thus

$$\int_0^\infty e^{-\omega(u) - x/u} (du)/u^2 \le \int_0^{\theta(x)} e^{-x/u} (du)/u^2$$
$$+ e^{-x/\theta(x)} \int_{\theta(x)}^\infty e^{-x/u} (du)/u^2 \le 2e^{-x/\theta(x)}/x.$$

Hence there is a constant $M' \ge M$ such that

$$|G(-x)| \le M' e^{-x/\theta(x)}/x \quad \text{for } x > 0. \tag{11.11}$$

STEP 3. Next we cancel the poles of G to obtain a holomorphic function H on the domain D, given by the plane slit along the negative real axis, $D = \mathbb{C} \setminus \mathbb{R}^-$. Let $B_1(\cdot)$ denote the special Blaschke product

$$B_1(z) = \prod_1^\infty \frac{\sqrt{\lambda_n} - \sqrt{z}}{\sqrt{\lambda_n} + \sqrt{z}}, \tag{11.12}$$

where we use the principal value of $\sqrt{z}$. Now define

$$H(z) = zG(z)B_1(z), \tag{11.13}$$

with values at the points λ_n defined by continuity. Notice that $|B_1(z)|$ is bounded by 1 on $\overline{D}$, so that by Step 1, $H(z) = \mathcal{O}(|z|^2)$ as $z \to \infty$ in D. Also, by Step 2, the boundary values of H on the two sides of the slit satisfy the inequalities

$$|H(-x \pm i0)| \le M, \quad |H(-x \pm i0)| \le M' e^{-x/\theta(x)} \quad (x > 0). \tag{11.14}$$

STEP 4. An extended maximum principle for unbounded domains, or so-called Phragmén–Lindelöf theorem, will show that $|H(z)| \le M$ throughout D. To verify this inequality, one may apply Proposition 12.1 below to the function $f(z) = H(z^2)$ in the right half-plane $\{\operatorname{Re} z > 0\}$.

We next use a harmonic-measure argument to obtain another bound for $|H(z)|$ on the circle $\{|z| = R\}$; cf. Section 9. Let $h_R(z)$ denote the harmonic measure of the two-sided half-line $L_R : \{-\infty < x \le -R,\ y = 0\}$ relative to D. By (11.14) the quotient $|H(z)|/M'$ is bounded by 1 in D, while its boundary values on L_R are majorized by $e^{-R/\theta(R)}$. Thus

$$\log\{|H(z)|/M'\} \le -\{R/\theta(R)\}h_R(z), \quad \forall z \in D.$$

On the circle $\{|z| = R\}$ the function $h_R(z)$ attains its (positive) minimum at the point $z = R$, and by similarity $h_R(R) = h_1(1)$. Setting $h_1(1) = b$, it follows that

$$|H(Re^{it})| \le M' e^{-bR/\theta(R)} \quad \text{for } R > 0,\ |t| < \pi. \tag{11.15}$$

The harmonic measure $h_1(z)$ may be computed with the aid of the conformal map $z' = \sqrt{z}$ (or $z' = 1/\sqrt{z}$), which takes D onto the right half-plane $\{\operatorname{Re} z > 0\}$. For the half-plane, the harmonic measure $h(z')$ of a boundary segment is equal to $(1/\pi)$ times the angle, under which the segment is seen from the point z'. However, there is a simple trick which shows directly that $h_1(1) = 1/2$. Indeed, the function $\phi(z) = 1/z$ maps D onto D and takes the half-line $(-\infty, -1]$ into the segment $[-1, 0)$. Thus the harmonic measure $h^*(z)$ of $[-1, 0)$ or $(-1, 0)$ relative to D is equal to $h_1(1/z)$. Now $h_1(z) + h^*(z) \equiv 1$ on D, hence $h_1(1) + h^*(1) = 2h_1(1) = 1$.

STEP 5. Finally observe that the sequence $\{\sqrt{\lambda_n}\}$ also is a Hadamard sequence, with constant $\sqrt{\rho} > 1$. Under these conditions one has the following essential boundedness result; see Proposition 12.2 below. There is a number $K_\rho = C_{\sqrt{\rho}}$, depending only on ρ, such that

$$1/|B_1(z)| \le K_\rho \tag{11.16}$$

at every point z in the union V of the annuli $\{\lambda_n\sqrt{\rho} \le |z| \le \lambda_{n+1}/\sqrt{\rho}\}, n = 1, 2, \ldots$. It follows that for $z = Re^{it} \in V$,

$$|G(z)| = \frac{|H(z)|}{|zB_1(z)|} \le M'K_\rho \frac{e^{-bR/\theta(R)}}{R}.$$

Hence by the residue theorem applied to G in the form (11.8),

$$|s(R)| = \left| \frac{1}{2\pi i} \int_{|z|=R} G(z)dz \right| \le Ce^{-bR/\theta(R)} \tag{11.17}$$

for $\lambda_n\sqrt{\rho} \le R \le \lambda_{n+1}/\sqrt{\rho}$, $n = 1, 2, \ldots$. Taking $R = \lambda_{n+1}/\sqrt{\rho}$ and using the monotonicity of $\theta(t)$ and $t/\theta(t)$, one concludes that for $\lambda_n \le u < \lambda_{n+1}$,

$$|s(u)| = |s(R)| \le Ce^{-(b/\sqrt{\rho})\lambda_{n+1}/\theta(\lambda_{n+1})} \le Ce^{-(b/\sqrt{\rho})u/\theta(u)}.$$

Since $b = h_1(1) = 1/2$, this inequality implies the desired estimate (11.5) with $c = 1/(2\sqrt{\rho})$.

12 Proof of Theorem 11.2, Continued

We begin with some relevant propositions of complex analysis, in which Ω denotes the right half-plane $\{\mathrm{Re}\, z > 0\}$.

Proposition 12.1. *Let $f(z)$ be holomorphic in the half-plane Ω and continuous on its closure. Suppose that $|f(iy)| \le M$ and $f(z) = \mathcal{O}(e^{\varepsilon|z|})$ for every $\varepsilon > 0$ as $z \to \infty$ in Ω. Then $|f(z)| \le M$ throughout Ω.*

Moreover, if $f \not\equiv 0$,

$$\int_{\mathbb{R}} \frac{\log|f(iy)|}{1+y^2} dy > -\infty. \tag{12.1}$$

For the first or Phragmén–Lindelöf part we refer to Titchmarsh [1939]. Formula (12.1) may be derived from Jensen's theorem for bounded holomorphic functions in a disc by conformal mapping; cf. Garnett [1981] (section 2.4).

Next we consider the standard Blaschke product for the half-plane Ω, associated with a Hadamard sequence $\{\lambda_k\}_1^\infty$ as in (11.4):

$$B(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z} = \prod_{k=1}^{\infty} \left(1 - \frac{2z}{\lambda_k + z}\right). \tag{12.2}$$

The convergence of $\sum 1/\lambda_k$ ensures that the product defines a meromorphic function with zeros at the points λ_k and poles at the points $-\lambda_k$. Such a product is in absolute value bounded by 1 on $\overline{\Omega}$, but in the case of a Hadamard sequence with constant ρ one can say more; cf. Gaier [1966], [1967].

Proposition 12.2. *There is a number C_ρ depending only on ρ such that for every point z in the union of the circles $\{|z| = R\}$, $\lambda_n\sqrt{\rho} \le R \le \lambda_{n+1}/\sqrt{\rho}$, $n = 1, 2, \ldots$, one has*

$$1/C_\rho \le |B(z)| \le C_\rho. \tag{12.3}$$

Proof. Observe that for $0 < a \le b < 1$

$$\frac{1+a}{1-a} = 1 + \frac{2a}{1-a} \le 1 + \frac{2}{1-b}a \le e^{ca}, \quad c = \frac{2}{1-b}. \tag{12.4}$$

Fixing $n \ge 1$ we take $|z| = R$ where $\lambda_n\sqrt{\rho} \le R \le \lambda_{n+1}/\sqrt{\rho}$. Then for $k \le n$ we have $\lambda_k/R \le \rho^{k-n}\lambda_n/R \le 1/\sqrt{\rho}$, hence by (12.4) with $c = 2/(1 - 1/\sqrt{\rho})$,

$$\begin{aligned}\prod_{k=1}^{n}\left|\frac{\lambda_k - z}{\lambda_k + z}\right| &\le \prod_{k=1}^{n}\frac{R+\lambda_k}{R-\lambda_k} = \prod_{k=1}^{n}\frac{1+\lambda_k/R}{1-\lambda_k/R} \\ &\le \exp\left(c\sum_{k=1}^{n}\frac{\lambda_k}{R}\right) \le \exp\left(\frac{c}{\sqrt{\rho}\,(1-1/\rho)}\right).\end{aligned}$$

For $k \ge n+1$ one similarly has $R/\lambda_k \le \rho^{n+1-k}R/\lambda_{n+1} \le 1/\sqrt{\rho}$, so that also

$$\prod_{k=n+1}^{\infty}\left|\frac{\lambda_k - z}{\lambda_k + z}\right| \le \prod_{k=n+1}^{\infty}\frac{1+R/\lambda_k}{1-R/\lambda_k} \le \exp\left(\frac{c}{\sqrt{\rho}\,(1-1/\rho)}\right).$$

Combination of the estimates establishes the upper bound in (12.3). The lower bound follows from the observation that $1/B(z) = B(-z)$. □

Theorem 11.2, Completion of the Proof. We still have to discuss the parts of the Theorem numbered (i) and (ii). Let F satisfy the conditions of Theorem 11.2 with $A = 0$ and let G and H be as in formulas (11.7), (11.13).

Part (i) follows easily from the considerations in Section 11. Indeed, if $F(u)$ is bounded and $\mathcal{O}(u^{-\beta})$ as $u \to \infty$, formula (11.10) shows that

$$|G(-x)| \le C\int_0^\infty u^{-\beta}e^{-x/u}(du)/u^2 = C'x^{-\beta-1} \qquad (x > 0). \tag{12.5}$$

Thus $H(-x) = \mathcal{O}(x^{-\beta})$. Proposition 12.1 now implies that $z^\beta H(z)$ is bounded on D, so that $G(z) = \mathcal{O}(|z|^{-\beta-1})$ on the family of annuli V. Application of (11.17) completes the proof.

We turn to *Part (ii)*. If $\limsup \omega(u)/u > \delta > 0$, it readily follows from the monotonicity of $\omega(u)$ that the integral in (11.6) is divergent. Suppose then that $F(u) = \mathcal{O}\{e^{-\omega(u)}\}$ with ω as in (11.6). By the proof in Section 11 the function $H(z)$ is bounded on D, and the boundary values $H(-x \pm i0)$ satisfy the final inequality (11.14). Define $f(z) = H(z^2)$ on the closure of the half-plane Ω and write $\sqrt{x} = y$. Then

$$\int_1^\infty \frac{\log|H(-x \pm i0)|}{x\sqrt{x}}dx = 2\int_1^\infty \frac{\log|f(\pm iy)|}{y^2}dy,$$

and by Proposition 12.1 these integrals must be finite if $H \not\equiv 0$. However, by (11.14) and the substitution $\theta(x) = u$, so that $x = u\omega(u)$,

$$\int_1^X \frac{\log|H(-x \pm i0)|}{x\sqrt{x}}dx \le C_1 - \int_1^X \frac{x}{\theta(x)x\sqrt{x}}dx$$
$$= C_1 - \int_{\theta(1)}^{\theta(X)} \frac{d\{u\omega(u)\}}{u\sqrt{u\omega(u)}} = C_1 - 2\int_{\theta(1)}^{\theta(X)} \frac{1}{u} d\sqrt{u\omega(u)}$$
$$\le C_2 - 2\int_{\theta(1)}^{\theta(X)} \frac{\sqrt{u\omega(u)}}{u^2}du \to -\infty \quad \text{as } X \to \infty;$$

cf. (11.6). Conclusion: $H \equiv 0,\ G \equiv 0,\ F \equiv 0$, so that $s(u) \equiv 0$. □

13 The Fourier Integral Method: Introduction

Our starting point is Wiener's Tauberian theorem in the form given by Pitt (Theorem II.8.4). Let K be a Wiener kernel, that is, K is in $L^1(\mathbb{R})$ and its Fourier transform

$$\hat{K}(u) = \int_{\mathbb{R}} K(x)e^{-iux}dx, \quad u \in \mathbb{R}, \tag{13.1}$$

is free of (real) zeros. Let S be bounded on $\mathbb{R}$ and slowly decreasing:

$$\liminf\{S(y) - S(x)\} \ge 0 \quad \text{as } x \to \infty \text{ and } 0 < y - x \to 0. \tag{13.2}$$

Then the limit relation

$$K * S(x) = \int_{\mathbb{R}} K(x-y)S(y)dy \to A\int_{\mathbb{R}} K(y)dy \quad \text{as } x \to \infty \tag{13.3}$$

implies that $S(x) \to A$ as $x \to \infty$.

Suppose now that the Fourier transform $\hat{K}(u)$ has a zero-free meromorphic extension $\hat{K}(w) = \hat{K}(u + iv)$ (an analytic extension which may have poles) to a strip Σ around the real axis. We write

$$\Sigma = \Sigma[-h, H) = \{w = u + iv : u \in \mathbb{R},\ -h \le v < H\}, \tag{13.4}$$

where $h > 0$ may be small and $H > 0$ may be $+\infty$. At the end of his article [1938], Beurling stated two remainder estimates related to Wiener's theorem. He assumed that $1/\hat{K}(w)$ is of at most polynomial growth in Σ. In his second result, the precise condition was that for $w \in \Sigma[-h, H)$,

$$\left|\frac{d}{dw}\frac{1}{\hat{K}(w)}\right| \le C(|w| + 1)^{p-1}. \tag{13.5}$$

One of Beurling's results may now be stated as follows.

Theorem 13.1. *Let $K \in L^1$ satisfy the conditions above with $p > 1/2$ and let S be bounded. Suppose that*

$$K * S(x) = A\int_{\mathbb{R}} K(y)dy + \mathcal{O}(e^{-\alpha x}) \quad \text{as } x \to \infty, \tag{13.6}$$

where $\alpha \in (0, H)$. Suppose also that S satisfies the Tauberian condition

$$S(y) - S(x) \geq -C(y-x) \quad \textit{for} \;\; y > x \;\; \textit{and} \;\; x \to \infty. \tag{13.7}$$

Then one has the remainder estimate

$$S(x) - A = \mathcal{O}(e^{-\alpha x/(p+1)}) \quad \textit{as} \;\; x \to \infty. \tag{13.8}$$

Beurling did not publish a proof, but later, various forms of the result were established by Lyttkens [1954–56], Ganelius [1962] and Frennemo [1965]. The theorem may be used to obtain remainder estimates for Cesàro and Riesz methods; cf. the Springer Lecture Notes by Ganelius [1971]. In [1996], Tammeraid used Beurling's theorem for one of his remainder theorems concerning Riesz methods.

However, the most important summability methods involve reciprocals $1/\hat{K}(w)$ that grow much faster than polynomials. In his papers [1958], [1962], [1964], Ganelius developed a general Fourier integral theory which allowed large functions $1/\hat{K}(w)$; cf. his Lecture Notes. In the following sections we discuss and illustrate a number of typical cases, for which detailed proofs are given. But first we show that the example of Abel summability leads to a large function $1/\hat{K}(w)$.

Example 13.2. (*Laplace–Stieltjes transform*) Let the series $\sum_0^\infty a_n$, or more generally, the integral $\int_{0-}^\infty ds(\cdot)$ with a 'standardized function' $s(\cdot)$ as in Section 2 or I.13, be Abel summable to A. Adjusting the notation to avoid multiple use of the same letter, one then has

$$F(\xi) = \mathcal{L}ds(1/\xi) = \int_{0-}^{\infty-} e^{-\eta/\xi} ds(\eta) = \int_0^\infty (1/\xi) e^{-\eta/\xi} s(\eta) d\eta \to A \tag{13.9}$$

as $\xi \to \infty$. Substituting $\xi = e^x$ and $\eta = e^y$, this becomes

$$F(e^x) = \int_{\mathbb{R}} \exp\{-(x-y) - e^{-(x-y)}\} s(e^y) dy \to A \quad \text{as } x \to \infty.$$

Writing $\exp(-x - e^{-x}) = K(x)$ and $s(e^y) = S(y)$, the limit relation assumes the Wiener form,

$$F(e^x) = \int_{\mathbb{R}} K(x-y) S(y) dy \to A = A \int_{\mathbb{R}} K(y) dy \quad \text{as } x \to \infty. \tag{13.10}$$

In the present case the Fourier transform $\hat{K}$ is given by

$$\hat{K}(w) = \int_{\mathbb{R}} \exp(-x - e^{-x}) e^{-iwx} dx = \int_0^\infty e^{-t} t^{iw} dt = \Gamma(1 + iw).$$

It follows that $1/\hat{K}$ is of exponential growth in every strip $\Sigma[-1/2, H)$ with *finite* $H > 0$.

Indeed, $\Gamma(z)$ is zero-free and meromorphic on $\mathbb{C}$; cf. Whittaker and Watson [1927/96] (section 12.1). Letting u and v denote the real and imaginary part of w, the functional equations $\Gamma(z)\Gamma(1-z) = \pi/(\sin \pi z)$ and $\Gamma(1+z) = z\Gamma(z)$ imply that

$$\frac{1}{\Gamma(1-v+iu)} = \frac{\sin\{\pi(v-iu)\}}{\pi(v-iu)}\Gamma(1+v-iu). \tag{13.11}$$

The first factor on the right is $\mathcal{O}(e^{\pi|u|})$ and for $v \geq -1/2$, the second factor is bounded by $\Gamma(1+v)$ (one may use the integral representation). Hence by Stirling's formula one has an inequality

$$\frac{1}{|\hat{K}(u+iv)|} = \frac{1}{|\Gamma(1-v+iu)|} \leq Ce^{\pi|u|+v\log(1+v)} \quad \text{for } u \in \mathbb{R},\ v \geq -1/2. \tag{13.12}$$

This inequality gives the correct order of $1/|\hat{K}(u+iv)|$ for large $|u|$ and v.

14 Fourier Integral Method: A Model Theorem

In this and the next two sections we illustrate Ganelius's method by treating a relatively simple case. Writing $w = u+iv$, let $\Omega = \Omega[-h, H, \gamma)$ denote the class of the Wiener kernels K such that $1/\hat{K}$ is analytic in the strip $\Sigma : \{-h \leq v < H\}$ and satisfies an inequality

$$\frac{1}{|\hat{K}(w)|} \leq Ce^{\gamma|w|} \quad \text{for } w \in \Sigma; \tag{14.1}$$

here γ must of course be positive.

For differences $K * S - A$, smallness at $+\infty$ will be measured by functions $e^{-\tau(x)}$, where $\tau(x)$ is nondecreasing on $\mathbb{R}$ and tends to ∞ as $x \to \infty$. For the time being we suppose that τ grows at most linearly, more precisely, that there is a constant $\beta > 0$ such that

$$\tau(x+1) \leq \tau(x) + \beta \quad \text{for } x \in \mathbb{R}. \tag{14.2}$$

Theorem 14.1. *Let the kernel K in $\Omega[-h, H, \gamma)$ and the bounded function S be such that for some constant A and a function $\tau(x)$ as described, with $0 < \beta < H$,*

$$\begin{aligned} T(x) \overset{\text{def}}{=} K * S(x) &= \int_{\mathbb{R}} K(x-y)S(y)dy \\ &= A\int_{\mathbb{R}} K(y)dy + \mathcal{O}\{e^{-\tau(x)}\} \quad \textit{on } \mathbb{R}. \end{aligned} \tag{14.3}$$

Suppose that S satisfies the Tauberian condition

$$\inf_{x\leq y\leq x+1/\tau(x)} \{S(y) - S(x)\} \geq -\mathcal{O}\{1/\tau(x)\} \quad \textit{as } x \to \infty. \tag{14.4}$$

Then the remainder $S - A$ satisfies the estimate

$$S(x) - A = \mathcal{O}\{1/\tau(x)\} \quad \textit{as } x \to \infty. \tag{14.5}$$

The result is in Ganelius [1962], [1971] (theorem 2.1); the proof in Sections 15, 16 is essentially a rearrangement of his. Here we illustrate the Theorem to discuss a remainder estimate related to Freud's Theorem 3.5 for the Laplace–Stieltjes transform.

Application 14.2. We saw in Example 13.2 that the Laplace–Stieltjes transform gives rise to the Wiener kernel $K(x) = \exp(-x - e^{-x})$; by inequality (13.12) the kernel K belongs to the class $\Omega[-1/2, H, \pi)$ for every number $H > 0$. Suppose now that the Laplace–Stieltjes transform $F(\xi) = \mathcal{L}ds(1/\xi)$ satisfies the condition

$$|F(\xi) - A| \le e^{-\omega(\xi)} \quad \text{for } 0 < \xi < \infty \tag{14.6}$$

of Section 2, but with Freud's restriction to the effect that $\omega(e\xi) \le \omega(\xi) + b$. The substitutions of Example 13.2 then show that $K * S(x) = F(e^x)$ satisfies relation (14.3) with $\tau(x) = \omega(e^x)$, and that τ satisfies condition (14.2) with $\beta = b$. Suppose also that $S(x) = s(e^x)$ satisfies the Tauberian condition (14.4). Repeated application of (14.4) will show that S is slowly decreasing on $\mathbb{R}$ (Section II.2). Indeed, there will be a constant C such that for large x and $y \ge x$,

$$S(y) - S(x) \ge -C(y - x) - 1/\tau(x),$$

and the right-hand side tends to 0 as $x \to \infty$ and $0 < y - x \to 0$. Thus $s(\cdot)$ is slowly decreasing on $\mathbb{R}^+$: $\liminf\{s(\eta) - s(\xi)\} \ge 0$ as $\xi \to \infty$, $1 < \eta/\xi \to 1$. It now follows from the boundedness of $F(\xi)$ as $\xi \to \infty$ that s, and hence S, is bounded; see Boundedness Theorem I.20.1. Applying Theorem 14.1, one concludes that the remainder $S(x) - A = s(e^x) - A$ satisfies the estimate (14.5).

In terms of $\xi = e^x$ the conclusion means that

$$s(\xi) - A = \mathcal{O}\{1/\omega(\xi)\} \quad \text{as } \xi \to \infty. \tag{14.7}$$

We have obtained this result from (14.6) under the Tauberian condition

$$\inf_{x \le y \le x + 1/\tau(x)} \{s(e^y) - s(e^x)\} \ge -\mathcal{O}\{1/\tau(x)\} \quad \text{as } x \to \infty.$$

In terms of ξ and η, the infimum must be taken over

$$\xi = e^x \le \eta = e^y \le e^{x+1/\tau(x)} \approx e^x + e^x/\tau(x) = \xi + \xi/\omega(\xi).$$

The Tauberian condition may thus be put into the equivalent form

$$\inf_{\xi \le \eta \le \xi + \xi/\omega(\xi)} \{s(\eta) - s(\xi)\} \ge -\mathcal{O}\{1/\omega(\xi)\} \quad \text{as } \xi \to \infty, \tag{14.8}$$

which is related to, but weaker than the standard condition $a_n \ge -C/n$ for series.

Remark 14.3. The result illustrates a special case of Ganelius's general Theorem 2.5. For a relatively slowly increasing function such as $\omega(u) = \beta \log u$, which corresponds to $\tau(x) = \beta x$, one finds that indeed, $\theta(u) \approx u/\omega(u)$. However, this is not true for rapidly increasing functions ω. The case of unrestricted functions ω is discussed in Section 19.

15 Auxiliary Inequality of Ganelius

In the proofs of the remainder theorems we will use an auxiliary result related to the Berry–Esseen inequality of Section IV.24. The theorem below is due to Ganelius; see Ganelius [1957] for a case of periodic functions, and Ganelius [1962] and Frennemo [1965] for the case of $\mathbb{R}$; cf. also Tenenbaum [1995].

For real functions W on $\mathbb{R}$ and $\delta > 0$, define

$$Q(\delta) = Q_W(\delta) = \sup_{x,y\in\mathbb{R},\, x\le y\le x+\delta} \{W(y) - W(x)\}. \tag{15.1}$$

Theorem 15.1. *Let W be real-valued and integrable over $\mathbb{R}$, with $Q_W(\delta)$ finite for some (and hence every) number $\delta > 0$. Then W is bounded, and there is an absolute constant C such that for any number $\lambda > 0$,*

$$\begin{aligned}\|W\|_\infty &\le CQ_W\left(\frac{1}{\lambda}\right) + 2\sup_{\xi\in\mathbb{R}}\left|\int_{\mathbb{R}} W(\xi - y)\frac{1-\cos\lambda y}{\pi\lambda y^2}dy\right| \\ &\le CQ_W\left(\frac{1}{\lambda}\right) + \frac{1}{\pi}\int_{-\lambda}^{\lambda} |\hat{W}(t)|dt. \end{aligned}\tag{15.2}$$

One may take $C = 20$.

Proof. The function W is locally bounded by the finiteness of $Q(\delta)$. It is also globally bounded:

$$\limsup_{|x|\to\infty} |W(x)| \le Q(\delta).$$

Indeed, if for example $\limsup_{x\to\infty} W(x)$ would be larger than $Q(\delta)$, there would be a number $\eta > 0$ and a sequence $x_n \to \infty$ such that $W(x_n) \ge Q(\delta) + \eta$. By (15.1) this inequality implies that $W(x) \ge \eta$ for $x_n - \delta \le x \le x_n$, $n = 1, 2, \ldots$, and this contradicts the integrability of W.

The proof of (15.2) involves the Fourier pair

$$D_\lambda(x) = \frac{1-\cos\lambda x}{\pi\lambda x^2}, \quad \Delta_\lambda(t) = \hat{D}_\lambda(t) = \begin{cases} 1 - |t|/\lambda & \text{for } |t| \le \lambda, \\ 0 & \text{for } |t| > \lambda; \end{cases} \tag{15.3}$$

cf. Example II.7.1. In terms of these functions,

$$\begin{aligned} Z(x) = Z_\lambda(x) &\overset{\text{def}}{=} \int_{\mathbb{R}} W(x-y)D_\lambda(y)dy \\ \text{is equal to } &\frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\Delta_\lambda(t)\hat{W}(t)dt. \end{aligned}\tag{15.4}$$

Let $\sup|W(x)| = \mu > 0$; we may suppose that $\mu = \sup W(x)$. Then for any number $\theta \in (0, 1)$ there is a number x_0 such that

$$W(x_0) > \theta\mu. \tag{15.5}$$

Writing

$$\begin{aligned}Z(x_0-\delta) &= \int_{-\delta}^{\delta} W(x_0-\delta-y)D_\lambda(y)dy + \int_{|y|>\delta} W(x_0-\delta-y)D_\lambda(y)dy \\ &= \int_{-\delta}^{\delta} W(x_0)D_\lambda(y)dy - \int_{-\delta}^{\delta}\{W(x_0)-W(x_0-\delta-y)\}D_\lambda(y)dy \\ &\quad + \int_{|y|>\delta} W(x_0-\delta-y)D_\lambda(y)dy,\end{aligned}$$

one finds that

$$\begin{aligned}Z(x_0-\delta) &> \{\theta\mu - Q(2\delta)\}\int_{-\delta}^{\delta} D_\lambda(y)dy - \mu\int_{|y|>\delta} D_\lambda(y)dy \\ &= \{(1+\theta)\mu - Q(2\delta)\}\int_{-\lambda\delta}^{\lambda\delta} D_1(z)dz - \mu. \qquad (15.6)\end{aligned}$$

For fixed $\lambda > 0$ we will take δ so large and θ so close to 1 that the coefficient of μ in the final member of (15.6) is positive. It is possible and convenient to take $\delta = 6/\lambda$. Then the inequality $D_1(x) \le 2/(\pi x^2)$ gives

$$\int_{\lambda\delta}^{\infty} D_1 \le 1/(3\pi); \quad \text{hence} \quad \int_{-\lambda\delta}^{\lambda\delta} D_1 \ge 1 - 2/(3\pi).$$

One now chooses θ in (0, 1) such that $(1+\theta)\{1-2/(3\pi)\} - 1 = 1/2$. If it is the case that

$$(1+\theta)\mu \le Q(2\delta) = Q(12/\lambda) \quad [\le 12Q(1/\lambda)],$$

then (15.2) holds with $C = 12$. We continue under the assumption that one has $(1+\theta)\mu > Q(12/\lambda)$. Then by (15.6)

$$\begin{aligned}Z(x_0-6/\lambda) &\ge \{(1+\theta)\mu - Q(12/\lambda)\}\{1-2/(3\pi)\} - \mu \\ &= \mu/2 - \{1-2/(3\pi)\}Q(12/\lambda).\end{aligned}$$

This implies

$$\begin{aligned}\sup|W(x)| = \mu &\le 24\{1-2/(3\pi)\}Q(1/\lambda) + 2Z(x_0-6/\lambda) \\ &\le 20Q(1/\lambda) + 2\sup_{x\in\mathbb{R}}|Z(x)|. \qquad (15.7)\end{aligned}$$

The proof of the inequalities (15.2) is completed by application of (15.4).

THE BERRY–ESSEEN INEQUALITY. Although we do not need it here, we describe how this inequality can be derived from Theorem 15.1. The inequality asserts that for arbitrary distribution functions S and U on $\mathbb{R}$,

$$\|S-U\|_\infty \le CQ_U\left(\frac{1}{\lambda}\right) + \frac{1}{\pi}\int_{-\lambda}^{\lambda} |\mathcal{F}dS(t) - \mathcal{F}dU(t)|\frac{dt}{|t|}, \quad \forall\lambda > 0; \qquad (15.8)$$

cf. Section IV.24. For the proof, set

$$W = U - S. \tag{15.9}$$

Then $W(-\infty+) = W(\infty-) = 0$ and

$$\begin{aligned} Q_W(\delta) &= \sup_{x,y\in\mathbb{R},\, x\le y\le x+\delta} \{W(y) - W(x)\} \\ &= \sup_{x\le y\le x+\delta} [U(y) - U(x) - \{S(y) - S(x)\}] \le Q_U(\delta). \end{aligned} \tag{15.10}$$

If W is in $L^1(\mathbb{R})$, integration by parts shows that

$$\mathcal{F}dW(t) = \int_{\mathbb{R}} e^{-itx} dW(x) = it \int_{\mathbb{R}} W(x) e^{-itx} dx.$$

In this case

$$|\hat{W}(t)| = |\mathcal{F}dU(t) - \mathcal{F}dS(t)|/|t|,$$

so that (15.8) is an immediate consequence of (15.2).

If W is not integrable over $\mathbb{R}$ one may apply the preceding argument to the auxiliary function

$$\begin{aligned} W_\varepsilon(x) &\overset{\text{def}}{=} -\int_0^\infty e^{-\varepsilon z} d_z W(x-z) = e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon z} dW(z) \\ &= W(x) - \varepsilon e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon z} W(z) dz, \end{aligned} \tag{15.11}$$

where $\varepsilon > 0$; cf. Tenenbaum. By the second representation and Fubini's theorem, W_ε is in L^1 and $\hat{W}_\varepsilon(t) = \mathcal{F}dW(t)/(\varepsilon + it)$. The final member of (15.11) shows that $W_\varepsilon(x) \to W(x)$ as $\varepsilon \searrow 0$ for every fixed x, because $W(-\infty+) = 0$ and $\sup |W(z)| \le 1$. Finally, by a differentiation which can be carried out for almost all x,

$$\frac{d}{dx}\left\{-\varepsilon e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon z} W(z) dz\right\} = \varepsilon^2 e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon z} W(z) dz - \varepsilon W(x) \le 2\varepsilon$$

almost everywhere. It follows that for $x \le y \le x + \delta$

$$W_\varepsilon(y) - W_\varepsilon(x) \le W(y) - W(x) + 2\varepsilon(y - x) \le Q_U(\delta) + 2\varepsilon\delta.$$

Hence by Theorem 15.1

$$|W_\varepsilon(x)| \le C\left\{Q_U\left(\frac{1}{\lambda}\right) + \frac{2\varepsilon}{\lambda}\right\} + \frac{1}{\pi}\int_{-\lambda}^{\lambda} \frac{|\mathcal{F}dW(t)|}{|\varepsilon + it|}$$

for every $x \in \mathbb{R}$ and every $\lambda > 0$. Inequality (15.8) follows by letting ε go to zero.

16 Proof of the Model Theorem

We begin with some propositions which will be used also in subsequent proofs.

Proposition 16.1. *Let K be a Wiener kernel, S bounded, $T = K * S$, and let D_λ, Δ_λ be the Fourier pair of* (15.3). *Finally, let M be an auxiliary function such that for real ξ, both*

$$\hat{R}(u) \stackrel{\text{def}}{=} \frac{1}{\hat{K}(u)} \int_{\mathbb{R}} \hat{M}(u-t)\Delta_\lambda(t)e^{-i\xi t}dt \tag{16.1}$$

and its inverse Fourier transform $R(\cdot) = R(\cdot; M, \lambda, \xi)$ are in L^1. Then

$$R * T(x) = 2\pi \int_{\mathbb{R}} S(x-y)M(y)D_\lambda(y-\xi)dy. \tag{16.2}$$

Proof. Since R and K are in L^1, Fubini's theorem gives

$$R * T = R * K * S = H * S,$$

with $H = R * K$ in L^1. By (16.1)

$$\hat{H}(u) = \hat{R}(u)\hat{K}(u) = \int_{\mathbb{R}} \hat{M}(u-t)\Delta_\lambda(t)e^{-i\xi t}dt.$$

Thus by Fourier inversion

$$H(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu}du \int_{\mathbb{R}} \hat{M}(u-t)\Delta_\lambda(t)e^{-i\xi t}dt = 2\pi M(x)D_\lambda(x-\xi).$$

It follows that

$$R * T(x) = S * H(x) = 2\pi \int_{\mathbb{R}} S(x-y)M(y)D_\lambda(y-\xi)dy.$$

□

For the proof of Theorem 14.1 we take

$$M(y) = e^{-y^2/2}, \quad \text{so that } \hat{M}(u) = \sqrt{2\pi}e^{-u^2/2}. \tag{16.3}$$

Proposition 16.2. *Let the kernel K be in the class $\Omega[-h, H, \gamma)$ of Section 14 and let $\hat{R}$ be the function given by* (16.1) *with $\hat{M}$ as in* (16.3). *Then $\hat{R}$ and its inverse Fourier transform $R = R(\cdot; M, \lambda, \xi)$ are in L^1. For every number $v \in [-h, H)$ there is a number $C(v)$ independent of λ and ξ such that*

$$|R(y)| \le C(v)e^{\gamma\lambda - vy}, \quad \forall y. \tag{16.4}$$

Proof. If the function $\hat{R}$ of (16.1) is in L^1, Fourier inversion will give

$$R(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{R}(u)e^{iyu}du = \int_{\mathbb{R}} I_y(u)du, \tag{16.5}$$

say. By (16.1), (16.3) and (15.3) the integrand $I_y(u)$ may be written as

$$I_y(u) = \frac{1}{\sqrt{2\pi}\hat{K}(u)} e^{iyu} \int_{-\lambda}^{\lambda} e^{-(u-t)^2/2} \left(1 - \frac{|t|}{\lambda}\right) e^{-i\xi t} dt. \tag{16.6}$$

The hypotheses on K imply that $I_y(u)$ has an analytic continuation $I_y(w)$ to the strip $\Sigma = \Sigma[-h, H)$ of (13.4). By (14.1) the continuation satisfies the inequality

$$|I_y(u+iv)| \le \frac{C}{\sqrt{2\pi}} e^{\gamma|u|+\gamma|v|-yv} \int_{-\lambda}^{\lambda} e^{-\{(u-t)^2-v^2\}/2} dt, \quad -h \le v < H. \tag{16.7}$$

For $v = 0$ this formula shows that $I_y(u)$ and hence $\hat{R}(u)$ is in L^1, so that we can define a function R by (16.5). Furthermore, by Cauchy's theorem and inequality (16.7), the path of integration in (16.5) may be moved from the real axis to any line $v =$ *constant* in the strip Σ. Doing this, it follows by inversion of the order of integration and the substitution $u - t = z$ that for certain numbers $C(v)$ and $C'(v)$ independent of y, λ, ξ,

$$\begin{aligned} |R(y)| &\le \int_{\mathbb{R}} |I_y(u+iv)| du \le C(v) e^{-vy} \int_{-\lambda}^{\lambda} dt \int_{\mathbb{R}} e^{\gamma|u|-(u-t)^2/2} du \\ &\le C(v) e^{-vy} \int_{-\lambda}^{\lambda} e^{\gamma|t|} dt \int_{\mathbb{R}} e^{\gamma|z|-z^2/2} dz \le C'(v) e^{\gamma\lambda - vy}. \end{aligned} \tag{16.8}$$

This proves (16.4); since v may be taken positive as well as negative, R will be in L^1.

Proof of Theorem 14.1. Let K, S and $T = K * S$ satisfy the conditions of the Theorem where we may assume $A = 0$. With M as in (16.3), let $\hat{R}$ be the function obtained from (16.1). By Proposition 16.2, $\hat{R}$ and its inverse Fourier transform R are in L^1. Moreover R satisfies an inequality (16.4) for any number $v \in [-h, H)$. Since the hypotheses of Proposition 16.1 are satisfied, we can also use formula (16.2).

For fixed $x > 0$ we now apply Ganelius's inequality (15.2) to the function

$$W(y) = W_x(y) \stackrel{\text{def}}{=} S(x-y)M(y) = S(x-y)e^{-y^2/2}. \tag{16.9}$$

Since

$$|S(x)| = |S(x)M(0)| \le \sup_y |S(x-y)M(y)|,$$

Theorem 15.1 shows that

$$|S(x)| \le C Q_{W_x}\left(\frac{1}{\lambda}\right) + 2 \sup_{\xi\in\mathbb{R}} \left| \int_{\mathbb{R}} W_x(\xi - y) \frac{1-\cos\lambda y}{\pi\lambda y^2} dy \right| \tag{16.10}$$

for every number $\lambda > 0$.

We first estimate the integral for appropriate λ. Since D_λ is even, formulas (16.9) and (16.2) show that the integral may be written as

$$\int_{\mathbb{R}} W_x(y) D_\lambda(y-\xi) dy = \frac{1}{2\pi} R * T(x) = \frac{1}{2\pi} \int_{\mathbb{R}} R(y) T(x-y) dy. \tag{16.11}$$

Here $R(y) = R(y; \lambda, \xi)$; by Proposition 16.2, $R(y)$ satisfies an inequality (16.4) for every number $v \in [-h, H)$.

To estimate the final convolution we will also use the hypothesis of Theorem 14.1 for T. By the boundedness of S and the estimate (14.3) with $A = 0$, T satisfies an inequality

$$|T(x)| \le Ce^{-\tau(x)}. \tag{16.12}$$

The properties of $\tau(\cdot)$, notably (14.2), imply that

$$\tau(x) \le \tau(x-y) + \beta y + \beta \ \text{ for } \ y > 0, \quad \tau(x-y) \ge \tau(x) \ \text{ for } \ y < 0. \tag{16.13}$$

Thus by (16.12) and by (16.4) with $v = -h$ and some number $v = b \in (\beta, H)$, there are constants C and C' such that

$$\sup_{\xi\in\mathbb{R}} |R * T(x)| \le C \int_{\mathbb{R}} e^{\gamma\lambda + \min\{hy, -by\} - \tau(x-y)} dy$$
$$\le Ce^{\gamma\lambda - \tau(x)} \left(\int_{-\infty}^{0} e^{hy} dy + \int_{0}^{\infty} e^{-by+\beta y+\beta} dy \right) = C' e^{\gamma\lambda - \tau(x)}.$$

From here on we take x *large*, so that in particular $\tau(x) > 0$. For such x we set $\lambda = \delta\tau(x)$, with a positive constant δ such that $\gamma\delta \le 1/2$; another condition on δ will be imposed later. Then by (16.11) the final term in (16.10) is bounded by

$$\frac{1}{\pi} \sup_{\xi\in\mathbb{R}} |R * T(x)| \le Ce^{-\tau(x)/2} < \frac{2C}{\tau(x)}. \tag{16.14}$$

It remains to estimate the term $Q_{W_x}(1/\lambda)$ in the second member of (16.10) for our λ. For this we will use the Tauberian condition. By formula (15.1) for Q_W and definition (16.9) for $W = W_x$, we need a suitable upper bound for

$$W_x(\zeta) - W_x(\eta) = S(x-\zeta)e^{-\zeta^2/2} - S(x-\eta)e^{-\eta^2/2} \tag{16.15}$$
$$= -\{S(x-\eta) - S(x-\zeta)\}e^{-\eta^2/2} + S(x-\zeta)(e^{-\zeta^2/2} - e^{-\eta^2/2})$$

when x is large and $\eta \le \zeta \le \eta + 1/\lambda$. We take x so large that $1/\lambda \le \delta\tau(x)/3$. If $|\eta| > 2\delta\tau(x)/3$ one has $|\zeta| \ge |\eta| - 1/\lambda > \delta\tau(x)/3$, and then both terms on the right of (16.15) are $\mathcal{O}(e^{-\delta^2\tau^2(x)/18}) = \mathcal{O}\{1/\tau(x)\}$ by the boundedness of S. Suppose now that $|\eta| \le 2\delta\tau(x)/3$, so that $|\zeta| \le \delta\tau(x)$. Observe that by (16.13), $\tau(x) \le \tau(0) + \beta x + \beta$ and

$$\tau(x) \le \tau\{x - \delta\tau(x)\} + \beta\delta\tau(x) + \beta.$$

We will require that $\beta\delta \le 1/2$, so that $x - \delta\tau(x) \to \infty$, and take x so large that

$$\tau(x) \le 2\tau\{x - \delta\tau(x)\} + 2\beta \le 3\tau\{x - \delta\tau(x)\}.$$

For large x and our present η and ζ, repeated application of (14.4) and the definition of λ now give

$$-\{S(x-\eta)-S(x-\zeta)\} \le C(\zeta-\eta)+\frac{C}{\tau(x-\zeta)}$$
$$\le \frac{C}{\lambda}+\frac{C}{\tau\{x-\delta\tau(x)\}} \le \frac{C'}{\tau(x)}.$$

Conclusion:

$$Q_{W_x}(1/\lambda) = \sup_{\eta\le\zeta\le\eta+1/\lambda} \{W_x(\zeta)-W_x(\eta)\} \le \mathcal{O}\{1/\tau(x)\} \quad \text{as } x\to\infty. \tag{16.16}$$

Together with (16.10) and (16.11), the inequalities (16.16) and (16.14) complete the proof of (14.5). □

17 A More General Theorem

Theorem 14.1 provides a good estimate for the remainder $S-A$ if the deviation $|K*S(x)-A|$ does not decrease more rapidly than an exponential $e^{-\beta x}$. As Ganelius [1964], [1971] (theorem 5.2) has shown, the Theorem can be extended to a situation of more rapid decrease provided $1/\hat{K}(w)$ satisfies an inequality (14.1) throughout a half-plane $\{v \ge -h\}$.

Theorem 17.1. *Let $\tau(x)$ be a nondecreasing function on $\mathbb{R}$ which tends to ∞ as $x\to\infty$ and for which there is a constant q such that $\tau(x+1)\le q\tau(x)$ when $x\ge x_0$ (so that in particular $\tau(x_0)>0$). Let the kernel K of class $\Omega[-h,\infty,\gamma)$ (Section 14) and the bounded function S be such that*

$$T(x) = K*S(x) = \int_{\mathbb{R}} K(x-y)S(y)dy$$
$$= A\int_{\mathbb{R}} K(y)dy + \mathcal{O}\{e^{-\tau(x)}\} \quad \text{on } \mathbb{R}. \tag{17.1}$$

Suppose that S satisfies the Tauberian condition

$$\inf_{x\le y\le x+1/\tau(x)} \{S(y)-S(x)\} \ge -\mathcal{O}\{1/\tau(x)\} \quad \text{as } x\to\infty. \tag{17.2}$$

Then the remainder $S-A$ satisfies the estimate

$$S(x)-A = \mathcal{O}\{1/\tau(x)\} \quad \text{as } x\to\infty. \tag{17.3}$$

Proof. The following proof is a refinement of the one in Section 16. We again take $A=0$, but in (16.3) and (16.9) need a new auxiliary function M, which involves a parameter $\mu\ge 1$:

$$M(y)=e^{-\mu y^2/2};\quad \hat{M}(u)=\sqrt{(2\pi/\mu)}e^{-u^2/(2\mu)};$$
$$W_x(y)=S(x-y)M(y)=S(x-y)e^{-\mu y^2/2}. \tag{17.4}$$

For $\hat{R}(u)$ we take the corresponding function (16.1), so that $R(y)$ is given by the integral $\int_{\mathbb{R}} I_y(u)du$ with

$$I_y(u) = \frac{1}{\sqrt{2\pi\mu}\hat{K}(u)} e^{iyu} \int_{-\lambda}^{\lambda} e^{-(u-t)^2/(2\mu)} \left(1 - \frac{|t|}{\lambda}\right) e^{-i\xi t} dt; \tag{17.5}$$

cf. (16.5), (16.6). The analytic continuation $I_y(u+iv)$ to the half-plane $v \geq -h$ now satisfies the inequality

$$|I_y(u+iv)| \leq \frac{C}{\sqrt{2\pi\mu}} e^{\gamma|u|+\gamma|v|-yv} \int_{-\lambda}^{\lambda} e^{-\{(u-t)^2-v^2\}/(2\mu)} dt \tag{17.6}$$

for $-h \leq v < \infty$. Thus the function

$$R(y) = R(y; \mu, \lambda, \xi) = \int_{\mathbb{R}} I_y(u+iv)du,$$

with parameters $\mu \geq 1$, $\lambda > 0$, $\xi \in \mathbb{R}$, satisfies the inequalities

$$\begin{aligned} |R(y)| &\leq C e^{\gamma|v|+v^2/(2\mu)-vy} \int_{-\lambda}^{\lambda} dt \int_{\mathbb{R}} e^{\gamma|u|-(u-t)^2/(2\mu)} du / \sqrt{2\pi\mu} \\ &\leq C' \exp\{\gamma|v| + v^2/(2\mu) - vy + \gamma\lambda + \gamma^2\mu/2\}, \quad \forall\, v \geq -h. \end{aligned} \tag{17.7}$$

Here and below $C, C', \cdots$ stand for numbers independent of y and of those parameters v, μ, λ, ξ that are still free; the 'constants' C may change from one formula to the next.

In the next step one chooses the parameters λ and v in a suitable manner. It is convenient to set $\lambda = \gamma\mu/2$. For $y \geq 3\gamma$ we take $v = (y-\gamma)\mu$, so that

$$|R(y)| \leq C e^{\gamma^2\mu - \mu(y-\gamma)^2/2} \quad \text{for } y \geq 3\gamma. \tag{17.8}$$

Taking $v = 0$ for $0 \leq y < 3\gamma$ and $v = -h$ for $y < 0$ we also obtain

$$|R(y)| \leq \begin{cases} Ce^{\gamma^2\mu} & \text{for } 0 \leq y < 3\gamma, \\ C'e^{hy+\gamma^2\mu} & \text{for } y < 0. \end{cases} \tag{17.9}$$

We now come to formulas (16.9)–(16.11) with the new function M. To estimate the integral in (16.10) or (16.11) we use the inequalities

$$|T(x-y)| \leq \begin{cases} Ce^{-\tau(x-y)} & \text{for } y < 3\gamma, \\ C' & \text{for } y \geq 3\gamma. \end{cases} \tag{17.10}$$

(Remember that S is bounded.) Combining (17.8)–(17.10) one finds that

$$\begin{aligned} |R * T(x)| &\leq C_1 \int_{-\infty}^{0} e^{hy+\gamma^2\mu-\tau(x-y)} dy + C_2 \int_0^{3\gamma} e^{\gamma^2\mu-\tau(x-y)} dy \\ &+ C_3 \int_{3\gamma}^{\infty} e^{\gamma^2\mu-\mu(y-\gamma)^2/2} dy \leq Ce^{\gamma^2\mu-\tau(x-3\gamma)} + C'e^{-\gamma^2\mu}. \end{aligned}$$

(Recall that $\mu \geq 1$.) From here on, let $x - 3\gamma \geq x_0$, so that $\tau(x - 3\gamma) > 0$ and $\tau(x) \leq q^{3\gamma+1}\tau(x - 3\gamma)$ or $-\tau(x - 3\gamma) \leq -q^{-3\gamma-1}\tau(x)$. We now take

$$\gamma^2\mu = \delta\tau(x) \quad \text{with } 0 < \delta < q^{-3\gamma-1}. \tag{17.11}$$

Then there is a number $\varepsilon > 0$ such that for our parameters λ and μ,

$$\int_{\mathbb{R}} W_x(y)D_\lambda(y - \xi)dy = \frac{1}{2\pi}R * T(x) = \mathcal{O}(e^{-\varepsilon\tau(x)}) = \mathcal{O}\{1/\tau(x)\}, \tag{17.12}$$

uniformly in ξ; cf. also (16.2) and (17.4).

For the application of Ganelius's inequality (15.2), it remains to estimate the quantity $Q_{W_x}(1/\lambda)$ in the second member of (16.10), now formed with the new function M. It is convenient to take $\lambda = \gamma\mu/2$ greater than 1. We need a suitable upper bound for

$$\begin{aligned} W_x(\zeta) - W_x(\eta) &= S(x - \zeta)e^{-\mu\zeta^2/2} - S(x - \eta)e^{-\mu\eta^2/2} \\ &= -\{S(x - \eta) - S(x - \zeta)\}e^{-\mu\eta^2/2} \\ &\quad + S(x - \zeta)(e^{-\mu\zeta^2/2} - e^{-\mu\eta^2/2}) \end{aligned} \tag{17.13}$$

when x is large and $\eta \leq \zeta \leq \eta + 1/\lambda$. If $|\eta| \geq 1$ and $|\zeta| \geq 1$, the second member of (17.13) (and hence the first) is

$$\mathcal{O}(e^{-\mu/2}) = \mathcal{O}(1/\mu) = \mathcal{O}\{1/\tau(x)\}.$$

From here on we concentrate on the case $\eta \leq \zeta \leq \eta + 1/\lambda$ and $|\zeta| \leq 2$. Observe that

$$|M'(s)| = \mu|s|e^{-\mu s^2/2} \leq \sqrt{\mu/e},$$

so that by the form of λ,

$$e^{-\mu\zeta^2/2} - e^{-\mu\eta^2/2} = \mathcal{O}\{(\zeta - \eta)\sqrt{\mu}\} = \mathcal{O}(1/\sqrt{\mu}).$$

It follows that

$$\begin{aligned} Q_{W_x}(1/\lambda) &= \sup_{\eta\leq\zeta\leq\eta+1/\lambda} \{W_x(\zeta) - W_x(\eta)\} \\ &\leq \sup_{|\zeta|\leq 2,\, \eta\leq\zeta\leq\eta+1/\lambda} [-\{S(x - \eta) - S(x - \zeta)\}] \\ &\quad + \mathcal{O}(1/\sqrt{\mu}) \sup_{|\zeta|\leq 2} |S(x - \zeta)| + \mathcal{O}(1/\mu). \end{aligned} \tag{17.14}$$

By the Tauberian condition, the relation $\lambda = \gamma\mu/2 = \delta\tau(x)/(2\gamma)$ and the restriction on the growth of $\tau(\cdot)$, the first term on the right is of the form $\mathcal{O}\{1/\tau(x - \zeta)\} = \mathcal{O}\{1/\tau(x)\}$. Hence by the revised formula (16.10) and (17.12), (17.14),

$$S(x) = \mathcal{O}\{1/\tau(x)\} + \mathcal{O}\{1/\sqrt{\tau(x)}\} \sup_{|\zeta|\leq 2} |S(x - \zeta)|. \tag{17.15}$$

Since S is bounded, this implies the preliminary result $S(x) = \mathcal{O}\{1/\sqrt{\tau(x)}\}$. Inserting this estimate into (17.15) one concludes that $S(x) = \mathcal{O}\{1/\tau(x)\}$.

18 Application to Stieltjes Transforms

In [1964] Ganelius obtained a Tauberian remainder theorem for general Stieltjes transforms of the type considered in Section IV.9. Taking $s(\cdot) = 0$ on $\mathbb{R}^-$ and renaming the variables, we write

$$F_\rho(\xi) = \int_{0-}^{\infty-} \frac{ds(\eta)}{(\xi+\eta)^\rho} = \lim_{B\to\infty} \int_{0-}^{B} \cdots. \tag{18.1}$$

It follows from formula (IV.9.12) that for $0 \le \alpha < \rho$,

$$s(\eta) = A\eta^\alpha \quad (\eta \ge 0) \quad \Leftrightarrow \quad F_\rho(\xi) = A'\xi^{\alpha-\rho} \quad (\xi > 0),$$
$$\text{where } A' = A\frac{\Gamma(\alpha+1)\Gamma(\rho-\alpha)}{\Gamma(\rho)}.$$

Let us now suppose that $F_\rho(\xi)/\xi^{\alpha-\rho}$ is close to A' for large ξ, as measured by an exponential $e^{-\omega(\xi)}$, where $\omega(\xi)$ is nondecreasing on $\mathbb{R}^+$, tends to ∞ as $\xi \to \infty$ and satisfies a growth condition of the form $\omega(e\xi) \le q\omega(\xi)$ when $\xi \ge \xi_0$.

Theorem 18.1. *Let $0 \le \alpha < \rho$, let $s(\eta)$ vanish for $\eta < 0$, be locally of bounded variation, continuous from the right and such that the Stieltjes transform $F_\rho(\xi)$ exists for $\xi > 0$. Let A and A' be related as above. Suppose that for a function ω as described,*

$$F_\rho(\xi) = [A' + \mathcal{O}\{e^{-\omega(\xi)}\}]\xi^{\alpha-\rho} \quad \textit{as } \xi \to \infty, \tag{18.2}$$

and that $s(\cdot)$ satisfies the Tauberian condition

$$\inf_{\xi\le\eta\le\xi+\xi/\omega(\xi)} \{s(\eta) - s(\xi)\} \ge -\mathcal{O}\{\xi^\alpha/\omega(\xi)\} \quad \textit{as } \xi \to \infty. \tag{18.3}$$

Then one has the remainder estimate

$$s(\xi) - A\xi^\alpha = \mathcal{O}\{\xi^\alpha/\omega(\xi)\} \quad \textit{as } \xi \to \infty. \tag{18.4}$$

Proof. Replacing $s(\eta)$ by $s(\eta) - A\eta^\alpha$ for $\eta \ge 0$, it may be assumed that $A = A' = 0$. For the time being we focus on

THE CASE $\alpha = 0$. In this case the Tauberian condition (18.3) implies that $s(\cdot)$ is slowly decreasing on $\mathbb{R}^+$; cf. Application 14.2. We can now appeal to Boundedness Theorem I.20.1 to conclude from the boundedness of $\xi^\rho F_\rho(\xi)$ as $\xi \to \infty$ that $s(\cdot)$ is bounded. Integrating by parts in formula (18.1), condition (18.2) gives

$$\xi^\rho F_\rho(\xi) = \rho\int_0^\infty \frac{\xi^\rho s(\eta)}{(\xi+\eta)^{\rho+1}} d\eta = \mathcal{O}\{e^{-\omega(\xi)}\} \quad \text{on } \mathbb{R}^+. \tag{18.5}$$

We next set $\xi = e^x$, $\eta = e^y$, $s(e^y) = S(y)$ and $\omega(e^x) = \tau(x)$, so that one has $\tau(x+1) \le q\tau(x)$ when $e^x \ge \xi_0$. This gives the relation

$$\int_{\mathbb{R}} \frac{e^{\rho x + y}}{(e^x+e^y)^{\rho+1}} S(y)dy = \int_{\mathbb{R}} \frac{e^{\rho(x-y)}}{(e^{x-y}+1)^{\rho+1}} S(y)dy = \mathcal{O}\{e^{-\tau(x)}\} \quad \text{on } \mathbb{R}.$$

Thus we have (17.1) with $K(x) = e^{\rho x}/(e^x + 1)^{\rho+1}$, bounded $S(\cdot)$ and $A = 0$. The Fourier transform of K is given by

$$\hat{K}(w) = \int_{\mathbb{R}} \frac{e^{\rho x - iwx}}{(e^x + 1)^{\rho+1}} dx = \int_0^\infty \frac{t^{iw}}{(1+t)^{\rho+1}} dt = \frac{\Gamma(1+iw)\Gamma(\rho - iw)}{\Gamma(\rho+1)};$$

cf. formula (IV.9.12). We verify that the kernel K is of class $\Omega[-1/2, \infty, \pi)$. The function $\hat{K}(w)$ is meromorphic and free of zeros, and one has

$$\frac{1}{\hat{K}(u+iv)} = C \sin\{\pi(v - iu)\} \frac{\Gamma(v - iu)}{\Gamma(\rho + v - iu)};$$

cf. formula (13.11). Since $\rho > 0$ and $v \geq -1/2$, the final quotient is bounded for $|u + iv| \geq \delta > 0$; see for example Titchmarsh [1939] (section 4.41). Thus $\hat{K}(w)$ satisfies an inequality (14.1) with $\gamma = \pi$ throughout the half-plane $\{v \geq -1/2\}$.

The Tauberian condition (18.3) can be restated as

$$\inf_{e^x \leq e^y \leq e^x + e^x/\tau(x)} \{S(y) - S(x)\} \geq -\mathcal{O}\{1/\tau(x)\} \quad \text{as } x \to \infty.$$

In terms of x and y themselves the infimum is over $x \leq y \leq x + \log\{1 + 1/\tau(x)\}$. For large x it is equivalent to take the infimum over $x \leq y \leq x + 1/\tau(x)$, hence we also have (17.2). All conditions of Theorem 17.1 being satisfied, conclusion (17.3) gives the desired remainder estimate (18.4) with $A = \alpha = 0$.

THE CASE $\alpha > 0$. In this case $s(\eta) = \mathcal{O}(1 + \eta^\alpha)$. One can now work with the function $S(y) = e^{-\alpha y} s(e^y)$, which is bounded at $+\infty$, although perhaps not at $-\infty$. Suitably adjusting the kernel and the Tauberian argument, one can again deduce (18.4); cf. Ganelius (loc. cit.).

Remarks 18.2. Using the polynomial approximation method, Subhankulov [1961a], [1961b], and An and Subhankulov [1964], obtained results for the Stieltjes transform under Freud's condition $\omega(e\xi) \leq \omega(\xi) + b$; cf. Subhankulov [1976] (chapter 4). Results involving 'rapidly decreasing' functions of the form $e^{-\omega(\xi)} = \exp(-c\xi^\delta)$ were obtained earlier by Vučković [1953], [1954], who used a complex method. For the Stieltjes transform, $\delta = 1/2$ is a *critical exponent*; if one has (18.2) with $\omega(\xi) = \xi^\delta$ and $\delta > 1/2$, the remainder $s(\xi) - A\xi^\alpha$ will vanish identically for $\xi \geq 0$.

An explanation for the critical exponent 1/2 can be obtained from complex analysis. As a function of the complex variable ζ, the Stieltjes transform $F_\rho(\zeta)$ corresponding to a bounded function $s(\cdot)$ is analytic and bounded in the domain D, consisting of the ζ-plane minus the semi-infinite strip $\{|\mathrm{Im}\,\zeta| \leq 1,\ \mathrm{Re}\,\zeta \leq 1\}$. Conformal mapping of D onto the right half-plane by what is roughly a square-root map makes the discussion of Section 9 applicable. Cf. also Hirschman and Widder [1955] (section 10.3).

A different kind of remainder estimate for the Stieltjes transform is given in Jordan [1976].

19 Fourier Integral Method: Laplace–Stieltjes Transform

Frennemo [1966–67] proved general remainder theorems for kernels K which correspond to n-dimensional Abel summability or Laplace transforms. We restrict ourselves to dimension 1. Keeping in mind what we have found in Sections 13, 14 for the kernel K corresponding to the Laplace–Stieltjes transform, we introduce the following notation. For $h \in (0, 1)$ and $\gamma > 0$, we let $\Lambda[-h, \gamma)$ denote the class of L^1 kernels K for which $1/\hat{K}(w)$ has an analytic continuation to the half-plane $\{\text{Im } w = v \geq -h\}$, such that for some constant C,

$$\frac{1}{|\hat{K}(u+iv)|} \leq Ce^{\gamma|u|+v\log(1+v)} \quad \text{for } u \in \mathbb{R},\ v \geq -h. \tag{19.1}$$

To describe the asymptotic behavior of transforms such as the Laplace–Stieltjes transform $\mathcal{L}ds$ on $\mathbb{R}^+$, we use an arbitrary (unbounded) positive, continuous, nondecreasing function ω as in Section 2. As before, the inverse function of $r\omega(r)$ is called $\theta(t)$; it is increasing and $o(t)$. For the estimation of $s(\cdot)$ we use the nondecreasing function

$$\omega_1(t) = t/\theta(t); \tag{19.2}$$

cf. Theorem 2.5 and formula (2.20). From s and the Laplace–Stieltjes transform $\mathcal{L}ds$ one can pass to a function S and a convolution $K * S(x)$ on $\mathbb{R}$ by the steps in Example 13.2. To describe the asymptotic behavior of such functions $K*S$ and S it is convenient to set

$$\omega(e^x) = \tau(x), \quad \omega_1(e^x) = e^x/\theta(e^x) = \tau_1(x). \tag{19.3}$$

Theorem 19.1. *Let the functions τ and τ_1 on $\mathbb{R}$ be as described above. Let the kernel K of class $\Lambda[-h, \gamma)$ and the bounded function S be such that the transform $T = K*S$ satisfies an estimate*

$$T(x) = K * S(x) = A \int_{\mathbb{R}} K(y)dy + \mathcal{O}\{e^{-\tau(x)}\} \quad \textit{on } \mathbb{R}. \tag{19.4}$$

Suppose that S satisfies the Tauberian condition

$$\inf_{x \leq y \leq x+1/\tau_1(x)} \{S(y) - S(x)\} \geq -\mathcal{O}\{1/\tau_1(x)\} \quad \textit{as } x \to \infty. \tag{19.5}$$

Then the remainder $S - A$ satisfies the estimate

$$S(x) - A = \mathcal{O}\{1/\tau_1(x)\} \quad \textit{as } x \to \infty. \tag{19.6}$$

Proof. The proof will again be based on Ganelius's Theorem 15.1. It combines the earlier approach involving formulas (16.10), (16.11) with the treatment of Frennemo (loc. cit.). We take $M(y)$ and $W_x(y)$ as in (17.4) and form the corresponding functions $\hat{R}$ and R with the aid of (16.1), using the new function $\hat{K}$. The function $R(y)$ is equal to $\int_{\mathbb{R}} I_y(u)du$, with I_y as in (17.5), but with the new $\hat{K}$. In the estimate (17.6) for $I_y(u+iv)$ we now have to replace $\gamma|v|$ by $v\log(1+v)$; cf. (19.1). It follows that

$$R(y) = R(y;\mu,\lambda,\xi) = \int_{\mathbb{R}} I_y(u+iv)du,$$

with $\mu \geq 1,\ \lambda > 0,\ \xi \in \mathbb{R}$, satisfies an inequality

$$|R(y)| \leq C\exp\{v\log(1+v) + v^2/(2\mu) - vy + \gamma\lambda + \gamma^2\mu/2\}, \quad \forall\, v \geq -h; \tag{19.7}$$

cf. (17.7). As before, $C,\ C', \cdots$ will denote numbers independent of y and the parameters that are still free. The parameter v is chosen as follows:

$$v = \begin{cases} -h & \text{for } y < 0, \\ h & \text{for } 0 \leq y < z, \\ v(y) & \text{for } y \geq z. \end{cases} \tag{19.8}$$

Conditions on $z\ (< x)$ and μ will be specified later, and for $y \geq z$ the function $v(y)$ is defined by the relation

$$v(y) + \mu\log\{1+v(y)\} = \mu y. \tag{19.9}$$

Thus

$$v\log(1+v) + v^2/(2\mu) - vy = -v^2/(2\mu) \quad \text{when } \ y \geq z. \tag{19.10}$$

We may assume that $A = 0$, so that by (19.4)

$$|T(x-y)| \leq Ce^{-\tau(x-y)} \leq Ce^{-\tau(x-z)} \quad \text{if } \ y < z. \tag{19.11}$$

For $y \geq z$ we simply use the boundedness of T. Hence

$$\begin{aligned} |R * T(x)| &\leq C\left\{\int_{y<z} |R(y)|e^{-\tau(x-z)}dy + \int_{y\geq z} |R(y)|dy\right\} \\ &= C(I_1 + I_2), \end{aligned} \tag{19.12}$$

say. Then by (19.7)–(19.12)

$$I_1/e^{\gamma\lambda+\gamma^2\mu/2} \leq Ce^{-\tau(x-z)}\int_{y<z} e^{-h|y|}dy \leq C'e^{-\tau(x-z)}. \tag{19.13}$$

For the estimation of I_2 we observe that by (19.9), $v(y)$ is increasing and

$$v(y) \geq y\sqrt{2\mu} \quad \text{for } \ y \geq \mu,$$

provided we take μ sufficiently large. Then by (19.7)–(19.12)

$$\begin{aligned} I_2/e^{\gamma\lambda+\gamma^2\mu/2} &\leq C\int_{y\geq z} e^{-v^2(y)/(2\mu)}dy \\ &\leq C\int_{z\leq y<\mu} e^{-v^2(z)/(2\mu)}dy + C\int_{y\geq\mu} e^{-y^2}dy \\ &\leq Ce^{-v^2(z)/(2\mu)+\log\mu} + Ce^{-\mu^2}. \end{aligned} \tag{19.14}$$

(Here the integral over $\{z \le y < \mu\}$ is taken equal to zero if $\mu \le z$.)

For large x one now chooses μ and z such that

$$\tau(x-z) = \omega(e^{x-z}) = c_1\mu, \quad v(z) = c_2\mu, \tag{19.15}$$

where $c_1 = (\gamma^2/2) + 1$, $c_2 = \sqrt{\gamma^2+4}$. Then μ will be large (see the comment below) and by (19.12)–(19.15)

$$\begin{aligned}|R * T(x)| &\le Ce^{\gamma\lambda+\gamma^2\mu/2}\{e^{-\tau(x-z)} + e^{-v^2(z)/(2\mu)+\mu} + e^{-\mu^2}\}\\ &= Ce^{\gamma\lambda+\gamma^2\mu/2}(e^{-c_1\mu} + e^{-\{(c_2^2/2)-1\}\mu} + e^{-\mu^2})\\ &\le C'e^{\gamma\lambda-\mu} \le 2C'/\mu\end{aligned}$$

when we take $\lambda = \mu/(2\gamma)$. The final inequality holds for all $\xi \in \mathbb{R}$, hence by (16.2),

$$\sup_{\xi\in\mathbb{R}} \left|\int_{\mathbb{R}} W_x(y)D_\lambda(y-\xi)dy\right| = \frac{1}{2\pi}\sup_{\xi\in\mathbb{R}} |R * T(x)| \le \frac{C}{\mu}. \tag{19.16}$$

Comment on the relation between x and μ, z, λ. In (19.15), the first relation and our choice of z make $x - z$ an increasing function of μ, while the second relation makes z an increasing function of μ. In fact, by (19.9) for $y = z$ and (19.15), one has

$$z = c_2 + \log(1 + c_2\mu).$$

As a result x becomes an increasing function of μ, or conversely, μ becomes an increasing function of x. The same holds for λ.

Before applying Ganelius's Inequality (15.2) we have to estimate $Q_{W_x}(1/\lambda)$. The first steps go as in (17.13), (17.14), but the relation between μ and τ will be different. Having (17.14) we will introduce the Tauberian condition. Observe that by the formula for z and (19.15)

$$e^z = e^{c_2}(1 + c_2\mu) \le C\omega(e^{x-z}), \quad e^x = e^{x-z}e^z \le Ce^{x-z}\omega(e^{x-z}),$$

where we may take $C \ge 1$. Hence by the properties of ω_1 and ω, by which in particular $\omega_1(Ct) \le C\omega_1(t)$ and $\omega_1\{r\omega(r)\} = \omega(r)$,

$$\omega_1(e^x) \le C\omega_1\{e^{x-z}\omega(e^{x-z})\} = C\omega(e^{x-z}) = C'\mu; \tag{19.17}$$

see (19.15). Thus by the definition of λ and (19.3),

$$1/\lambda = 2\gamma/\mu \le C/\omega_1(e^x) = C/\tau_1(x). \tag{19.18}$$

Hence by condition (19.5)

$$\begin{aligned}&\sup_{|\zeta|\le 2,\, \eta\le\zeta\le\eta+1/\lambda} [-\{S(x-\eta) - S(x-\zeta)\}]\\ &\quad\le C \sup_{|\zeta|\le 2,\, \eta\le\zeta\le\eta+1/\tau_1(x)} [-\{S(x-\eta) - S(x-\zeta)\}]\\ &\quad\le C'/\tau_1(x-2) \le C'e^2/\tau_1(x).\end{aligned} \tag{19.19}$$

Indeed, (19.3) implies that $\tau_1(t+2) = \omega_1(e^2 e^t) \le e^2\tau_1(t)$. Combining (19.19) with (17.14) and (19.16)–(19.18) one obtains the analog of (17.15):

$$S(x) = \mathcal{O}\{1/\tau_1(x)\} + \mathcal{O}\{1/\sqrt{\tau_1(x)}\} \sup_{|\zeta|\le 2} |S(x-\zeta)|. \tag{19.20}$$

Thus $S(x) = \mathcal{O}\{1/\sqrt{\tau_1(x)}\}$; iteration gives the desired result (19.6). □

From Theorem 19.1 one obtains the following Corollary for the *Laplace–Stieltjes transform* $\mathcal{L}ds$.

Corollary 19.2. *Let* $s(\cdot)$ *satisfy the standard conditions listed after* (2.2), *so that in particular* $F(\xi) = \mathcal{L}ds(1/\xi)$ *exists for* $0 < \xi < \infty$. *Let* ω *and* θ *be related positive nondecreasing functions as above. Suppose that*

$$|F(\xi) - A| \le e^{-\omega(\xi)} \quad \textit{on} \;\; \mathbb{R}^+, \tag{19.21}$$

and that

$$\inf_{\xi\le\eta\le\xi+\theta(\xi)} \{s(\eta) - s(\xi)\} \ge -\mathcal{O}\{\theta(\xi)/\xi\} \quad \textit{as} \;\; \xi \to \infty. \tag{19.22}$$

Then

$$s(\xi) - A = \mathcal{O}\{\theta(\xi)/\xi\} \quad \textit{as} \;\; \xi \to \infty. \tag{19.23}$$

The result follows from Theorem 19.1 by the method of Application 14.2. Writing $\exp(-x - e^{-x}) = K(x)$ and $s(e^y) = S(y)$, the hypotheses (19.21), (19.22) give

$$|K * S(x) - A| = |F(e^x) - A| \le e^{-\omega(e^x)} \quad \text{on} \;\; \mathbb{R},$$

$$\inf_{e^x\le e^y\le e^x+\theta(e^x)} \{S(y) - S(x)\} \ge -\mathcal{O}\{\theta(e^x)/e^x\} \quad \text{as} \;\; x \to \infty.$$

The first relation implies (19.4) because $\omega(e^x) = \tau(x)$. The infimum in the second relation is taken over

$$x \le y \le x + \log\{1 + \theta(e^x)/e^x\} \quad \text{or} \quad x \le y \le x + \log\{1 + 1/\tau_1(x)\};$$

it may be replaced by the infimum over $x \le y \le x + 1/\tau_1(x)$, so that one obtains (19.5). As in Application 14.2, the functions S and s will be slowly decreasing on $\mathbb{R}$ and $\mathbb{R}^+$, respectively, hence relation (19.4) implies that they are bounded. Conclusion (19.23) now follows from (19.6).

Remarks 19.3. The Corollary establishes the important case $\phi = 1$ of Theorem 2.5, and is equivalent to that case: one can replace $s(t)$ by $s(t) - A$ for $t \ge 0$.

The case $\phi(t) = t^\alpha$ of Theorem 2.5 can be treated in a similar way provided $\alpha > -1$. One may now start with the relation

$$F_\alpha(\xi) = \xi^{-\alpha}F(\xi) = \int_0^\infty \eta^{-\alpha}s(\eta)(\eta/\xi)^\alpha e^{-\eta/\xi} d_\eta(\eta/\xi) = \mathcal{O}\{e^{-\omega(\xi)}\}$$

as $\xi \to \infty$. This relation can be rewritten in the form

$$\int_{\mathbb{R}} K_\alpha(x-y)S_\alpha(y)dy = \mathcal{O}\{e^{-\tau(x)}\} \quad \text{as } x \to \infty,$$

with

$$K_\alpha(x) = \exp\{-(\alpha+1)x - e^{-x}\}, \qquad S_\alpha(y) = e^{-\alpha y}s(e^y), \qquad \tau(x) = \omega(e^x).$$

The Fourier transform $\hat{K}_\alpha(w) = \Gamma(\alpha + 1 + iw)$ is in a class $\Lambda[-h, \gamma)$ with $h > 0$. The estimate for $K_\alpha * S_\alpha$ and the Tauberian condition (2.27) imply that S_α is bounded at $+\infty$. The earlier Tauberian argument may now be adjusted to obtain the desired conclusion $S_\alpha(x) = \mathcal{O}\{1/\tau_1(x)\}$ as $x \to \infty$; cf. Frennemo (loc. cit.).

20 Related Results

In [1971] (chapter 3), Ganelius considered remainder theorems in Wiener form. The following is a special case for dimension 1. Let H and K be Wiener kernels whose Fourier transforms $\hat{H}$, $\hat{K}$ have holomorphic extensions to strips around the real axis. Suppose that these satisfy inequalities of the form

$$\begin{aligned} c_1 \le |\hat{H}(w)|e^{\alpha|w|} \le c_2, &\qquad |\mathrm{Im}\, w| \le \delta, \\ c_3 \le |\hat{K}(w)|e^{\beta|w|} \le c_4, &\qquad |\mathrm{Im}\, w| \le \varepsilon, \end{aligned}$$

with positive constants $\alpha < \beta$, δ, ε, c_j. Let S be a bounded function such that

$$S(x) = \mathcal{O}\{e^{-\sigma(x)}\} \text{ and } K * S(x) = \mathcal{O}\{e^{-\tau(x)}\} \quad \text{as } x \to \infty,$$

where σ and τ are nonnegative subadditive functions which satisfy the conditions

$$\begin{aligned} \sup\,(\sigma(x) - \delta|x|) + \int_{\mathbb{R}} e^{\sigma(y)-\delta|y|}dy < \infty, \\ \sup\,(\tau(x) - \varepsilon|x|) + \int_{\mathbb{R}} e^{\tau(y)-\varepsilon|y|}dy < \infty. \end{aligned}$$

Then

$$H * S(x) = \mathcal{O}(1)\exp\left\{-\frac{\alpha}{\beta}\tau(x) - \left(1 - \frac{\alpha}{\beta}\right)\sigma(x)\right\} \quad \text{as } x \to \infty.$$

Work of Lyttkens. In her early papers, Lyttkens [1954–56] sharpened Beurling's Theorem 13.1. It turns out that the condition on $\hat{K}$ is required only in the strip given by $\{0 < \mathrm{Re}\, w < H\}$ and it is enough to require $p > 0$; cf. also Ganelius [1962].

In her papers [1974], [1975] and [1978], Lyttkens replaced the earlier analyticity and growth conditions on $g(w) = 1/\hat{K}(w)$ in a strip around the real axis by growth conditions on a number of derivatives of $g(\cdot)$ on the axis itself. In her article [1978] these conditions are of the following form:

$$\left\{\int_{-X}^{X} |g^{(n)}(u)|^2 du\right\}^{1/2} \le P_n V(X), \quad X \ge X_0,\ n = 0, 1, 2, \ldots. \tag{20.1}$$

Here $V(\cdot)$ is a positive nondecreasing function, and $\{P_n\}$ is a logarithmically convex sequence with $P_0 = 1,\ P_1$ finite, such that $P_n^{1/n} \to \infty$. [In some results $P_n = \infty$ for $n > m$.] In terms of the numbers P_n one defines

$$p(x) = \sup_n \frac{x^n}{P_n}, \quad x \ge 0. \tag{20.2}$$

Suppose now that for a bounded function S,

$$T(x) = K * S(x) = \mathcal{O}\{e^{-\tau(x)}\} \quad \text{as } x \to \infty. \tag{20.3}$$

It is required that τ satisfy a certain regularity condition and that

$$e^{\tau(x)} \le p(cx) \quad \text{for some constant } c > 0 \text{ and } x \ge x_0.$$

For simplicity, Lyttkens imposed the Tauberian condition

$$S(x) + Cx \quad \text{nondecreasing for some constant } C. \tag{20.4}$$

Then one has the remainder estimate

$$S(x) = \mathcal{O}\left(\frac{1}{W^{\leftarrow}\{e^{\tau(x)}\}}\right) \quad \text{as } x \to \infty, \tag{20.5}$$

where $W^{\leftarrow}$ denotes the inverse function of $W(X) = XV(X)$. We quote the following concrete example.

Theorem 20.1. *Let k and q be positive constants such that $kq < k + q$. Let K be in L^1 and let $g = 1/\hat{K}$ satisfy condition* (20.1) *with*

$$V(X) = C_1 \exp(C_2 X^q) \quad \text{and} \quad P_n = n^{n/k} C_3^n \quad (C_j > 0). \tag{20.6}$$

Let S be a bounded function such that

$$K * S(x) = \mathcal{O}\{\exp(-x^\alpha)\} \quad \text{as } x \to \infty, \tag{20.7}$$

where $0 < \alpha < k$. Then under the Tauberian condition (20.4),

$$S(x) = \mathcal{O}(x^{-\alpha/q}) \quad \text{as } x \to \infty. \tag{20.8}$$

If g is entire and satisfies a growth condition

$$|g(w)| \le C_4 \exp(C_5 |w|^q),$$

then (20.1) and (20.6) are fulfilled for any number k such that $kq \le k + q$ and by the Theorem, (20.7) implies (20.8) for any number α which satisfies the condition $\alpha q < \alpha + q$. The special case $q = 2$, which corresponds to the 'Weierstrass kernel'

$K(x) = \exp(-x^2/2)$, was considered by Ganelius (loc. cit., section 5.3). He proved that in this case, (20.7) implies (20.8) whenever $0 < \alpha \le 2$.

In [1974], Lyttkens used her method to obtain remainder estimates for the case where $g = 1/\hat{K}$ is analytic in a domain around the real axis which becomes very narrow as $\operatorname{Re} w \to \pm\infty$. Such results can be applied to Lambert summability, for which

$$\hat{K}(w) = iw\Gamma(1+iw)\zeta(1+iw);$$

there are applications to number theory. A predecessor of this work is Strube [1974]. In [1975] the latter used the Fourier integral method to obtain remainder estimates for Borel summability.

In [1986], Lyttkens obtained remainder theorems which strengthen a ratio theorem of Korenblyum [1955].

Remarks 20.2. In [1954c], [1954e], [1954f], Korevaar used complex Fourier transforms to study relations between remainder estimates for power series, Dirichlet series and Lambert series. In particular the Riemann hypothesis, on the zeros of the zeta function, can be restated as a (speculative) remainder theorem for Lambert series.

21 Nonlinear Problems of Erdős for Sequences

Quadratic Tauberian theorems for sequences were first considered by Erdős [1949b] in connection with the elementary proof of the prime number theorem (Selberg [1949], Erdős [1949a]). It is convenient to formulate the problems in terms of convolutions.

For sequences $a = \{a_0, a_1, a_2, \cdots\}$ and $b = \{b_0, b_1, b_2, \cdots\}$, the *convolution* $a*b$ is defined by the formula

$$(a*b)_n = \sum_{k=0}^{n} a_k b_{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{21.1}$$

Under such convolution, the constant sequence

$$\sigma \stackrel{\text{def}}{=} \{1, 1, 1, \cdots\} \tag{21.2}$$

acts as a summation operator:

$$(\sigma * a)_n = a_0 + a_1 + \cdots + a_n = s_n; \qquad (\sigma*\sigma)_n = n+1. \tag{21.3}$$

Roughly speaking, the basic Tauberian remainder problem is as follows:

Question 21.1. *Assuming that $a*a$ is close to $\sigma*\sigma$, how close must a be to σ when $a \ge 0$?*

It is often convenient to consider the average of the deviation $a - \sigma$ or to measure the 'summed difference'

$$\rho \stackrel{\text{def}}{=} \sigma * (a-\sigma), \quad \text{that is,} \quad \rho_n = s_n - (n+1), \ \forall n. \tag{21.4}$$

In our discussion we consider three MODEL PROBLEMS.

Problem 21.2. ('Square-root problem') What can one say about ρ if

$$a * a = \sigma * \sigma + b \quad \text{with } a \geq 0 \text{ and } b = \mathcal{O}(\sigma)\,?$$

In other words,

$$\sum_{k=0}^{n} a_k a_{n-k} = (n+1) + b_n, \quad a_n \geq 0, \;\; b_n = \mathcal{O}(1). \tag{21.5}$$

Observe that $b = 0$ implies $a = \sigma$.

Natural tools for this problem are power series and complex analysis. These tools give

Theorem 21.3. *The hypotheses* (21.5) *in Problem 21.2 imply that*

$$\rho_n = s_n - (n+1) = \mathcal{O}\{(n+1)^{1/3}\}.$$

The conclusion is valid also if the condition $b_n = \mathcal{O}(1)$ is relaxed to the mean-square condition

$$\frac{1}{n+1} \sum_{k=0}^{n} b_k^2 = \mathcal{O}(1), \tag{21.6}$$

and then it is optimal as an order estimate.

See Sections 22–24 for the proof and for related results; cf. Korevaar [1999]. It is not known what the optimal estimate is for ρ_n under the condition $b_n = \mathcal{O}(1)$; cf. Remarks 24.2.

The second problem involves the 'derived' sequence a' [one may think of the series $\sum_0^\infty a_n e^{nt}$ associated with a] and the difference operator $\Delta = \sigma^{\leftarrow}$:

$$a' \stackrel{\text{def}}{=} \{0, a_1, 2a_2, \cdots, na_n, \cdots\}, \quad \Delta b = \{b_0, b_1 - b_0, \cdots, b_n - b_{n-1}, \cdots\}. \tag{21.7}$$

In our terminology, Erdős's principal question was

Problem 21.4. What can one say about $\rho = \sigma * (a - \sigma)$ if

$$a' + a * a = \sigma' + \sigma * \sigma + \Delta b \quad \text{with } a \geq 0 \text{ and } b = \mathcal{O}(\sigma * \sigma)\,?$$

Again, $b = 0$ implies $a = \sigma$. In the following it is convenient to take $a_0 = 0$. Convolving with σ, the relations above can then be written as

$$\Sigma_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k (k + s_{n-k}) = n^2 + \mathcal{O}(n) \quad \text{and } a_n \geq 0. \tag{21.8}$$

Here Erdős [1949b] proved the surprising

Theorem 21.5. *The hypotheses* (21.8) *imply that*

$$r_n \stackrel{\text{def}}{=} s_n - n = \mathcal{O}(1).$$

The striking aspect is the *strength* of the remainder estimate. In his paper, Erdős described a direct application to (generalized) prime number theory. Suppose that real numbers p_i, with $1 < p_1 < p_2 < \cdots$, satisfy the relation

$$\sum_{p_i \le x} \log^2 p_i + \sum_{p_i p_j \le x} \log p_i \log p_j = 2x \log x + \mathcal{O}(x). \tag{21.9}$$

Selberg established this relation for the ordinary primes; cf. the exposition in Nathanson [2000]. Erdős observed that this formula and Theorem 21.5, with

$$a_k = \sum_{e^{k-1} < p_i \le e^k} \frac{\log p_i}{p_i},$$

together imply the relation

$$\sum_{p_i \le x} \frac{\log p_i}{p_i} = \log x + \mathcal{O}(1). \tag{21.10}$$

If one has (21.9) and (21.10), the final part of the Selberg–Erdős proofs readily implies the analog of the prime number theorem for the sequence $\{p_i\}$.

Theorem 21.5 is contained in Theorem 25.1, which will be proved in Sections 25–28; cf. Korevaar [2001a].

More recently one has considered a third, more fundamental question:

Problem 21.6. What can one say about $a - \sigma$ *itself* if

$$a' + a * a = \sigma' + \sigma * \sigma + c \quad \text{with } a \ge 0 \text{ and } c = \mathcal{O}(\sigma)\,?$$

For $c = 0$ one obtains $a = \sigma$, but $c = -\sigma$ gives the completely different solution $a_n = 1 - (-1)^n$! We will again take $a_0 = 0$, so that the relations above can be written in the form

$$na_n + \sum_{k=1}^{n-1} a_k a_{n-k} = 2n + \mathcal{O}(1) \quad \text{and } a_n \ge 0. \tag{21.11}$$

For the third problem Hildebrand and Tenenbaum [1994] obtained the following amazing theorem:

Theorem 21.7. *The hypotheses* (21.11) *imply that*

$$\text{EITHER } a_n = 1 + \mathcal{O}(1/n) \quad \text{OR} \quad a_n = 1 - (-1)^n + \mathcal{O}(1/n).$$

For the proof and connections with prime number theory we refer to the paper.

Remarks 21.8. In the Fall of 1948, Erdős gave several lectures in Amsterdam on the elementary proof of the prime number theorem. These lectures resulted in the first publication on the proof (van der Corput [1948]). At that time, Erdős asked the author to listen to, and comment on, his very complicated proof of Theorem 21.5. In the Spring of 1950, at Purdue University, Erdős showed the author a letter from Siegel [1950], which sketched how one could simplify the final part of the proof. Siegel's main tool was an ingenious 'Fundamental Relation'; see Section 26. Several years later, H.N. Shapiro [1959] obtained another proof of Erdős's result. Erdős considered the Tauberian theorem as one of his main accomplishments; cf. his article [1997].

It is interesting that a related remainder theorem for a nonlinear differential equation is much easier to prove; cf. Section 25 below.

In a Supplementary Note at the end of his paper [1949b], Erdős considered a problem related to (our later) Problem 21.2. He observed that the condition

$$\sum_{k=0}^{n} a_k s_{n-k} - n^2/2 = \mathcal{O}(n) \quad \text{as} \quad n \to \infty \tag{21.12}$$

for nonnegative a_n implies that $\rho_n = o(n)$ as $n \to \infty$, but not $\rho_n = o(n^{1/2})$. Several authors later found better results for this problem, which is also known as the Erdős–Hua problem. We mention Avakumović [1954], Bojanić, Jurkat and Peyerimhoff [1956], Lindberg [1962] and Subhankulov [1972]. The latter two authors obtained $\rho_n = \mathcal{O}(n^{2/3})$. In [1962/63] the present author used an integral form and Laplace transformation to show that (21.12) implies $\rho_n = \mathcal{O}(n^{3/5})$. That estimate holds also under a weaker mean-square condition and then it is optimal. The power series method used below for Theorem 21.3 would give the same result.

22 Introduction to the Proof of Theorem 21.3

Our method would prove a little more for the 'square-root problem', namely, the following.

Theorem 22.1. *Let*

$$\sum_{k=0}^{n} a_k a_{n-k} = n + 1 + b_n, \quad n = 0, 1, 2, \ldots \tag{22.1}$$

with $a_n \geq 0$. Suppose that

$$\sum_{k=0}^{n} b_k^2 = \mathcal{O}(n) \quad \text{as} \quad n \to \infty, \tag{22.2}$$

or more generally, $\sum_0^n b_k^2 = \mathcal{O}(n^{2\beta+1})$ with $-1/2 < \beta < 1$. Then one has the optimal order estimate

$$\rho_n = s_n - (n+1) = \sum_0^n a_k - (n+1) = \mathcal{O}(n^{1/3}) \quad as \ n \to \infty, \tag{22.3}$$

or $\rho_n = \mathcal{O}(n^{(2\beta+1)/3})$, *respectively.*

PRELIMINARIES. For simplicity we restrict ourselves to the case of (22.2) and (22.3); cf. Korevaar [1999] for the case $\beta \neq 0$. It follows from (22.1) with $2n$ instead of n and (22.2) that $a_n = \mathcal{O}(\sqrt{n})$. We now introduce the generating functions

$$A(z) = \sum_0^\infty a_n z^n, \quad B(z) = \sum_0^\infty b_n z^n \quad (|z| < 1). \tag{22.4}$$

Then (22.1) can be written in the equivalent form

$$A^2(z) = \frac{1}{(1-z)^2} + B(z). \tag{22.5}$$

Observe that $(1-z)^{-1}A(z) = \sum_0^\infty s_n z^n$. Hence by the definition of ρ_n in (22.3),

$$R(z) \stackrel{\text{def}}{=} \sum_0^\infty \rho_n z^n = \frac{A(z)}{1-z} - \frac{1}{(1-z)^2}. \tag{22.6}$$

How can one use (22.2) to estimate the coefficients ρ_n of $R(z)$ in terms of the coefficients b_n of $B(z)$?

As a start we note several consequences of (22.2) for the functions B and B'. Taking $|z| = r < 1$ we obtain from Cauchy–Schwarz and partial summation that

$$\begin{aligned}\max_t |B(re^{it})|^2 &= \max_t \left| \sum_0^\infty b_n r^n e^{int} \right|^2 \le \sum_0^\infty b_n^2 r^n \sum_0^\infty r^n \\ &= (1-r) \sum_{n=0}^\infty \left(\sum_{k=0}^n b_k^2 \right) r^n \cdot \frac{1}{1-r} = \mathcal{O}\left\{ \frac{1}{(1-r)^2} \right\}.\end{aligned} \tag{22.7}$$

Furthermore, by Parseval's relation and additional partial summation,

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} |B(re^{it})|^2 dt &= \sum_0^\infty b_n^2 r^{2n} = (1-r^2) \sum_{n=0}^\infty \left(\sum_{k=0}^n b_k^2 \right) r^{2n} \\ &= \mathcal{O}\left\{ \frac{1}{1-r} \right\}, \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} |B'(re^{it})|^2 dt &= (1-r^2) \sum_{n=1}^\infty \left(\sum_{k=1}^n k^2 b_k^2 \right) r^{2n-2} \\ &= \mathcal{O}\left\{ \frac{1}{(1-r)^3} \right\}.\end{aligned} \tag{22.8}$$

USE OF REAL TAUBERIAN THEORY. What would real-variable methods give us? By (22.7)

$$B(x) = \mathcal{O}\{1/(1-x)\} \quad \text{as } x \nearrow 1.$$

Hence by (22.5) and the condition $a_n \geq 0$,

$$A(x) = \frac{1}{1-x}\{1+(1-x)^2 B(x)\}^{1/2} \sim \frac{1}{1-x} \quad \text{as } x \nearrow 1. \tag{22.9}$$

By the classical Theorem I.7.4 of Hardy and Littlewood, relation (22.9) and the condition $a_n \geq 0$ imply that $s_n = \sum_{k=0}^{n} a_k \sim n$. The Tauberian remainder theory of Section 2 gives only a little more. By (22.7) and (22.9)

$$A(x) = \frac{1}{1-x}\{1 + \mathcal{O}(1-x)\} \quad \text{as } x \nearrow 1,$$

which gives $s_n - n = \mathcal{O}(n/\log n)$ as $n \to \infty$. However, we aim for a much stronger conclusion! – and turn to complex analysis.

COMPLEX METHOD. To derive (22.3) we will use *Cauchy's formula*. Let $C(0, r)$ denote the positively oriented circle $\{z = re^{it},\ -\pi < t \leq \pi\}$. Then

$$\rho_n = \frac{1}{2\pi i}\int_{C(0,r)} \frac{R(z)}{z^{n+1}}dz = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} R(re^{it})e^{-int}dt. \tag{22.10}$$

Later we take $r = 1 - 1/n$, so that $1/r^n$ remains bounded as $n \to \infty$. For a good estimate the domain of integration has to be split up. To this end we distinguish between various sets of points z in the unit disc $\Delta = \{|z| < 1\}$.

The relations (22.6) and (22.5) give

$$R(z) = \frac{\omega_z}{(1-z)^2}\{1+(1-z)^2 B(z)\}^{1/2} - \frac{1}{(1-z)^2}, \tag{22.11}$$

with $\omega_z = \pm 1$ and some determination of the fractional power. It will be convenient to give names to subsets of Δ where $(1-z)^2 B(z)$ is relatively 'large', and relatively 'small', respectively:

$$\begin{aligned} L &\stackrel{\text{def}}{=} \{z \in \Delta : |1-z|^2 |B(z)| \geq 1/3\}, \\ S &\stackrel{\text{def}}{=} \{z \in \Delta : |1-z|^2 |B(z)| < 2/3\}. \end{aligned} \tag{22.12}$$

On the open set S we will use the holomorphic fractional power in (22.11) which is given by the binomial series. Thus since $R(z)$ is holomorphic in Δ, the function ω_z becomes locally constant on S. The union of the components of S where $\omega_z = 1$ in (22.11) will be called S_+, the components where $\omega_z = -1$ will jointly form S_-.

We then have the following inequalities:

$$|R(z)| \leq \begin{cases} 2/|1-z|^2 + (1/|1-z|)|B(z)|^{1/2} & \text{throughout } \Delta, \\ \mathcal{O}(|B(z)|) & \text{uniformly in } L \text{ and } S_+. \end{cases} \tag{22.13}$$

Furthermore by (22.11)

$$R(z) = -2/(1-z)^2 + \mathcal{O}(|B(z)|) \quad \text{uniformly in } S_-. \tag{22.14}$$

23 Proof of Theorem 21.3, Continued

Several steps will be required to estimate the final integral in (22.10). By symmetry it will be sufficient to deal with the upper half of the circle $C(0, r)$, the arc given by $z = re^{it}$, $0 \le t < \pi$. Here the important part is the arc $C(r, \delta)$ given by $0 \le t < \delta$, with small $\delta > 0$. We take r close to 1, so that on $C(r, \delta)$

$$|1 - re^{it}|^2 = (1 - r)^2 + 4r \sin^2(t/2) \approx (1 - r)^2 + t^2. \tag{23.1}$$

By (22.7)

$$|1 - re^{it}|^2 |B(re^{it})| = \mathcal{O}\left\{\frac{(1 - r)^2 + t^2}{1 - r}\right\}. \tag{23.2}$$

Hence by (22.12) the real points $z = x$ close to 1 will be in S; since by (22.6) $R(x) \ge -1/(1 - x)^2$, it follows from (22.11) that these points are in S^+. Thus (23.2) shows that for r sufficiently close to 1, there will be an arc of the form

$$\{z = re^{it},\ 0 \le t < c(1 - r)^{1/2}\}$$

which belongs to S_+.

It will be convenient to use the notations $L(r, \delta)$, $S(r, \delta)$, $S_+(r, \delta)$ and $S_-(r, \delta)$ for the parts of $C(r, \delta)$ in L, S, S_+ and S_-, respectively.

THE MOST DELICATE STEP. We begin with the contributions to the integral in (22.10), due to the first term for $R(z)$ in (22.14), when $z = re^{it}$ runs over arcs of $S_-(r, \delta)$ given by $\lambda < t < \mu$. Here λ must be $\ge c(1 - r)^{1/2}$, hence much larger than $1 - r$, so that $|1 - re^{it}|^2 \approx t^2$. Integration by parts shows that

$$\int_\lambda^\mu \frac{1}{(1 - re^{it})^2} e^{-int} dt = \left[\frac{1}{(1 - re^{it})^2} \frac{e^{-int}}{-in}\right]_\lambda^\mu + \frac{2r}{n} \int_\lambda^\mu \frac{1}{(1 - re^{it})^3} e^{-i(n-1)t} dt.$$

The first term on the right is bounded by $(2/n)|1 - re^{i\lambda}|^{-2}$ and the same is true for the final term since $|1 - re^{it}|^3 \approx t^3$. Thus

$$\left|\int_\lambda^\mu (1 - re^{it})^{-2} e^{-int} dt\right| \le (4/n)|1 - re^{i\lambda}|^{-2}. \tag{23.3}$$

It remains to estimate the sum $\sum_\lambda |1 - re^{i\lambda}|^{-2}$ corresponding to the initial points $re^{i\lambda}$ of certain maximal arcs $(re^{i\lambda}, re^{i\mu})$ of $S_-(r, \delta)$. Notice that such an initial point must belong to the boundary of S, where $|1 - z|^2 |B(z)| = 2/3$. We will ignore maximal arcs of $S_-(r, \delta)$ which lie entirely in L. A maximal arc $(re^{i\lambda}, re^{i\mu})$ that is not a subset of L must contain a point $z = re^{i\tau}$ where $|1 - z|^2 |B(z)| < 1/3$. Hence

$$|B(re^{i\lambda})| = (2/3)|1 - re^{i\lambda}|^{-2},$$
$$|B(re^{i\tau})| < (1/3)|1 - re^{i\tau}|^{-2} < (1/3)|1 - re^{i\lambda}|^{-2},$$

so that

$$(1/3)|1 - re^{i\lambda}|^{-2} < |B(re^{i\lambda}) - B(re^{i\tau})| < \int_\lambda^\tau |B'(re^{it})| dt. \tag{23.4}$$

From the preceding we obtain

Lemma 23.1. *For r close to* 1 *and small* $\delta > 0$, *let* $S^*(r,\delta)$ *denote the union of the maximal arcs* $(re^{i\lambda}, re^{i\mu})$ *of* $S_-(r,\delta)$ *which are not in L. Then the contribution of the term* $-2/(1-z)^2$ *of* $R(z)$ *in* (22.14) *to the integral in* (22.10) *over* $S^*(r,\delta)$ *can be estimated as follows:*

$$\begin{aligned}\left|-2\int_{S^*(r,\delta)}(1-z)^{-2}z^{-n-1}dz\right| &= 2\left|\sum_\lambda\int_\lambda^\mu(1-re^{it})^{-2}e^{-int}dt\right|\\ \leq (8/n)\sum_\lambda|1-re^{i\lambda}|^{-2} &< (24/n)\int_0^\delta|B'(re^{it})|dt. \end{aligned}\tag{23.5}$$

Proof of the Estimate (22.3). The starting point is formula (22.10) with $r = 1-1/n$ and (large) $n \geq 2$. It will be sufficient to obtain an upper bound for

$$\left|\int_0^\pi R(re^{it})e^{-int}dt\right|.$$

Using small $\delta > 0$ we will split the integral at $t = \delta$, and define

$$L^*(r,\delta) = L(r,\delta)\setminus S^*(r,\delta) = L(r,\delta)\cap\{C(r,\delta)\setminus S^*(r,\delta)\}.$$

Notice that the arc $C(r,\delta)$ of the circle $C(0,r)$ is the union of the nonoverlapping parts $L^*(r,\delta)\cup S_+(r,\delta)$ and $S^*(r,\delta)$. Abusing the notation, we write

$$\int_{S^*(r,\delta)}\cdots dt \quad\text{for}\quad \int_{re^{it}\in S^*(r,\delta)}\cdots dt,$$

etc. With this convention,

$$\int_0^\pi = \int_\delta^\pi + \int_{L^*(r,\delta)\cup S_+(r,\delta)} + \int_{S^*(r,\delta)} = I_1 + I_2 + I_3, \tag{23.6}$$

say. The integral I_1 is estimated with the aid of the first formulas in (22.13) and (22.8), combined with Hölder's inequality for $p = 4/3,\ q = 4$:

$$\begin{aligned}|I_1| &\leq 2\int_\delta^\pi|1-re^{it}|^{-2}dt + \int_\delta^\pi|1-re^{it}|^{-1}|B(re^{it})|^{1/2}dt\\ &\leq \mathcal{O}(\delta^{-1}) + \left(\int_\delta^\pi|1-re^{it}|^{-4/3}dt\right)^{3/4}\left(\int_\delta^\pi|B(re^{it})|^2dt\right)^{1/4}\\ &= \mathcal{O}(\delta^{-1}) + \mathcal{O}\{\delta^{-1/4}(1-r)^{-1/4}\}. \end{aligned}\tag{23.7}$$

For I_2 formulas (22.13), (22.8) and Cauchy–Schwarz give

$$I_2 = \mathcal{O}\left(\int_0^\delta|B(re^{it})|dt\right) = \mathcal{O}\{\delta^{1/2}(1-r)^{-1/2}\}. \tag{23.8}$$

It remains to estimate I_3. Using (22.14), (23.5) in Lemma 23.1 and (22.8) one obtains

$$\begin{aligned} I_3 &= \mathcal{O}\left((1/n)\int_0^\delta |B'(re^{it})|dt\right) + \mathcal{O}\left(\int_0^\delta |B(re^{it})|dt\right) \\ &= \mathcal{O}\{(1/n)\delta^{1/2}(1-r)^{-3/2}\} + \mathcal{O}\{\delta^{1/2}(1-r)^{-1/2}\}. \end{aligned} \tag{23.9}$$

Recall now that $1-r = 1/n$ where $2 \le n \to \infty$. Then by (22.10), (23.6)–(23.9) and corresponding results for $\int_{-\pi}^0$,

$$\begin{aligned} |\rho_n \le & \left|\int_{-\pi}^{\pi} R(re^{it})e^{-int}dt\right| \\ &= \mathcal{O}(\delta^{-1}) + \mathcal{O}(\delta^{-1/4}n^{1/4}) + \mathcal{O}\{\delta^{1/2}n^{1/2}\}. \end{aligned} \tag{23.10}$$

The choice $\delta = n^{-1/3}$ *miraculously* makes all $\mathcal{O}$-terms *of the same order*, and provides the desired estimate $\rho_n = \mathcal{O}(n^{1/3})$ of (22.3). □

24 An Example and Some Remarks

The optimality of the order estimate in Theorem 22.1 can be deduced from an example. What one needs is a sequence $a = \{a_n\}_{n=0}^\infty$ with $a_n \ge 0$, for which (22.1) holds with a sequence $b = \{b_n\}$ that satisfies (22.2), but for which the order estimate (22.3) cannot be improved:

$$\limsup_{n\to\infty} |\rho_n|/n^{1/3} = 1, \tag{24.1}$$

say. For the construction we set

$$a = \sigma + d, \quad \text{so that } b = 2\sigma * d + d * d, \quad \rho = \sigma * d. \tag{24.2}$$

By the support of a sequence $c = \{c_n\}_{n=0}^\infty$ we mean the set of the integers $n \ge 0$ for which $c_n \ne 0$. The supporting interval is the minimal interval containing the support. Our final sequence d will be obtained by combining sequences with well-separated supporting intervals. Using integers p and q with $p \ge 2q,\ q \ge 2$, the basic building block $e(p,q)$ is the sequence with generating function

$$E(z) = z^p(1 + z + \cdots + z^{q-1})(1 - z^q). \tag{24.3}$$

THE CASE OF ONE TERM. For $d = e(p,q)$, let a and b be given by (24.2). Then $a \ge 0$ and the generating functions for ρ and b will be

$$\begin{aligned} R(z) &= \sum \rho_n z^n = (1-z)^{-1}E(z) = z^p(1 + z + \cdots + z^{q-1})^2, \\ B(z) &= \sum b_n z^n = z^p(1 + z + \cdots + z^{q-1})^2\{2 + z^p(1-z^q)^2\}. \end{aligned} \tag{24.4}$$

From this one obtains

Lemma 24.1. *Suppose $d = e(p,q)$ with $p \ge 2q,\ q \ge 2$. Then the support of $b = \{b_n\}$ in $\mathbb{N}_0$ belongs to the union of the disjoint intervals $[p, p+2q-2]$ and $[2p, 2p+4q-2]$, which together contain fewer than $6q$ integers. In this case $|b_n| \le 2q$ for all n, so that $\sum_{k=0}^n b_k^2 < 24q^3$ for all n. At the same time,* $\max |\rho_n| = \rho_{p+q-1} = q$.

Taking $q = [p^{1/3}]$ with large p, one concludes that for the special sequence $d = e(p, q)$,

$$\sum_{k=0}^{n} b_k^2 < 24n, \quad \forall n, \quad \text{while} \quad \sup |\rho_n|/n^{1/3} \approx 1. \tag{24.5}$$

The desired example is obtained by adding infinitely many sequences $e(p_i, q_i)$ with well-separated supporting intervals:

$$d = \sum_{i=1}^{\infty} e(p_i, q_i), \quad q_i = [p_i^{1/3}] \text{ and } p_i \geq 2q_i,$$
$$p_{i+1} \geq 4p_i, \quad q_{i+1} \geq 2q_i, \quad q_1 \geq 2. \tag{24.6}$$

With such a sequence d, the sequences a and b obtained from (24.2) will satisfy the conditions of Theorem 22.1 for $\beta = 0$, while ρ will satisfy relation (24.1). The latter follows immediately from (24.5), since the supporting intervals of the terms $\sigma * e(p_i, q_i)$ are disjoint. Because of nonlinearity, the verification of condition (22.2) requires some care; cf. Korevaar [1999].

Remarks 24.2. In the general case of Theorem 22.1, involving an arbitrary number β in $(-1/2, 1)$, the proof is essentially the same as in the case $\beta = 0$ above.

The theorem can be extended to convolution roots of order $m > 2$. Suppose that

$$a^{*m} = \sigma^{*m} + b \quad \text{with } a \geq 0 \text{ and } \sum_{0}^{n} b_k^2 = \mathcal{O}(n^{2\beta+1})$$

as $n \to \infty$, where $-1/2 < \beta < m - 1$. Then one has the optimal estimate

$$\rho_n = \mathcal{O}(n^{(2\beta+1)/(2m-1)}) \quad \text{as } n \to \infty;$$

see Korevaar (loc. cit.).

We end with a comment on the original Problem 21.2. Complex analysis is better suited to handle the mean-square condition (21.6) on b than the straight order condition $b_n = \mathcal{O}(1)$. It is an *open problem* whether boundedness of b implies boundedness or near-boundedness of ρ. The example above may support such a conjecture. Indeed, looking at relation (24.2) it is hard to see how b could be bounded while $\sigma * d$ is not. In that case

$$2\sigma * d \text{ and } -d * d$$

would have to become large in exactly the same places and in the same way. Could one devise a combinatorial method for the problem, related to the one which Erdős used for Problem 21.4? One might also take a clue from the proof of the Erdős–Fuchs theorem [1956], which likewise asserts that certain sequences cannot fully compensate each other; cf. Halberstam and Roth [1983], Newman [1998].

25 Introduction to the Proof of Theorem 21.5

In the case of Problem 21.4 it is convenient to assume that the sequence $\{a_n\}$ starts with $a_0 = 0$; more generally, we set $a_\nu = 0$ and $s_\nu = 0$ for any indices $\nu < 1$ that may occur. The following result contains Theorem 21.5; the case $\gamma > 0$ was stated by Erdős [1949b] without proof.

Theorem 25.1. *Let* $a_n \geq 0$ *and* $s_n = \sum_{k=1}^{n} a_k$, $n = 1, 2, \ldots$.

(i) Suppose that

$$\Sigma_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k(s_{n-k} + k) = n^2 + \mathcal{O}(n^{1+\gamma}), \tag{25.1}$$

where $0 \leq \gamma < 1$. *Then*

$$r_n \stackrel{\text{def}}{=} s_n - n = \mathcal{O}(n^\gamma), \tag{25.2}$$

and

$$\Delta \sum_{k=1}^{n} r_{n-k} r_k = \mathcal{O}(n^{1+\gamma}). \tag{25.3}$$

(ii) Conversely, the relations (25.2) *and* (25.3) *together imply* (25.1). *In the case* $\gamma = 0$, *the relations* (25.1) *and* (25.2) *are equivalent.*

One can quickly dispose of the converse part. Indeed, $s_{n-k} + k = n + r_{n-k}$ and $a_k = \Delta s_k = 1 + \Delta r_k$, hence

$$\begin{aligned} \Sigma_n &= \sum_{k=1}^{n} a_k(n + r_{n-k}) = n s_n + \sum_{k=1}^{n} (1 + \Delta r_k) r_{n-k} \\ &= n s_n + \sum_{k=1}^{n} r_{n-k} + \Delta \sum_{k=1}^{n} r_{n-k} r_k. \end{aligned}$$

If one assumes (25.2) and (25.3), the final member takes the form required for Σ_n, namely, $n^2 + \mathcal{O}(n^{1+\gamma})$. One may observe also that (25.3) follows from (25.1) and (25.2). For $\gamma = 0$ relation (25.1) follows from (25.2).

The real problem is to derive (25.2) from (25.1). For this we will combine Erdős's method for the case $\gamma = 0$ with a 'Fundamental Relation' due to Siegel [1950]; see Section 26. We begin with a weaker hypothesis.

Proposition 25.2. *Suppose one has* (25.1) *or just*

$$\Sigma_n = n^2 + o(n^2) \quad \text{as } n \to \infty. \tag{25.4}$$

Then

$$r_n = s_n - n = o(n). \tag{25.5}$$

Proof. One has to show that $\alpha = \liminf s_n/n$ and $\beta = \limsup s_n/n$ are both equal to 1. By the definition of Σ_n and partial summation,

$$\Sigma_n = \sum_{k=1}^{n} a_k(s_{n-k} + k) \geq \sum_{k=1}^{n} k a_k = n s_n - \sum_{k=1}^{n-1} s_k. \tag{25.6}$$

We first prove that $\beta < \infty$. Supposing for a moment that $\beta = \infty$, one focuses on large n such that $s_n/n \geq s_k/k$ for all $k \leq n$. Then by (25.6) and (25.4)

$$n s_n \leq \Sigma_n + (s_n/n) \sum_{k=1}^{n-1} k \leq n^2 + o(n^2) + n s_n/2,$$

so that $s_n/n \leq 2 + o(1)$. Thus $\beta \leq 2$ which contradicts the assumption $\beta = \infty$.

If $\alpha \geq 1$ then $s_{n-k} \geq n - k - o(n)$ for $k \leq n$, hence

$$n^2 + o(n^2) \geq \Sigma_n \geq \{n - o(n)\} \sum_{k=1}^{n} a_k,$$

so that $\beta \leq 1$ and therefore $\beta = \alpha = 1$. Similarly the assumption $\beta \leq 1$ implies $\alpha \geq 1$ and $\alpha = \beta = 1$. It remains to deal with the possibility that, in fact,

$$\alpha < 1 < \beta, \tag{25.7}$$

which we assume to be the case until we reach a contradiction.

For $k \leq n$ one has $s_{n-k} + k \geq \alpha n + (1 - \alpha)k - o(n)$, so that by (25.6)

$$\Sigma_n \geq \{\alpha n - o(n)\} s_n + (1 - \alpha) \left(n s_n - \sum_{k=1}^{n-1} s_k \right).$$

Choosing n such that $s_n = \beta n + o(n)$, one concludes that

$$n^2 + o(n^2) \geq \{\alpha\beta + (1 - \alpha)\beta/2\} n^2 - o(n^2),$$

which implies $\alpha\beta < 1$.

Finally take n such that $s_n = \alpha n + o(n)$. Then for sufficiently small $\delta > 0$ and $k \leq \delta n$,

$$s_{n-k} + k \leq \alpha n + k + o(n) \leq (\alpha + \delta) n + o(n) \leq (\beta - \varepsilon) n + o(n)$$

with $\varepsilon > 0$. Also, for $k > \delta n$,

$$s_{n-k} + k \leq \beta n + (1 - \beta)k + o(n) \leq \beta n - (\beta - 1)\delta n + o(n) \leq (\beta - \varepsilon) n + o(n),$$

provided ε had been chosen small enough. It follows that

$$n^2 - o(n^2) \leq \Sigma_n \leq (\beta - \varepsilon) n s_n + o(n^2) \leq (\beta - \varepsilon)\alpha n^2 + o(n^2),$$

which would imply $\alpha\beta > 1$.

This contradiction proves that (25.7) is false, so that $\alpha = 1$ or $\beta = 1$, and by the preceding it follows that indeed $\alpha = \beta = 1$.

Remarks 25.3. Relation (25.5) could also be derived with the aid of power series and the theory of differential equations. It is interesting to speculate how much more one could get in that way under the stronger hypotheses of Theorem 25.1. Taking $\gamma = 0$ for simplicity, one can derive from (25.1) that the function $y(x) = \sum a_n e^{-nx}$ satisfies a differential equation

$$y' - y^2 = -\frac{2}{x^2} + \mathcal{O}\left(\frac{1}{x}\right) \quad \text{for } 0 < x \le 1. \tag{25.8}$$

Under the positivity condition $a_n \ge 0$ the function $y(x)$ will be nonnegative and nonincreasing. By the standard substitution $y = -u'/u$ where we take $u > 0$, the Riccati equation (25.8) goes over into the linear equation

$$u'' + \left\{-\frac{2}{x^2} + \mathcal{O}\left(\frac{1}{x}\right)\right\} u = 0,$$

which can be treated as a perturbation problem. In our case of negative u' one can use a method from Bellman's book [1953] to show that

$$u(x) = \frac{c}{x} + \mathcal{O}(1), \quad u'(x) = -\frac{c}{x^2} + \mathcal{O}\left(\frac{1}{x}\right);$$

cf. Korevaar [1956] (meetings abstract). It follows that

$$y(x) = \frac{1}{x} + \mathcal{O}(1) \quad \text{as } x \searrow 0.$$

Hence by the Hardy–Littlewood Theorem I.7.4 one has $s_n \sim n$. In fact, a Tauberian remainder estimate of Freud (Section 2) would show that $r_n = \mathcal{O}(n/\log n)$. A complex method might give more; cf. Section IV.21 or Ingham [1941], and Hayman [1956], Bender [1974], Flajolet [2002].

Open Problem. Could one treat Erdős's Problem 21.4 by complex analysis, or by a discrete analog of the method, sketched here for the Riccati equation (25.8)?

26 The Fundamental Relation and a Reduction

Replacing n by q and substituting $s_k = k + r_k$, the definition of r_n in (25.2) and relation (25.1) can be rewritten as

$$\sum_{0 \le k < q} a_{q-k} = q + r_q, \qquad \sum_{0 \le k < q} a_{q-k} \cdot (-r_k) = q r_q + \mathcal{O}(q^{1+\gamma}). \tag{26.1}$$

We multiply the first relation by $1 - R/D$ and the second by η/D. Adding, one obtains an important relation due to C.L. Siegel [1950]:

Theorem 26.1. (Fundamental Relation) *For* $0 < D < R$ *and* $\eta = \pm 1$,

$$\sum_{k<q} a_{q-k}\left(1 - \frac{R + \eta r_k}{D}\right) = q\left(1 - \frac{R - \eta r_q}{D}\right) + \left(1 - \frac{R}{D}\right) r_q + \mathcal{O}\left(\frac{q^{1+\gamma}}{D}\right). \tag{26.2}$$

This relation will be applied several times with q large and $R = R_n$, where R_n is given by the formula

$$\frac{R_n}{n^\gamma} \stackrel{\text{def}}{=} \max_{1 \le k \le n} \frac{|r_k|}{k^\gamma}. \tag{26.3}$$

In the proofs it may then be assumed that

$$\textit{the nondecreasing sequence } \{R_n/n^\gamma\} \textit{ is unbounded,} \tag{26.4}$$

or there would be nothing to prove.

Proposition 26.2. *The conclusion* $r_n = \mathcal{O}(n^\gamma)$ *of Theorem 25.1 is valid at least in the case* $0 < \gamma < 1$.

Proof. Let condition (25.1) be satisfied for some number $\gamma \in (0, 1)$ but suppose that (25.2) is false, so that $R_n/n^\gamma \nearrow \infty$ by (26.3). We will focus on the *special sequence* of integers $n \to \infty$ (occasionally called n^*) for which R_n/n^γ exceeds $R_{n-1}/(n-1)^\gamma$, so that $R_n = |r_n|$. Observe that

$$|r_k| \le (k/n)^\gamma R_n < 2^{-\gamma} R_n, \quad \forall k < n/2. \tag{26.5}$$

We now apply the Fundamental Relation with $q = n$, $R = R_n$, $D = (1 - 2^{-\gamma})R$, $\eta = \pm 1 = R/r_n$. The result is

$$\begin{aligned}\sum_{k<n/2} a_{n-k}\left(1 - \frac{R_n \pm r_k}{(1 - 2^{-\gamma})R_n}\right) &+ \sum_{k \ge n/2} a_{n-k}\left(1 - \frac{R_n \pm r_k}{(1 - 2^{-\gamma})R_n}\right) \\ &= n - \frac{r_n}{2^\gamma - 1} + \mathcal{O}\left(\frac{n^{1+\gamma}}{R_n}\right).\end{aligned}$$

Hence by (26.5), (25.5) and (26.4),

$$\sum_{k<n/2} a_{n-k} \cdot \text{neg} + \sum_{k \ge n/2} a_{n-k}(1 + \text{neg}) = n + o(n).$$

Here 'neg' stands for numbers ≤ 0. It follows that for the integers $n = n^*$,

$$s_{n/2} = \sum_{k \ge n/2} a_{n-k} \ge n - o(n) \qquad (s_\nu = s_{[\nu]}).$$

However, by Proposition 25.2 the left-hand side is asymptotic to $n/2$ when $n \to \infty$. This contradiction establishes the desired estimate (25.2) in the case $0 < \gamma < 1$. □

Henceforth we take $\gamma = 0$ and assume that the hypotheses of Theorem 25.1 are satisfied for this case. The proof that the sequence $\{r_n\}$ is bounded will require several steps. Following Erdős [1949b] we carry out an initial reduction.

Proposition 26.3. *The sequence $\{a_k\}$ is bounded and for the proof of Theorem 25.1 in the case $\gamma = 0$, it may and will be assumed that for a suitable arbitrarily small positive number ε (which will be specified later),*

$$a_k < 2 + \varepsilon \quad \text{for all } k \geq k_0. \tag{26.6}$$

Proof. Subtracting Σ_n in (25.1) from Σ_{n+1}, one finds that

$$(n+1)a_{n+1} \leq \Sigma_{n+1} - \Sigma_n = 2n + \mathcal{O}(n).$$

Hence there is a constant C such that

$$a_k \leq C, \quad \forall k. \tag{26.7}$$

Suppose now that the boundedness of the sequence $\{r_n\}$ has been proved under the additional condition (26.6). Then the general case can be handled as follows. For a positive integer p to be determined below, define

$$a_k^* = \frac{1}{p} \sum_{m=(k-1)p+1}^{kp} a_m, \quad k = 1, 2, \ldots. \tag{26.8}$$

Then by (25.1) with $\gamma = 0$,

$$\begin{aligned}(k-1)p^2 a_k^* &= (k-1)p\{a_{(k-1)p+1} + \cdots + a_{kp}\} \\ &\leq \Sigma_{kp} - \Sigma_{(k-1)p} = (2k-1)p^2 + \mathcal{O}(kp).\end{aligned}$$

Hence for any small number $\varepsilon > 0$ we can choose an integer $p = p(\varepsilon)$ so large that the numbers a_k^* satisfy an inequality (26.6). From here on we keep p fixed.

We next show that the sequence $\{a_k^*\}$ and the corresponding partial sums s_n^* also satisfy condition (25.1). Indeed, since $a_k \leq C$,

$$\begin{aligned}\sum_{k=1}^{n} a_k^*(s_{n-k}^* + k) &= \frac{1}{p^2} \sum_{k=1}^{n} \{a_{(k-1)p+1} + \cdots + a_{kp}\}(s_{(n-k)p} + kp) \\ &= \frac{1}{p^2} \sum_{k=1}^{n} \sum_{m=(k-1)p+1}^{kp} a_m\{s_{np-m} + m + \mathcal{O}(kp - m)\} \\ &= \frac{1}{p^2} \Sigma_{np} + \mathcal{O}(n) = n^2 + \mathcal{O}(n).\end{aligned}$$

Thus by the supposition that the Theorem has been proved under condition (26.6) we obtain $\sum_{k=1}^{n} a_k^* = n + \mathcal{O}(1)$. This implies (25.2) with $\gamma = 0$ by the definition of a_k^* in (26.8) and the boundedness of the sequence $\{a_k\}$.

27 Proof of Theorem 25.1, Continued

Continuing with the case $\gamma = 0$ and assuming that $a_k < 2 + \varepsilon$ for $k \geq k_0$ (26.6), we proceed under the *supposition that the remainder sequence* $\{r_n\}_1^\infty$ *is unbounded.* In accordance with (26.3) we set

$$R_n = \max_{k \leq n} |r_k|, \quad \text{so that } R_n \nearrow \infty. \tag{27.1}$$

In the following we let n run through the *special subsequence* of the positive integers (occasionally called n^*) for which $|r_n|$ exceeds all preceding numbers $|r_k|$, hence $R_n = |r_n|$. For definiteness we assume that $r_n > 0$ (see Remark 27.4 for the case $r_n < 0$).

Definitions 27.1. Taking n equal to a large index n^*, set

$$r_n = R_n = R \quad \text{and} \quad \log R = \rho; \tag{27.2}$$

it is assumed that $\rho > 2$. As in Erdős's article [1949b], indices $k \leq n = n^*$ for which r_k is relatively close to $\pm R$ are called u's and v's according to the conditions

$$r_v < -R + \rho, \quad \text{and } r_u > R - \rho^2, \text{ respectively.} \tag{27.3}$$

We also need the indices u^* and $v^* \leq n = n^*$ which are defined by the conditions

$$r_{v^*} < -R + \rho^3, \quad \text{and } r_{u^*} > R - \rho^4, \text{ respectively.} \tag{27.4}$$

The indices $u, v, \cdots$ form subsets $U, V, \cdots$ of the interval $[1, n]$ which depend on $n = n^*$. In order to keep the terminology light we will speak simply of indices $u, v, \cdots$, without mentioning the dependence on n^* all the time. In the following it is shown that there are 'many' u's and 'many' v's.

By the preceding the index $n = n^*$ is a number u.

Proposition 27.2. *For $n = n^* \to \infty$, the number of indices v $(< n)$ is at least equal to $n/(2+\varepsilon) - o(n)$, and the largest index v is of the form $n - o(n)$.*

Proof. If $k < n$ is a number v one has $R + r_k < \rho$. With $q = n$, $D = \rho$ and $\eta = 1$, the Fundamental Relation 26.1 with $\gamma = 0$ thus shows that

$$\begin{aligned}\sum_{k<n} a_{n-k}\left(1 - \frac{R + r_k}{\rho}\right) &= \sum_{k \in V} a_{n-k}(1 + \text{neg}) + \sum_{k \notin V} a_{n-k} \cdot \text{neg} \\ &= n + \left(1 - \frac{R}{\rho}\right) R + \mathcal{O}\left(\frac{n}{\rho}\right).\end{aligned}$$

Since we have condition (25.1) with $\gamma = 0$ we may apply Proposition 26.2 for the case $0 < \gamma \leq 1/2$ to conclude that $R^2/\rho = o(n)$. It follows that

$$\sum_{k \in V} a_{n-k} \geq n + o(n) \quad [\text{in fact, } = n + o(n)].$$

Hence by the bound $2+\varepsilon$ on the numbers a_k with $k \geq k_0$, the number of indices v is at least $n/(2+\varepsilon) - o(n)$.

To prove the second statement in the Proposition, suppose that max $v < \theta n$ for some number $\theta \in (0,1)$. Then $R + r_k \geq \rho$ for $k \geq \theta n$, hence by the Fundamental Relation with q, D and η as above,

$$\sum_{k \geq \theta n} a_{n-k} \cdot \text{neg} + \sum_{k<\theta n} a_{n-k}(1+\text{neg}) = n + o(n).$$

This would imply that

$$s_n - s_{(1-\theta)n} = \sum_{k<\theta n} a_{n-k} - \mathcal{O}(1) \geq n - o(n),$$

in contradiction to the relation $s_\nu \sim \nu$ (Proposition 25.2).

Proposition 27.3. *There are also many indices $u < n = n^*$. More precisely, let q be a (large) index v, so that $R + r_q < \rho$. Then the number of indices $u < q$ is at least $q/(2+\varepsilon) - o(q)$ and the largest index $u < q$ is of the form $q - o(q)$.*

Proof. If k is a u one has $R - r_k < \rho^2$. With $D = \rho^2$ and $\eta = -1$, the Fundamental Relation with $\gamma = 0$ now shows that

$$\begin{aligned}\sum_{k<q} a_{q-k}\left(1 - \frac{R-r_k}{\rho^2}\right) &= \sum_{k\in U} a_{q-k}(1+\text{neg}) + \sum_{k\notin U} a_{q-k}\cdot \text{neg}\\ &= q\left(1 - \frac{R+r_q}{\rho^2}\right) + \left(\frac{R}{\rho^2} - 1\right)\cdot(-r_q) + o(q).\end{aligned}$$

Thus

$$\sum_{k\in U} a_{q-k} \geq q + \left(\frac{R}{\rho^2} - 1\right)(R-\rho) - o(q),$$

while

$$\sum_{k<q} a_{q-k} = q + r_q < q - R + \rho.$$

It follows that

$$\frac{R^2}{\rho^2} = o(q) \quad \text{and} \quad \sum_{k\in U} a_{q-k} \geq q - o(q).$$

Hence the number of indices $u < q$ is at least $q/(2+\varepsilon) - o(q)$.

That the largest index $u < q$ is of the form $q - o(q)$ follows as in the proof of Proposition 27.2.

Remark 27.4. If $r_n < 0$ one may interchange ρ and ρ^2 in the definition of u's and v's; cf. (27.3). For the v-number n, the argument of Proposition 27.2 (but with $\eta = -1$) would then give a good result on the u-numbers. Similarly, the argument of Proposition 27.3 (but with $\eta = 1$) would now give a good result on the v-numbers.

By Proposition 27.2 one may take the element $q \in V$ in Proposition 27.3 of the form $n - o(n)$. One thus has the following corollary.

Proposition 27.5. *The total number of indices u and v up to $n = n^*$ is at least equal to $\{2/(2+\varepsilon)\}n - o(n)$.*

The method of Propositions 27.2 and 27.3 also gives the following

Proposition 27.6. *If q is a 'large' index u – larger than $n/\log n$, say – the number of indices $v^* < q$ is at least $q/(2+\varepsilon) - o(q)$ and the largest such number v^* is of the form $q - o(q)$. Similarly, if q is a large number v^*, the number of indices $u^* < q$ is at least $q/(2+\varepsilon) - o(q)$ and the largest such number u^* is of the form $q - o(q)$.*

28 The End Game

We continue under the assumptions and with the notations of Section 27, fixing a value $\varepsilon \leq 1/10$. Here Siegel's ideas [1950] will be used to show that there are many indices less than $n = n^*$ which are neither u's nor v's; enough to give a *contradiction.*

Proposition 28.1. *Set $L = 5R$, let t be an integer satisfying the condition*

$$(n/\log n) + L \leq t \leq n$$

and let I_t denote the interval $\{t - L < q \leq t\}$.

(i) If I_t contains at least one index u (or at least one index v, respectively), then

$$r_q < R/2 \;\; (\text{or} \;\; r_q > -R/2, \;\; \text{resp.}) \;\; \textit{for some index} \;\; q \in I_t. \tag{28.1}$$

(ii) In any case the number of integers in I_t which are neither u's nor v's is at least equal to

$$R/(2+2\varepsilon) - o(R). \tag{28.2}$$

Remark 28.2. The constant 5 in $L = 5R$ and the denominator 2 in (28.1) have been chosen experimentally to ensure that for small ε, there are enough non-u's, non-v's both here and in Proposition 28.3 below.

Proof of Proposition 28.1. If I_t is free of indices u and v there is nothing to prove. Also, the proof of parts (i) and (ii) is easy if I_t contains both a number u and a number v. Indeed, one has $r_u = R - o(R)$ and $r_v = -R + o(R)$, while

$$|r_k - r_{k-1}| = |a_k - 1| < 1 + \varepsilon \;\; \text{for} \;\; k > k_0.$$

From here on we assume that I_t contains at least one number u but no v (the proof is similar if I_t contains a number v but no u).

(α) Suppose now that (28.1) is false, that is, $r_q \geq R/2$ throughout I_t. Then we can show that up to t, there are too many indices which are outside the set U^*. For the indices $k \leq t$ in U^* one has $R + r_k > 2R - \rho^4$. Hence by the Fundamental Relation 26.1 with $\gamma = 0,\; D = 2R - \rho^4$ and $\eta = 1$,

$$\sum_{k<q} a_{q-k}\left(1-\frac{R+r_k}{2R-\rho^4}\right) = \sum_{k\in U^*} a_{q-k}\cdot \text{neg} + \sum_{k\notin U^*} a_{q-k}(1+\text{neg})$$
$$= q\left(1-\frac{R-r_q}{2R-\rho^4}\right) + o(q).$$

Recall that $a_\nu = 0$ and $s_\nu = 0$ when $\nu < 1$. Then since $R - r_q \le R/2$,

$$\sum_{k\le t,\, k\notin U^*} a_{q-k} = \sum_{k<q,\, k\notin U^*} a_{q-k} \ge \frac{3}{4}q - o(q) \tag{28.3}$$

whenever $t - L < q \le t$. Defining $\chi(k) = 1$ if $k \ge 1$ is not in U^* and $\chi(k) = 0$ otherwise, we sum over q. Then the left-hand side of (28.3) gives the sum

$$\sum_{t-L<q\le t}\ \sum_{k\le t} \chi(k) a_{q-k} = \sum_{k\le t} \chi(k) \sum_{t-L<q\le t} a_{q-k}.$$

Now the final inner sum is equal to $s_{t-k} - s_{t-k-L} \le L + 2R$, so that the repeated sum on the left is at most equal to $(L+2R)\sum_{k\le t}\chi(k)$. Also summing over q on the right-hand side of (28.3), one thus obtains the inequality

$$(L+2R)\sum_{k\le t}\chi(k) \ge \frac{3}{4}Lt - o(Lt) \quad \text{or} \quad \sum_{k\le t,\, k\notin U^*} 1 \ge \frac{15}{28}t - o(t). \tag{28.4}$$

Recall that the given interval I_t contains an index u, hence if we use both parts of Proposition 27.6, one after the other, we find that the number of indices $u^* \le t$ is at least $t/(2+\varepsilon) - o(t)$. But since $\varepsilon \le 1/10$ this is inconsistent with (28.4):

$$\frac{15}{28} + \frac{1}{2+\varepsilon} \ge \frac{85}{84} > 1.$$

The contradiction establishes the relevant part of (28.1).

(β) We continue under the assumption that I_t contains an index u but no v. For part (ii) we now choose an index $q \in I_t$ for which $r_q < R/2$. Let u' be the (or a) number u in I_t closest to q; there will be no number u (and no number v) between u' and q. We have $r_{u'} > R - \rho^2$, hence $|r_{u'} - r_q| > R/2 - o(R)$. Since $|r_{u'} - r_q| < (1+\varepsilon)|u' - q|$, it follows that $|u' - q| > R/(2+2\varepsilon) - o(R)$. Thus the total number of non-u's, non-v's in I_t is at least $R/(2+2\varepsilon) - o(R)$.

For the ultimate contradiction it remains to sum over appropriate intervals I_t.

Proposition 28.3. *Let $a_k < 2+\varepsilon$ for $k \ge k_0$ with $\varepsilon \le 1/10$, let the numbers $R = R_n$ become arbitrarily large and let $n = n^*$ be a special index as in Section 27. Then the number of integers $\le n$ which are neither u's nor v's is at least*

$$\frac{n}{10(1+\varepsilon)} - o(n). \tag{28.5}$$

Proof. We continue with the notation of Proposition 28.1 so that in particular $(n/\log n) + L \le t \le n$. Observe that the intervals $I_t = (t - L, t]$ belong to this range for $t = n, n - L, \cdots, t - (N - 1)L$, where $NL \approx n - (n/\log n)$. Summing over these intervals I_t, we conclude from part (ii) of Proposition 28.1 that the total number of non-u's, non-v's up to n is at least

$$\begin{aligned}\left\{\frac{R}{2(1+\varepsilon)} - o(R)\right\}\frac{NL}{L} &\ge \left\{\frac{R}{2(1+\varepsilon)} - o(R)\right\}\frac{n - o(n)}{5R} \\ &= \frac{n}{10(1+\varepsilon)} - o(n).\end{aligned}$$

Conclusion 28.4. *For $\varepsilon \le 1/10$, Proposition 28.3 contradicts Proposition 27.5. This contradiction shows that the supposition $R_n \to \infty$ of* (27.1) *is false, in other words,*

$$r_n = s_n - n = \mathcal{O}(1).$$

Indeed, for $\varepsilon \le 1/10$,

$$\frac{2}{2+\varepsilon} + \frac{1}{10(1+\varepsilon)} \ge \frac{20}{21} + \frac{1}{11} > 1.$$

Hence if (27.1) would be true, Propositions 27.5 and 28.3 would show that for large $n = n^*$, the total number of indices u and v up to n, together with the indices up to n that are neither u's nor v's, would exceed n.

References

The Bibliography doubles as *Author Index*. The numbers in italics, placed in square brackets at the end of each reference, refer to the Sections in which an item is mentioned.

van Aardenne–Ehrenfest, T., see Korevaar

Abel, N.H.

[1826] Recherches sur la série $1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots$. J. für Math. **1**, 311–339. Oeuvres complètes, vol. I, pp 219–250. [*I.1, 2; III.1*]

Agmon, S.

[1953] Complex variable Tauberians. Trans. Amer. Math. Soc. **74**, 444–481. [*III.4*]

[1965] Lectures on elliptic boundary value problems. Van Nostrand, New York. [*III.19*]

Agmon, S. and Kannai, P.

[1967] On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators. Israel J. Math. **5**, 1–30. [*III.16, 19*]

Agnew, R.P.

[1954] Tauberian relations among partial sums, Riesz transforms, and Abel transforms of series. J. Reine Angew. Math. **193**, 94–118. [*I.5*]

Ahlfors, L. and Heins, M.

[1949] Questions of regularity connected with the Phragmén-Lindelöf principle. Ann. of Math. (2) **50**, 341–346. [*VII.9*]

Aljančić, S. and Arandelović, D.

[1977] O-regularly varying functions. Publ. Inst. Math. Beograd (N.S.) **22(36)**, 5–22. [*IV.6*]

Aljančić, S., Bojanić, R. and Tomić, M.

[1954] Sur la valeur asymptotique d'une classe des intégrales définies. Acad. Serbe Sci. Publ. Inst. Math. **7**, 81–94. [*IV.5*]

[1974] Slowly varying functions with remainder term and their applications in analysis. Serbian Academy of Sciences and Arts, Monographs CDLXVII, no. 41, Belgrade, pp 1–51. [*IV.1*]

Allan, G.R., O'Farrell, A.G. and Ransford, T.J.

[1987] A Tauberian theorem arising in operator theory. Bull. London Math. Soc. **19**, 537–545. [*III.1, 13, 15*]

Alvarez, T. and González, M.

[1991] Some examples of Tauberian operators. Proc. Amer. Math. Soc. **111**, 1023–1027. [*V.18*]

Amir, A.: see Jakimovski, A.

An, F.I. and Subhankulov, M.A.

[1964] A Tauberian theorem and its application to the rapidity of convergence of Fourier series (Russian). Izv. Akad. Nauk UzSSSR Ser. Fiz.-Mat. **64**, 5–13. [*VII.18*]

Ananda-Rau, K.

[1921] On Lambert's series. Proc. London Math. Soc. (2) **19**, 1–20. [*I.2*]

[1928] On the converse of Abel's theorem. J. London Math. Soc. **3**, 200–205. [*I.22*]

Andrews, G.E.

[1976] The theory of partitions. Encycl. Math. Appl. vol. 2. Addison–Wesley, Reading. [*IV.23; VII.10*]

Apostol, T.M.

[2000] A centennial history of the prime number theorem. Number theory, pp 1–14. Trends in Math., Birkhäuser, Basel. [*III.1*]

Aramaki, J.

[1988] On an extension of the Ikehara Tauberian theorem. Pacific J. Math. **133**, 13–30. [*III.4*]

[1996] An extension of the Ikehara Tauberian theorem and its application. Acta Math. Hungar. **71**, 297–326. [*III.4*]

Arandelović, D., see also Aljančić

[1976] Sur un théorème mercerien asymptotique général pour des fonctions à comportement régulier. Publ. Inst. Math. Beograd (N.S.) **20**, 29–36. [*IV.13*]

Arendt, W. and Batty, C.J.K.

[1988] Tauberian theorems and stability of one-parameter semigroups. Trans. Amer. Math. Soc. **306**, 837–852. [*III.1, 13, 15*]

[1995] A complex Tauberian theorem and mean ergodic semigroups. Semigroup Forum **50**, 351–366. [*III.13*]

Arendt, W., Batty, C.J.K., Hieber, M. and Neubrander, F.

[2001] Vector-valued Laplace transforms and Cauchy problems. Birkhäuser, Basel. [*III.1, 13, 15*]

Arendt, W. and Prüss, J.

[1992] Vector-valued Tauberian theorems and asymptotic behavior of linear Volterra equations. SIAM J. Math. Anal. **23**, 412–448. [*III.13*]

Ash, J.M., Erdős, P. and Rubel, L.A.

[1974] Very slowly varying functions. Aequationes Math. **10**, 1–9. [*IV.4*]

Atzmon, A. and Olevskiĭ, A.

[1996] Completeness of integer translates in function spaces on $\mathbb{R}$. J. Approx. Theory **87**, 291–327. See also Math. Rev. **98a**: 46035. [*II.8*]

Avakumović, V.G.

[1935] Sur une extension de la condition de convergence des théorèmes inverses de sommabilité. C. R. Acad. Sci. Paris **200**, 1515–1517. [*IV.6*]

[1936] Über Laplacesche Integrale, deren Wachstum von iteriertem Exponentialcharakter ist. Acad. R. Serbe, Bull. Acad. Sci. math. nat. A **3**, 173–181. [*IV.16, 19*]

[1940a] Bemerkungen über Laplacesche Integrale, deren Wachstum von Exponentialcharakter ist. I and II, Math. Z. **46**, 62–69; III, Math. Z. **47**, 141–152. [*III.4; IV.18, 21*]

[1940b] Neuer Beweis eines Satzes von G.H. Hardy und S. Ramanujan über das asymptotische Verhalten der Zerfällungskoeffizienten. Amer. J. Math. **62**, 877–880. [*III.4; IV.23*]

[1940c] Über das Verhalten Dirichletscher Reihen am Rande des Konvergenzgebietes. Math. Z. **46**, 650–664. [*III.4; IV.21*]

[1950a] Bemerkung über einen Satz des Herrn T. Carleman. Math. Z. **53**, 53–58. [*III.19; IV.15; VII.2*]

[1950b] Einige Sätze über Laplacesche Integrale. Acad. Serbe Sci. Publ. Inst. Math. **3**, 287–304. [*IV.15; VII.2*]

[1954] A note on a question set by P. Erdős and L.K. Hua. Acad. Serbe Sci. Publ. Inst. Math. **6**, 47–56. [*VII.21*]

Avakumović, V.G. and Karamata, J.

[1936] Über einige Taubersche Sätze, deren Asymptotik von Exponentialcharakter ist, I. Math. Z. **41**, 345–356. [*IV.15*]

Badiozzaman, A.J. and Thorpe, B.

[1996] Some best possible Tauberian results for Abel and Cesàro summability. Bull. London Math. Soc. **28**, 283–290. [*I.24*]

Baernstein, A.

[1969] A nonlinear Tauberian theorem in function theory. Trans. Amer. Math. Soc. **146**, 87–105. [*IV.1*]

Báez-Duarte, L.

[1995] General Tauberian theorems on the real line. Adv. Math. **112**, 56–119. [*IV.24*]

[1997] Hardy–Ramanujan's asymptotic formula for partitions and the central limit theorem. Adv. Math. **125**, 114–120. [*IV.23*]

Balkema, A.A.

[2002] On the Laplace transforms of logconcave densities. Publ. Inst. Math. Beograd (N.S.) **71(85)**, 9–19. [*IV.1, 24*]

Balkema, A.A., Geluk, J.L. and de Haan, L.

[1979] An extension of Karamata's Tauberian theorem and its connection with complementary convex functions. Quart. J. Math. Oxford (2) **30**, 385–416. [*IV.15, 24*]

[1995] Measuring asymptotic convexity. Publ. Inst. Math. Beograd (N.S.) **58(72)**, 106–116. [*IV.24*]

Balkema, A.A., Klüppelberg, C. and Resnick, S.I.

[1993] Densities with Gaussian tails. Proc. London Math. Soc. (3) **66**, 568–588. [*IV.24*]

[1999] Limit laws for exponential families. Bernoulli **5**, 951–968. [*IV.24*]

[2003] Domains of attraction for exponential families. Stochastic Process. Appl. **107**, 83–103. [*IV.1, 24*]

Balkema, A.A., Klüppelberg, C. and Stadtmüller, U.

[1995] Tauberian results for densities with Gaussian tails. J. London Math. Soc. (2) **51**, 383–400. [*IV.24*]

Banach, S.

[1932] Théorie der opérations linéaires, Warsaw. [*V.14–16*]

Baran, M.

[1991] A Karamata method. I. Elementary properties and applications. Canad. Math. Bull. **34**, 147–157. [*I.15*]

Baron, S.A.

[1966/77] Introduction to the theory of summability of series (Russian). Second ed. 'Valgus', Tallinn. [*I.1, 2, 24; VI.20*]

Baron, S. and Leviatan, D.

[1991] Professor Amnon Jakimovski – on his sixty-fifth birthday. In: Approximation, interpolation and summability, Israel Math. Conf. Proc. 4, Bar-Ilan Univ., pp 1–24.

[1997] G.G. Lorentz and the theory of summability. In: Mathematics from Leningrad to Austin (Selected works of G.G. Lorentz), vol. 1, Birkhäuser, Boston, pp 41–57.

Bateman, P.T. and Diamond, H.G.

[1996] A hundred years of prime numbers. Amer. Math. Monthly **103**, 729–741. [*III.1*]

Batty, C.J.K., see also Arendt

[1990] Tauberian theorems for the Laplace–Stieltjes transform. Trans. Amer. Math. Soc. **322**, 783–804. [*III.13*]

[1994a] Some Tauberian theorems related to operator theory. Functional analysis and operator theory (Warsaw 1992), Banach Center Publ. 30, Warsaw, pp 21–34. [*III.13*]

[1994b] Asymptotic behaviour of semigroups of operators. Functional analysis and operator theory (Warsaw 1992), Banach Center Publ. 30, Warsaw, pp 35–52. [*III.15*]

Batty, C.J.K., van Neerven, J. and Räbiger, F.

[1998] Tauberian theorems and stability of solutions of the Cauchy problem. Trans. Amer. Math. Soc. **350**, 2087–2103. [*III.13*]

Batty, C.J.K. and Yeates, S.B.

[2000] Weighted and local stability of semigroups of operators. Math. Proc. Cambridge Philos. Soc. **129**, 85–98. [*III.15*]

Beekmann, W., see Zeller

Beke, M.

[1916] Differenciál és Integrálszámitás (Hungarian; 'Differential and integral calculus'), vol. 2. Budapest. [*VI.21*]

Bellman, R.

[1953] Stability theory of differential equations. Mc-Graw-Hill, New York. (Reprinted by Dover, New York, 1969.) [*VII.25*]

Bender, E.A.

[1974] Asymptotic methods in enumeration. SIAM Rev. **16**, 485–515. [*IV.21; VII.25*]

Beneš, V.E.

[1961] Extensions of Wiener's Tauberian theorem for positive measures. J. Math. Anal. Appl. **2**, 1–20. [*II.13*]

Bennett, G. and Kalton, N.J.

[1972] FK-spaces containing c_0. Duke Math. J. **39**, 561–582. [*V.14*]

Berg, L.

[1960] Über das asymptotische Verhalten der inversen Laplace-Transformation. Math. Nachr. **22**, 87–91. [*IV.24*]

Berry, A.C.

[1941] The accuracy of the Gaussian approximation to the sum of independent variates. Trans. Amer. Math. Soc. **49**, 122-136. [*IV.24*]

Beurer, J., Borwein, D. and Kratz, W.

[1999] Two Tauberian theorems for Dirichlet series methods of summability. Acta Sci. Math. (Szeged) **65**, 139–168. [*VI.24*]

Beurling, A.

[1938] Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle. C. R. du 9^e Congrès des Math. Scand., Helsingfors, pp 345–366. [*II.8; V.1, 4, VII.13*]

[1945] Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel. Acta Math. **77**, 127–136. [*II.1, 3, 8*]

[1947] Sur une classe de fonctions presque-périodiques. C. R. Acad. Sci. Paris **225**, 326–328. [*II.8*]

[1957] Lecture Notes (unpublished). [*IV.11*]

[1989] The collected works of Arne Beurling, vol. 2. Harmonic analysis. Birkhäuser Boston, Boston.

Binder, C.

[1984] Alfred Tauber (1866–1942). Ein österreichischer Mathematiker. Jahrb. Überblicke Math. 1984, Math. Surv. **17**, 151–166. [*I.1*]

Bingham, N.H.

[1981] Tauberian theorems and the central limit theorem. Ann. Probab. **9**, 221–231. [*IV.11; VI.1, 13, 22*]

[1984a] On Valiron and circle convergence. Math. Z. **186**, 273–286. [*VI.1, 13, 18, 22*]

[1984b] On Borel and Euler summability. J. London Math. Soc. (2) **29**, 141–146. [*VI.1, 13, 18, 22*]

[1984c] Tauberian theorems for summability methods of random-walk type. J. London Math. Soc. (2) **30**, 281–287. [*VI.1, 13, 22*]

[1985] On Tauberian theorems in probability theory. Nieuw Arch. Wisk. (4) **3**, 157–166. [*I.7, 24; VI.1, 13, 18, 22*]

[1988] Tauberian theorems for Jakimovski and Karamata–Stirling methods. Mathematika **35**, 216–224. [*VI.13, 18, 22*]

[1989] Tauberian theorems in probability theory. In: Probability measures on groups, IX. Lecture Notes in Math. nr. 1379, Springer, Berlin, pp 6–20. [*II.13; V.22; VI.13*]

Bingham, N.H. and Goldie, C.M.

[1982] Extensions of regular variation. I: Uniformity and quantifiers; II: Representation and indices. Proc. London Math. Soc. (3) **44**, 473–496; 497–534. [*IV.6*]

[1983] On one-sided Tauberian conditions. Analysis **3**, 159–188. [*IV.11; VI.13, 18, 22*]

[1988] Riesz means and self-neglecting functions. Math. Z. **199**, 443–454. [*VI.13*]

Bingham, N.H., Goldie, C.M. and Teugels, J.L.

[1987] Regular variation. Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge Univ. Press, Cambridge. (Paperback edition with additions, 1989.) [*I.1; III.1, 19; IV.1, 2, 4–6, 8–16, 23; V.22*]

Bingham, N.H. and Inoue, A.

[1997] The Drasin–Shea–Jordan theorem for Fourier and Hankel transforms. Quart. J. Math. Oxford (2) **48**, 279–307. [*IV.13; V.9*]

[1999] Ratio Mercerian theorems with application to Hankel and Fourier transforms. Proc. London Math. Soc. (3) **79**, 626–648. [*IV.13; V.9*]

[2000a] Extension of the Drasin–Shea–Jordan theorem. J. Math. Soc. Japan **52**, 545–559. [*IV.13*]

[2000b] Abelian, Tauberian and Mercerian theorems for arithmetic sums. J. Math. Anal. Appl. **250**, 465–493. [*IV.13*]

[2000c] Tauberian and Mercerian theorems for systems of kernels. J. Math. Anal. Appl. **252**, 177–197. [*IV.13; V.9*]

Bingham, N.H. and Stadtmüller, U.

[1990] Jakimovski methods and almost-sure convergence. In: Disorder in physical systems, pp 5–17. Oxford Univ. Press. [*VI.13*]

Bingham, N.H. and Tenenbaum, G.

[1986] Riesz and Valiron means and fractional moments. Math. Proc. Cambridge Philos. Soc. **99**, 143–149. [*VI.18, 22*]

Bingham, N.H. and Teugels, J.L.

[1975] Duality for regularly varying functions. Quart. J. Math. Oxford (2) **26**, 333–353. [*IV.15*]

[1979] Tauberian theorems and regular variation. Nieuw Arch. Wisk. (3) **27**, 153–186. [*IV.9*]

[1980] Mercerian and Tauberian theorems for differences. Math. Z. **170**, 247–262. [*IV.13*]

Blackwell, D.

[1948] A renewal theorem. Duke Math. J. **15**, 145–150. [*V.22*]

[1953] Extension of a renewal theorem. Pacific J. Math. **3**, 315–320. [*II.13; V.22*]

Bloom, S.

[1976] A characterization of B-slowly varying functions. Proc. Amer. Math. Soc. **54**, 243–250. [*IV.11*]

Boas, R.P.

[1939] A Tauberian theorem connected with the problem of three bodies. Amer. J. Math. **61**, 161–164. [*I.17*]

[1954] Entire functions. Academic Press, New York. [*III.19; VII.9*]

Boas, R.P. and Bochner, S.

[1938] Closure theorems for translations. Ann. of Math. (2) **39**, 287–300. [*II.8*]

Bochner, S., see also Boas

[1933a] Ein Satz von Landau und Ikehara. Math. Z. **37**, 1–9. [*III.4, 6*]

[1933b] Umkehrsätze für allgemeine Limitierungsverfahren. Sitz.-Ber. Preuss. Akad. Wiss., Math.-Phys. Klasse **3/4**, 126–144. [*II.8, 14, 15*]

[1934] An extension of a Tauberian theorem on series with positive terms. J. London Math. Soc. **9**, 141–148. [*IV.9*]

[1969] Selected mathematical papers of S. Bochner. Benjamin, New York.

Bochner, S. and Chandrasekharan, K.

[1949] Fourier transforms. Ann. of Math. Studies vol. 19, Princeton Univ. Press, Princeton. [*I.25; IV.9*]

Bochner, S. and Hardy, G.H.

[1926] Note on two theorems of Norbert Wiener. J. London Math. Soc. **1**, 240–242. [*II.6*]

Bojanić (= Bojanic), R., see also Aljančić

Bojanic, R. and DeVore, R.

[1966] On polynomials of best one sided approximation. Enseignement Math. **12**, 139–164. [*VII.5*]

Bojanic, R., Jurkat, W. and Peyerimhoff, A.

[1956] Über einen Taubersatz für Faltungen. Math. Z. **65**, 195–199. [*VII.21*]

Bojanic, R. and Karamata, J.

[1963a] On slowly varying functions and asymptotic relations. Math. Res. Center Tech. Report 432, Madison, WI. [*IV.2, 5*]

[1963b] On a class of functions of regular asymptotic behavior. Math. Res. Center Tech. Report 436, Madison, WI. [*IV.2, 6*]

Bojanic, R. and Seneta, E.

[1971] Slowly varying functions and asymptotic relations. J. Math. Anal. Appl. **34**, 302–315. [*IV.2, 3*]

Boos, J.

[2000] Classical and modern methods in summability. Oxford Univ. Press, Oxford. [*I.1, 12, 18; V.1, 14, 15; VI.4, 24, 25*]

Boos, J. and Parameswaran, M.R.

[1997] A theorem of Mazur–Orlicz type. J. Anal. **5**, 43–52. [*V.19*]

Boos, J. and Tietz, H.

[1992] Convexity theorems for the circle methods of summability. J. Comput. Appl. Math. **40**, 151–155. [*VI.22*]

Borel, É.

[1895] Sur la sommation des séries divergentes. C. R. Acad. Sci. Paris **121**, 1125–1127. [*VI.1, 2*]

[1899] Mémoire sur les séries divergentes. Ann. Sci. École Norm. Sup. (3) **16**, 9–136. [*I.2, 3; VI.1, 2*]

[1901] Le prolongement analytique et les séries sommables. Math. Ann. **55**, 74–80. [*VI.1, 2*]

[1901/28] Leçons sur les séries divergentes. Gauthier–Villars, Paris. [*I.2, 3; VI.1*]

Borichev, A.A. and Hedenmalm, H.

[1995] Completeness of translates in weighted spaces on the half-line. Acta Math. **174**, 1–84. [*V.9*]

Borwein, D., see also Beurer, Robinson

[1957] On a scale of Abel-type summability methods. Proc. Cambridge Philos. Soc. **53**, 318–322. [*V.21*]

[1958] On Borel-type methods of summability. Mathematika (London) **5**, 128–133. [*VI.21*]

[1960] Relations between Borel-type methods of summability. J. London Math. Soc. **35**, 65–70. [*VI.21*]

[1981] Tauberian conditions for the equivalence of weighted mean and power series methods of summability. Canad. Math. Bull. **24**, 309–316. [*VI.24*]

[1990] Asymptotic relationships between Dirichlet series. J. Math. Anal. Appl. **147**, 604–609. [*I.25*]

[1992] A Tauberian theorem concerning Borel-type and Riesz summability methods. Canad. Math. Bull. **35**, 14–20. [*VI.22*]

Borwein, D. and Kiesel, R.

[1994] Weighted means and summability by generalized Nörlund and other methods. J. Math. Anal. Appl. **184**, 607–619. [*VI.24*]

Borwein, D. and Kratz, W.

[1989] On relations between weighted mean and power series methods of summability. J. Math. Anal. Appl. **139**, 178–186. [*VI.24*]

[1994] An $\mathcal{O}$-Tauberian theorem and a high-indices theorem for power series methods of summability. Math. Proc. Cambridge Philos. Soc. **115**, 365–375. [*VI.24*]

Borwein, D., Kratz, W. and Stadtmüller, U.

[2001] One-sided Tauberian theorems for Dirichlet series methods of summability. Rocky Mountain J. Math. **31**, 797–830. [*VI.24*]

Borwein, D. and Markovich, T.

[1988] A Tauberian theorem concerning Borel-type and Cesàro methods of summability. Canad. J. Math. **40**, 228–247. [*VI.22*]

[1992] Weighted means and summability by the circle and other methods. J. Approx. Theory **68**, 49–55. [*VI.22*]

Borwein, D. and Meir, A.

[1987] A Tauberian theorem concerning weighted means and power series. Math. Proc. Cambridge Philos. Soc. **101**, 283–286. [*VI.24*]

Borwein, D. and Smet, E.

[1974] Tauberian theorems for Borel-type methods of summability. Canad. Math. Bull. **17**, 167–173. Addendum, same journal **19** (1976), 245–246. [*VI.2*]

Borwein, P. and Erdélyi, T.

[1995] Polynomials and polynomial inequalities. Grad. Texts in Math. vol. 161, Springer, New York. [*I.24*]

Bosanquet, L.S.

[1943] Note on convexity theorems. J. London Math. Soc. **18**, 239–248. [*I.17*]

Bourion, G.

[1937] L'ultraconvergence dans les séries de Taylor. Act. Sci. et Ind. nr. 472, Hermann, Paris. [*VI.7*]

de Branges, L.

[1985] A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152. [*VI.24*]

Brossard, J. and Chevalier, L.

[1990] Problème de Fatou ponctuel et dérivabilité des mesures. Acta Math. **164**, 237–263. [*II.19*]

[1995] Une réciproque optimale du théorème de Fatou ponctuel. Adv. Math. **115**, 300–318. [*II.19*]

de Bruijn, N.G., see also Korevaar

[1948] On Mahler's partition problem. Nederl. Akad. Wetensch. Proc. **51**, 659–669 = Indag. Math. **10**, 210–220. [*IV.23*]

[1958/81] Asymptotic methods in analysis. North-Holland Publ. Co., Amsterdam. (Reprinted by Dover Publ., New York, 1981.) [*I.7; IV.16*]

[1959] Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform. Nieuw Arch. Wisk. (3) **7**, 20–26. [*IV.2, 15, 16*]

Burkill, J.C.

[1969] Albert Edward Ingham. Bull. London Math. Soc. **1**, 109–124. [*II.17*]

Burris, S.N.

[2001] Number theoretic density and logical limit laws. Math. Surveys and Monographs vol. 86, Amer. Math. Soc., Providence, RI. [*IV.19*]

Butzer, P.L. and Neuheuser, H.-G.

[1965] Auf Cesàro-Limitierungsverfahren eingeschränkte Tauberbedingungen. Monatsh. Math. **69**, 1–17. [*I.24*]

Carleman, T.

[1934] Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes. C. R. du 8^e Congrès des Math. Scand., Stockholm, pp 34–44. [*III.19*]

[1944] L'intégrale de Fourier et questions qui s'y rattachent. Uppsala. [*II.1, 4, 8*]

Carleson, L.

[1997] Wiener's Tauberian theorem. In: The legacy of Norbert Wiener: A centennial symposium (Cambridge, MA, 1994). Proc. Symp. Pure Math. vol. 60, Amer. Math. Soc., Providence, pp 65–70. [*V.9*]

Cartan, H.

[1940] Sur les classes de fonctions définies par des inégalités portant sur leurs dérivées successives. Act. Sci. et Ind. nr 867, Hermann, Paris. [*I.17*]

Cartwright, M.L.

[1930] On the relation between the different types of Abel summation. Proc. London Math. Soc. (2) **31**, 81–96. [*I.25*]

Cauchy, A.L.

[1821] Analyse algébrique. Paris. [*I.2*]

Cesàro, E.
[1890] Sur la multiplication des séries. Bull. Sci. Math. (2) **14**, 114–120. [*I.2, 3*]

Chandrasekharan, K., see also Bochner
[1968] Introduction to analytic number theory. Grundl. math. Wiss. vol. 148, Springer, Berlin. [*III.4*]

Chandrasekharan, K. and Minakshisundaram, S.
[1952] Typical means. Oxford Univ. Press. [*I.17, 18; VI.13*]

Chatterji, S.D.
[1984] Tauber's theorem – a few historical remarks. Jahrb. Überblicke Math. 1984, Math. Surv. **17**, 167–175. [*I.5*]

Chevalier, L., see Brossard

Chill, R.
[1998] Tauberian theorems for vector-valued Fourier and Laplace transforms. Studia Math. **128**, 55–69. [*III.13*]

Chung, Kai Lai and Pollard, H.
[1952] An extension of renewal theory. Proc. Amer. Math. Soc. **3**, 303–309. [*V.22*]

Chung, Kai Lai and Wolfowitz, J.
[1952] On a limit theorem in renewal theory. Ann. of Math. (2) **55**, 1–6. [*V.22*]

Čížek, J.
[1999] On the Tauberian constant in the Ikehara theorem. Czechoslovak Math. J. **49(124)**, 673–682. [*III.4*]

Connor, J.
[1993] Gap Tauberian theorems. Bull. Austral. Math. Soc. **47**, 385–393. [*V.21*]

van der Corput, J.G.
[1948] Démonstration élémentaire du théorème sur la distribution des nombres premiers. Scriptum **1**, Mathematisch Centrum, Amsterdam. [*VII.21*]

Cross, R.W.
[1992] On a theorem of Kalton and Wilansky concerning Tauberian operators. J. Math. Anal. Appl. **171**, 156–170. [*V.18*]

Dales, H.G.
[2000] Banach algebras and automatic continuity. Clarendon Press, Oxford. [*V.5*]

Dales, H.G. and Hayman, W.K.
[1981] Esterle's proof of the Tauberian theorem for Beurling algebras. Ann. Inst. Fourier (Grenoble) **31**, 141–150. [*V.7, 8*]

Delange, H.
[1949] The converse of Abel's theorem on power series. Ann. of Math. (2) **50**, 94–109. [*I.7; II.6*]
[1950] Sur les théorèmes inverses des procédés de sommation des séries divergentes, I, II. Ann. Sci. École Norm. Sup. (3) **67**, 99–160, 199–242. [*I.20; II.3; IV.9*]
[1952] Encore une nouvelle démonstration du théorème de Littlewood. Bull. Sci. Math. (2) **76**, 179–189. [*I.7; III.1*]

[1954] Généralisation du théorème de Ikehara. Ann. Sci. École Norm. Sup. (3) **71**, 213–242. [*III.4*]

[1955a] Sur un théorème de Karamata. Bull. Sci. Math. **79**, 9–12. [*IV.2*]

[1955b] Théorèmes taubériens et applications arithmétiques. Mém. Soc. Roy. Liège (4) **16**, 1–87. [*III.4*]

[1997] Un théorème taubérien donnant des résultats équivalents au théorème des nombres premiers. Gaz. Math. No. 71, 39–58. [*III.6*]

DeVore, R., see Bojanic

Diaconis, P.

[2002] G. H. Hardy and probability??? Bull. London Math. Soc. **34**, 385–402. [*VI.6*]

Diaconis, P. and Stein, C.

[1978] Some Tauberian theorems related to coin tossing. Ann. Probability **6**, 483–490. [*VI.1, 13*]

Diamond, H.G., see also Bateman

[1972] On a Tauberian theorem of Wiener and Pitt. Proc. Amer. Math. Soc. **31**, 152–158. [*III.9*]

[1975] Changes of sign of $\pi(x) - \operatorname{li} x$. Enseignement Math. (2) **21**, 1–14. [*III.4*]

[1982] Elementary methods in the study of the distribution of prime numbers. Bull. Amer. Math. Soc. (N.S.) **7**, 553–589. [*III.1*]

[2001] A proof of the H-L-K Tauberian theorem by extension into the complex plane. Preprint. [*IV.21*]

Diamond, H.G. and Essén, M.

[1978] One-sided Tauberian theorems for kernels which change sign. Proc. London Math. Soc. (3) **36**, 463–489. [*II.5*]

Doetsch, G.

[1920] Ein Konvergenzkriterium für Integrale. Math. Ann. **82**, 68–82. [*I.15*]

[1930] Sätze von Tauberschem Charakter im Gebiet der Laplace- und Stieltjes-Transformation. Sitz.-Ber. Preuss. Akad. Wiss., Phys.-Math. Klasse 1930, 144–157. [*I.15, 21*]

[1931] Über den Zusammenhang zwischen Abelscher and Borelscher Summabilität. Math. Ann. **104**, 403–414. [*I.25; VI.22*]

[1937] Theorie und Anwendung der Laplace-Transformation. Grundl. math. Wiss. vol. 47, Springer, Berlin. [*I.15; III.4*]

[1950] Handbuch der Laplace-Transformation, vol. I. Birkhäuser, Basel. [*I.15; III.4*]

Domar, Y.

[1981] Translation invariant subspaces of weighted l^p and L^p spaces. Math. Scand. **49**, 133–144. [*V.7*]

[1983] A solution of the translation-invariant subspace problem for weighted L^p on $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{Z}$. Radical Banach algebras and automatic continuity. Lect. Notes in Math. nr. 975, Springer, Berlin, pp 214–226. [*V.9*]

Drasin, D.

[1968] Tauberian theorems and slowly varying functions. Trans. Amer. Math. Soc. **133**, 333–356. [*IV.1, 13*]

Drasin, D. and Seneta, E.
[1986] A generalization of slowly varying functions. Proc. Amer. Math. Soc. **96**, 470–472. [*IV.2*]

Drasin, D. and Shea, D.F.
[1976] Convolution inequalities, regular variation and exceptional sets. J. Analyse Math. **29**, 232–293. [*IV.13*]

Drozhzhinov, Yu.N., see Vladimirov

Dunford, N. and Schwartz, J.T.
[1963/88] Linear operators. Part II: Spectral theory. Self-adjoint operators in Hilbert space. Interscience, New York. [*V.5, 8*]

Duren, P.L.
[1970] Theory of H^p spaces. Academic Press, New York. [*III.13*]
[1983] Univalent functions. Grundl. math. Wiss. vol. 259, Springer, Berlin. [*V.1, 24*]

Edrei, A.
[1964] The deficiencies of meromorphic functions of finite lower order. Duke Math. J. **31**, 1–21. [*IV.14*]
[1965] Sums of deficiencies of meromorphic functions. J. Analyse Math. **14**, 79–107. [*IV.14*]

Edrei, A. and Fuchs, W.H.J.
[1966] Tauberian theorems for a class of meromorphic functions with negative zeros and positive poles. In: Contemporary Problems in the Theory of Analytic Functions (Internat. Conf., Erevan, 1965), pp 339–358. Izdat. Nauka, Moscow. [*IV.1, 13*]

Edwards, H.M.
[1974] Riemann's zeta function. Academic Press, New York. [*I.26*]

Edwards, R.E.
[1958] Comments on Wiener's Tauberian theorems. J. London Math. Soc. **33**, 462–466. [*II.13*]
[1967] Fourier series, vol. 2. Holt, Rinehart and Winston, New York. [*II.8*]

Elliott, P.D.T.A.
[1979] Probabilistic number theory, vol. 1. Grundl. math. Wiss. vol. 239, Springer, Berlin. [*III.4; IV.3, 4, 24*]

Embrechts, P.
[1978] A second-order theorem for Laplace transforms. J. London Math. Soc. (2) **17**, 102–106. [*IV.13*]

Erdélyi, T., see Borwein, P.

Erdős, P., see also Ash
[1942] On an elementary proof of some asymptotic formulas in the theory of partitions. Ann. of Math. (2) **43**, 437–450. [*IV.23*]
[1949a] On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. Proc. Nat. Acad. Sci. U.S.A. **35**, 374–384. [*I.26; III.1; VII.21*]
[1949b] On a Tauberian theorem connected with the new proof of the prime number theorem and Supplementary Note. J. Indian Math. Soc. (N.S.) **13**, 131–147. [*VII.21, 25–27*]

[1952] On a Tauberian theorem for Euler summability. Publ. Inst. Math. Belgrade (N.S.) **4**, 51–56. [*VI.15*]

[1956] On a high-indices theorem in Borel summability. Acta Math. Acad. Sci. Hungar. **7**, 265–281. [*VI.15*]

[1997] Some of my favorite problems and results. In: The Mathematics of Paul Erdős, vol. I (R.L. Graham and J. Nešetřil, Eds.), Algorithms and Combinatorics **13**, Springer, Berlin, pp 47–67. [*VII.21*]

Erdős, P., Feller, W. and Pollard, H.

[1949] A property of power series with positive coefficients. Bull. Amer. Math. Soc. **55**, 201–204. [*V.1, 22, 23*]

Erdős, P. and Fuchs, W.H.J.

[1956] On a problem of additive number theory. J. London Math. Soc. **31**, 67–73. [*VII.24*]

Esseen, C.G.

[1945] Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law. Acta Math. **77**, 1–125. [*IV.24*]

Essén, M., see Diamond

Esterle, J.

[1980] A complex-variable proof of the Wiener Tauberian theorem. Ann. Inst. Fourier (Grenoble) **30**, 91–96. [*V.3, 8*]

[1983] Quasimultipliers, representations of H^{∞}, and the closed ideal problem for commutative Banach algebras. In: Radical Banach algebras and automatic continuity, Lecture Notes in Math. nr. 975, Springer, Berlin, pp 66–162. [*III.15*]

Esterle, J., Strouse, E. and Zouakia, F.

[1990] Theorems of Katznelson–Tzafriri type for contractions. J. Funct. Anal. **94**, 273–287. [*III.15*]

Eymard, P.

[1964] L'algèbre de Fourier d'un groupe localement compact. Bull. Soc. Math. France **92**, 181–236. [*V.3*]

Faĭnleĭb, A.S.

[1968] A generalization of Esseen's inequality and its application in probabilistic number theory (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **32**, 859–879. [*IV.24*]

Fast, H.

[1951] Sur la convergence statistique. Colloquium Math. **2**, 241–244. [*I.24*]

Fatou, P.

[1905] Sur quelques théorèmes de Riemann. C. R. Acad. Sci. Paris **140**, 569–570. [*III.1*]

[1906] Séries trigonométriques et séries de Taylor. Acta Math. **30**, 335–400. [*II.19; III.1, 12*]

Faulhaber, G.

[1956] Äquivalenzsätze für die Kreisverfahren der Limitierungstheorie. Math. Z. **66**, 34–52. [*VI.22*]

Faulstich, K. (= Stadtmüller, K.) and Luh, W.

[1982] Summability of power series on prescribed sets by regular Riesz methods. Analysis **2**, 253–265. [*VI.4*]

Faulstich, K., Luh, W. and Tomm, L.

[1981/82] Universelle Approximation durch Riesz-Transformierte der geometrischen Reihe. Manuscripta Math. **36**, 309–321. [*VI.4*]

[1983] On equiconvergent matrix transforms of power series. In: Constructive function theory (Varna 1981), pp 317–320. Bulgar. Acad. Sci., Sofia. [*VI.4*]

Feichtinger, H.G. and Schmeisser, H.-J.

[1986] Weighted versions of Beurling's Tauberian theorem. Math. Ann. **275**, 353–363. [*VI.11*]

Feigin, P.D. and Yashchin, E.

[1983] On a strong Tauberian result. Z. Wahrsch. Verw. Gebiete **65**, 35–48. [*IV.1, 24*]

Fejér, L.

[1904] Untersuchungen über Fouriersche Reihen. Math. Ann. **58**, 51–69. [*I.3*]

[1914] Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene. Math. Abhandlungen H.A. Schwarz gewidmet, Berlin, pp 42–53. [*I.5*]

Fekete, M.

[1958] New methods of summability. J. London Math. Soc. **33**, 466–470. [*VI.21*]

Feller, W., see also Erdős

[1963] On the classical Tauberian theorems. Arch. Math. **14**, 317–322. [*I.7; IV.1, 10*]

[1950/68] An introduction to probability theory and its applications, vol. 1. Wiley, New York. [*V.1, 22, 23; VI.6*]

[1966/71] An introduction to probability theory and its applications, vol. 2. Wiley, New York. [*I.7; IV.1, 2, 7, 8; V.22*]

Flajolet, P.

[2002] Singular combinatorics. Proceedings of the International Congress of Mathematicians (Beijing 2002), vol. 3, pp 561–572. [*IV.21; VII.25*]

Foiaş, C., see Sz.-Nagy

Frennemo, L.

[1965] On general Tauberian remainder theorems. Math. Scand. **17**, 77–88. [*VII.13, 15*]

[1966–67] Tauberian problems for the n-dimensional Laplace transform. I, Math. Scand. **19**, 41–53; II, ibid. **20**, 225–239. [*VII.2, 19*]

[1967] Studies on the remainder in Tauberian problems. Göteborg Diss. in Science **5**, 12 p. [*VII.2*]

[1999] A generalization of Littlewood's theorem for the Laplace transform. Preprint, Dept. of Mathematics, Chalmers University of Technology, Gothenburg. [*IV.10; V.9*]

[2002] General n-dimensional Tauberian problems with application to the Laplace and Stieltjes transforms. Math. Scand. **91**, 269–284. [*IV.10; V.9*]

Freud, G.

[1951] Restglied eines Tauberschen Satzes I. Acta Math. Acad. Sci. Hung. **2** (1951), 299–308. [*III.18; VII.2*]

[1952/53] Restglied eines Tauberschen Satzes II. Acta Math. Acad. Sci. Hung. **3**, 299–307. [*III.18; VII.2*]

[1954] Restglied eines Tauberschen Satzes III. Acta Math. Acad. Sci. Hung. **5**, 275–288. [*III.18; VII.2, 3, 11*]

[1955] Über einseitige Approximation durch Polynome I. Acta Sci. Math. (Szeged) **16**, 12–28. [*VII.3, 5, 7*]

Freud, G. and Ganelius, T.
[1957] Some remarks on one-sided approximation. Math. Scand. **5**, 276–284. [*VII.7*]

Fridy, J.A.
[1985] On statistical convergence. Analysis **5**, 301–313. [*I.24*]

Fridy, J.A. and Roberts, K.L.
[1980] Some Tauberian theorems for Euler and Borel summability. Internat. J. Math. Sci. **3**, 731–738. [*VI.22*]

Frobenius, G.
[1880] Über die Leibnitzsche Reihe. J. Reine Angew. Math. **89**, 262–264. [*I.2, III.1*]

Fuchs, W.H.J., see Edrei, Erdős

Gaier, D., see also Landau
[1952] Über die Summierbarkeit beschränkter und stetiger Potenzreihen an der Konvergenzgrenze. Math. Z. **56**, 326–334. [*VI.22*]
[1953a] Complex Tauberian theorems for power series. Trans. Amer. Math. Soc. **75**, 48–68. [*III.12*]
[1953b] Zur Frage der Indexverschiebung beim Borel-Verfahren. Math. Z. **58**, 453–455. [*VI.2*]
[1965] Der allgemeine Lückenumkehrsatz für das Borelverfahren. Math. Z. **88**, 410–417. [*I.9; VI.1, 15*]
[1966] On the coefficients and growth of gap power series. J. SIAM Numer. Anal. **3**, 248–265. [*VI.1, 15, 17, 19; VII.12*]
[1967] Complex variable proofs of Tauberian theorems. Matscience Report, No. 56, Inst. of Math. Sciences, Madras. [*VI.15; VII.12*]

Ganelius, T.H., see also Freud
[1954/55] On the remainder in a Tauberian theorem. Kungl. Fysiog. Sällsk. i Lund Förh. **24** (1954), no. 20, 6pp. (1955). [*VII.2*]
[1956a] Un théorème taubérien pour la transformation de Laplace. C. R. Acad. Sci. Paris **242**, 719–721. [*VII.2*]
[1956b] On one-sided approximation by trigonometrical polynomials. Math. Scand. **4**, 247–258. [*VII.5, 7*]
[1957] Some applications of a lemma on Fourier series. Acad. Serbe Sci. Publ. Inst. Math. **11**, 9–18. [*VII.15*]
[1958] General and special Tauberian remainder theorems. Treizième congrès des mathématiciens scandinaves, Helsinki 1957, pp 102–103. [*VII.2, 13*]
[1962] The remainder in Wiener's tauberian theorem. Mathematica Gothoburgensia, **1**. Acta Universitatis Gothoburgensis, Göteborg, 13 pp. [*VII.2, 13–15, 20*]
[1964] Tauberian theorems for the Stieltjes transform. Math. Scand. **14**, 213–219. [*VII.2, 13, 17, 18*]
[1970] Regularly varying functions and Poincaré's theorem on difference equations. Symposia on theoretical physics and mathematics, vol. 10, pp 7–17. Plenum, New York. [*IV.13*]
[1971] Tauberian remainder theorems. Lecture Notes in Math. nr. 232, Springer, Berlin. [*II.8; III.1, 16, 19; V.23; VII.1–3, 13–15, 17, 20*]

[1972] Some approximation theorems related to Wiener's. Proc. Conf. Constructive Theory of Functions (Budapest, 1969), Akad. Kiadó, Budapest. pp 173–181. [*II.8*]

[1986] Géza Freud's work on Tauberian remainder theorems. J. Approx. **46**, 42–50. [*VII.2*]

Garling, D.J.H. and Wilansky, A.

[1972] On a summability theorem of Berg, Crawford and Whitley. Proc. Cambridge Philos. Soc. **71**, 495–497. [*V.18*]

Garnett, J.B.

[1981] Bounded analytic functions. Academic Press, New York. [*VII.12*]

Garsia, A.M.

[1963] A new proof of Beurling's theorem on the spectrum of bounded functions on the real axis. J. Math. Anal. Appl. **7**, 436–439. [*II.8*]

Gehring, F.W.

[1957] The Fatou theorem and its converse. Trans. Amer. Math. Soc. **85**, 106–121. [*II.19*]

[1960] Harmonic functions and Tauberian theorems. Proc. London Math. Soc. (3) **10**, 88–106. [*II.19*]

Geĭsberg, S.P. and Konjuhovskiĭ, V.S.

[1971] Description of certain systems of primary ideals in rings of functions that are integrable with weight (Russian). Leningrad Gos. Ped. Inst. Učen. Zap. **404**, 349–361. [*V.9*]

Gel'fand, I.M. (= Gelfand, I.)

[1939] On normed rings. Dokl. Akad. Nauk SSSR **23**, 430–432. Parts II and III, same journal **25**, 570–572 and 573–574. [*II.8; V.1*]

[1941a] Normierte Ringe. Mat. Sbornik, N.S. **9** (51), 3–24. [*II.8; V.1*]

[1941b] Zur Theorie der Charaktere der abelschen topologischen Gruppen. Mat. Sbornik, N.S. **9** (51), 49–50. [*III.15*]

[1941c] Über absolut konvergente trigonometrische Reihen und Integrale. Mat. Sbornik **9**, 51–66. [*V.5*]

[1995–96] Mathematical publications of I. M. Gelfand vols 1, 2. Birkhäuser, Boston.

Gelfand, I., Raikov, D. and Shilov, G.

[1964/01] Commutative normed rings (translated from the Russian). Chelsea, New York, 1964; Amer. Math. Society, Providence, RI, 1999, 2001. [*II.8; V.5, 8*]

Geluk, J.L., see also Balkema

[1979] An Abel–Tauber theorem on convolutions with the Möbius function. Proc. Amer. Math. Soc. **77**, 201–209. [*II.18*]

[1981a] Π-regular variation. Proc. Amer. Math. Soc. **82**, 565–570. [*IV.12*]

[1981b] An Abel–Tauber theorem for partitions. Proc. Amer. Math. Soc. **82**, 571–575. [*IV.23*]

[1985] Abelian and Tauberian theorems for O-regularly varying functions. Proc. Amer. Math. Soc. **93**, 235–241. [*IV.8*]

[1989] An Abel–Tauber theorem for partitions, II. J. Number Theory **33**, 170–181. [*IV.23*]

Geluk, J.L. and de Haan, L.

[1981] On functions with small differences. Nederl. Akad. Wetensch. Indag. Math. **43**, 187–194. [*IV.6, 12*]

[1987] Regular variation: Extensions and Tauberian theorems. Math. Centre Tract **40**, CWI, Amsterdam. [*IV.6, 12, 20*]

Geluk, J.L., de Haan, L. and Stadtmüller, U.

[1986] A Tauberian theorem of exponential type. Canad. J. Math. **38**, 697–718. [*IV.1, 15, 16, 20*]

Godement, R.

[1947] Théorèmes taubériens et théorie spectrale. Ann. Sci. École Norm. Sup. (3) **64**, 119–138. [*V.3*]

Goldberg, R.R.

[1961] Fourier transforms. Cambridge Tracts in Math. nr. 52, Cambridge Univ. Press. [*II.8; V.3*]

Goldie, C.M., see Bingham

González, M., see Alvarez

Gordon, B.

[1958] On a Tauberian theorem of Landau. Proc. Amer. Math. Soc. **9**, 693–696. [*II.17*]

Gorin, E.A.

[1978] G.E. Šilov's investigations in the theory of commutative Banach algebras and their subsequent development (Russian). Uspekhi Mat. Nauk **33**, no. 4 (202), 169–188, 256. [English translation: Russian Math. Surveys 33 (1978), no. 4, 193–218.] [*V.7*]

Gorny, A.

[1939] Contribution à l'étude des fonctions dérivables d'une fonction réelle. Acta Math. **71**, 317–358. [*I.17*]

Graham, S.W. and Vaaler, J.D.

[1981] A class of extremal functions for the Fourier transform. Trans. Amer. Math. Soc. **265**, 283–302. [*III.1, 4, 5*]

Grimmett, G. and Stirzaker, D.

[1992] Probability and random processes. Oxford Univ. Press. (First edition 1982.) [*V.23*]

Grosse-Erdmann, K.-G.

[1993a] On the Borel–Okada theorem and the Hadamard multiplication theorem. Complex Variables Theor. Appl. **22**, 101–112. [*VI.4*]

[1993b] The Borel–Okada theorem revisited. Habilitationsschrift, FernUniversität Hagen. [*VI.4*]

Gurariĭ, V.P.

[1976] Harmonic analysis in spaces with a weight (Russian). Trudy Moskov. Mat. Obšč. **35**, 21–76. English transl. in Trans. Moscow Math. Soc. **35** (1979), 21–75. [*V.9*]

[1998] Group methods in commutative harmonic analysis. In: Commutative harmonic analysis, II. Encyclopaedia Math. Sci. vol. 25 (V. P. Havin and N. K. Nikolski, eds), Springer, Berlin, pp 1–325. [*V.9*]

de Haan, L., see also Balkema, Geluk

[1970] On regular variation and its application to the weak convergence of sample extremes. Math. Centre Tract **32**, CWI, Amsterdam. [*IV.2, 5, 6*]

[1976] An Abel–Tauber theorem for Laplace transforms. J. London Math. Soc. (2) **13**, 537–542. [*IV.1, 6, 12*]

de Haan, L. and Stadtmüller, U.

[1985] Dominated variation and related concepts and Tauberian theorems for Laplace transforms. J. Math. Anal. Appl. **108**, 344–365. [*IV.8, 10*]

[1996] Generalized regular variation of second order. J. Austral. Math. Soc. (Series A) **61**, 381–395. [*IV.12*]

Hadamard, J.

[1892] Essai sur l'étude des fonctions données par leur développement de Taylor. J. de Math. (4) **8**, 101–186. [*VI.7*]

[1896] Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. Bull. Soc. Math. France **24**, 199–220. [*I.10, 26; III.1*]

Halász, G.

[1967a] Remarks to a paper of D. Gaier on gap theorems. Acta Sci. Math. (Szeged) **28**, 311–322. [*I.23; III.1; VII.11*]

[1967b] On the sequence of generalized partial sums of a series. Studia Sci. Math. Hungar. **2**, 435–439. [*I.23*]

[1968] Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen. Acta Math. Acad. Sci. Hungar. **19**, 365–403. [*III.4*]

[1969] Tauberian theorems for univalent functions. Studia Sci. Math. Hungar. **4**, 421–440. [*III.12*]

[1971] On the distribution of additive and the mean values of multiplicative arithmetic functions. Studia Sci. Math. Hungar. **6**, 211–233. [*VI.14*]

Halberstam, H. and Roth, K.F.

[1983] Sequences (second edition). Springer, New York. [*VII.24*]

Hardy, G.H., see also Bochner

[1904a] On differentiation and integration of divergent series. Cambridge Phil. Soc. Trans. **19**, 297–321. [*VI.1, 2*]

[1904b] Researches in the theory of divergent series and divergent integrals. Quart. J. Math. **35**, 22–66. [*VI.1, 2, 18*]

[1910] Theorems relating to the summability and convergence of slowly oscillating series. Proc. London Math. Soc. (2) **8**, 301–320. [*I.1, 6, 18, 25*]

[1911a] Properties of logarithmico-exponential functions. Proc. London Math. Soc. (2) **10**, 54–90. [*IV.1*]

[1911b] Notes on some points in the integral calculus (31): The uniform convergence of Borel's integral. Messenger of Math. **40**, 161–165. [*VI.1, 2*]

[1914a] Note on Lambert's series. Proc. London Math. Soc. (2) **13**, 192–198. [*I.2*]

[1914b] Notes on some points in the integral calculus (37): On the region of convergence of Borel's integral. Messenger of Math. **43**, 22–24. [*VI.3*]

[1949] Divergent Series. Clarendon Press, Oxford. (Corrected edition 1963, also published 1992 in Éditions Jacques Gabay, Sceaux; annotated edition, Oxford 2004?) [*Pref; I.1–3, 10, 15, 18, 19, 21, 22, 25; II.5, 6, 8, 18; III.1; IV.13; V.10; VI.1–4, 6, 9, 18, 21, 22, 24, 25; VII.3*]

[1974] Collected papers of G.H. Hardy (including joint papers with J. E. Littlewood and others). Vol. 6: Series. Clarendon Press, Oxford. [*I.1, 7*]

Hardy, G.H. and Littlewood, J.E.

[1913a] The relations between Borel's and Cesàro's methods of summation. Proc. London Math. Soc. (2) **11**, 1–16. [*VI.1, 8, 22*]

[1913b] Contributions to the arithmetic theory of series. Proc. London Math. Soc. (2) **11**, 411–478. [*I.17*]

[1913c] Tauberian theorems concerning series of positive terms. Messenger of Math. **42**, 191–192. [*I.1*]

[1914a] Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive. Proc. London Math. Soc. (2) **13**, 174–191. [*I.1, 7, 15, 17; III.1, 18; IV.1, 8; VI.1*]

[1914b] Some theorems concerning Dirichlet's series. Messenger of Math. **43**, 134–147. [*I.8, 22*]

[1916] Theorems concerning the summability of series by Borel's exponential method. Rend. Palermo **41**, 36–53. [*I.1, 9; VI.1, 8, 9, 11, 21, 22*]

[1918] Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Math. **41**, 119–196. [*III.1*]

[1921] On a Tauberian theorem for Lambert's series, and some fundamental theorems in the analytic theory of numbers. Proc. London Math. Soc. (2) **19**, 21–29. [*I.1, 10; II.1, 12*]

[1926] A further note on the converse of Abel's theorem. Proc. London Math. Soc. (2) **25**, 219–236. [*I.1, 8, 23*]

[1929] Notes on the theory of series (11): On Tauberian theorems. Proc. London Math. Soc. (2) **30**, 23–37. [*I.15, 21; III.19; IV.8, 9*]

[1931] Notes on the theory of series (16): Two Tauberian theorems. J. London Math. Soc. **6**, 281–286. [*I.7*]

[1936] Notes on the theory of series (20): On Lambert series. Proc. London Math. Soc. (2) **41**, 257–270. [*I.10*]

[1943] Notes on the theory of series (22): On the Tauberian theorem for Borel summability. J. London Math. Soc. **18**, 194–200. [*I.9; VI.1, 9, 11, 13, 21*]

Hardy, G.H. and Ramanujan, S.

[1917] Asymptotic formulae for the distribution of integers of various types. Proc. London Math. Soc. (2) **16**, 112–132. [*IV.1, 15, 23*]

[1918] Asymptotic formulae in combinatory analysis. Proc. London Math. Soc. (2) **17**, 75–115. [*IV.23*]

Hardy, G.H. and Riesz, M.

[1915] The general theory of Dirichlet's series. Cambridge Tracts no. 18, Cambridge Univ. Press. (Second ed. 1952, Stechert-Hafner reprint, New York, 1964.) [*I.18; VI.13*]

Hardy, G.H. and Wright, E.M.

[1979] An introduction to the theory of numbers, fifth edition. Clarendon Press, Oxford. (First edition 1938.) [*I.4; VII.10*]

Hayman, W.K., see also Dales

[1953] La régularité des fonctions univalentes. C. R. Acad. Sci. Paris **237**, 1624–1625. [*V.24*]

[1955] The asymptotic behaviour of p-valent functions. Proc. London Math. Soc. (3) **5**, 257–284. [*V.24*]

[1956] A generalisation of Stirling's formula. J. Reine Angew. Math. **196**, 67–95. [*IV.21; VII.25*]

[1970] Tauberian theorems for multivalent functions. Acta Math. **125**, 269–298. [*III.12*]

Hayman, W.K. and Korenblum, B.
[1976] An extension of the Riesz–Herglotz formula. Ann. Acad. Sci. Fenn. Ser. A1 Math. **2**, 175–201. [*V.8*]

Hedenmalm, H., see also Borichev
[1985] On the primary ideal structure at infinity for analytic Beurling algebras. Ark. Mat. **23**, 129–158. [*V.9*]

Heilbronn, H. and Landau, E.
[1933a] Bemerkung zur vorstehenden Arbeit von Herrn Bochner. Math. Z. **37**, 10–16. [*III.4, 6*]
[1933b] Ein Satz über Potenzreihen. Math. Z. **37**, 17. [*III.17*]
[1933c] Anwendungen der N. Wienerschen Methode. Math. Z. **37**, 18–21. [*III.4, 6*]

Heins, M., see also Ahlfors
[1962] Selected topics in the classical theory of functions of a complex variable. Holt, Rinehart and Winston, New York. [*VII.9*]
[1968] Complex function theory. Academic Press, New York. [*III.4*]

Hellerstein, S., Shea, D.F. and Williamson, J.
[1970] A tauberian theorem characterizing the regularity of growth of a class of entire functions. Duke Math. J. **37**, 489–499. [*IV.1*]

Hernando, B.
[1998] Some properties of the second conjugate of a Tauberian operator. J. Math. Anal. Appl. **228**, 60–65. [*V.18*]

Hewitt, E. and Ross, K.A.
[1970] Abstract harmonic analysis, vol. 2. Grundl. math. Wiss. vol. 152, Springer, Berlin. [*II.8, 13; V.3*]

Hieber, M., see Arendt

Hildebrand, A. and Tenenbaum, G.
[1994] On some Tauberian theorems related to the prime number theorem. Compositio Math. **90**, 315–349. [*VII.21*]

Hildebrand, F.B.
[1974] Introduction to numerical analysis (second edition). McGraw–Hill, New York. (Reprinted by Dover Publ., New York, 1987.) [*VI.20*]

Hill, J.D.
[1937] On perfect methods of summability. Duke Math. J. **3**, 702–714. [*V.16*]

Hille, E.
[1962] Analytic function theory, vol. 2. Ginn and Co., New York. (Reprinted by Chelsea, New York, 1987.) [*VI.4*]

Hille, E. and Phillips, R.
[1957/74] Functional analysis and semi-groups, revised edition. Amer. Math. Soc. Colloq. Publ., vol. 31, Providence, RI. [*II.8; V.4, 5*]

Hirschman, I.I. and Widder, D.V.

[1955] The convolution transform. Princeton Univ. Press. [*VII.9, 18*]

Holland, F.

[1988] Extensions of Milin's Tauberian theorem. J. London Math. Soc. (2) **37**, 128–140. [*V.24*]

Holt, J.J. and Vaaler, J.D.

[1996] The Beurling–Selberg extremal functions for a ball in Euclidean space. Duke Math. J. **83**, 203–248. [*III.5*]

Holub, J.R.

[1993] Characterizations of Tauberian and related operators on Banach spaces. J. Math. Anal. Appl. **178**, 280–288. [*V.18*]

Hörmander, L.

[1983–85] The analysis of linear partial differential operators, vols 1–4. Grundl. math. Wiss. vols 256, 257, 274, 275, Springer, Berlin. [*II.11; III.19*]

Hyslop, J.M.

[1936a] The generalisation of a theorem on Borel summability. Proc. London Math. Soc. **41**, 243–256. [*VI.22*]

[1936b] On the summability of series by a method of Valiron. Proc. Edinburgh Math. Soc. (2) **4**, 218–223. [*VI.22*]

Ibragimov, I.A. and Linnik, Yu.V.

[1971] Independent and stationary sequences of random variables. Wolters- Noordhoff, Groningen. [*VI.6*]

Ikehara, S.

[1931] An extension of Landau's theorem in the analytic theory of numbers. J. Math. and Phys. M.I.T. **10**, 1–12. [*II.1; III.1, 2, 4*]

Ingham, A.E., see also Burkill [1969]

[1932] The distribution of prime numbers. Cambridge Tracts in Math. nr. 30, Cambridge Univ. Press. (Reprinted 1990.) [*III.1, 2*]

[1935] On Wiener's method in Tauberian theorems. Proc. London Math. Soc. (2) **38**, 458–480. [*III.1, 4, 6, 7, 9, 10, 12, 13*]

[1936] Some trigonometrical inequalities with applications to the theory of series. Math. Z. **41**, 367–379. [*III.1, 10, 11, 12*]

[1937] On the 'high-indices theorem' of Hardy and Littlewood. Quart. J. Math. Oxford (2) **8**, 1–8. [*I.1, 23*]

[1941] A Tauberian theorem for partitions. Ann. of Math. (2) **42**, 1075–1090. [*III.1, 4; IV.1, 15, 21, 23; VII.25*]

[1945] Some Tauberian theorems connected with the prime number theorem. J. London Math. Soc. **20**, 171–180. [*II.17, 18*]

[1950] A further note on trigonometrical inequalities. Proc. Cambridge Phil. Soc. **46**, 535–537. [*III.12*]

[1965] On Tauberian theorems. Proc. London Math. Soc. (3) **14A**, 157–173. [*I.1, 7; VII.3, 11*]

[1968a] On the high-indices theorem for Borel summability. In: Abhandlungen aus Zahlentheorie und Analysis, zur Erinnerung an Edmund Landau (1877–1938) (P. Turan, ed.), V.E.B. Deutscher Verlag der Wiss., Berlin, pp 119–135. (Also published as Number theory and analysis, a collection of papers in honor of Edmund Landau, Plenum Press, New York, 1969.) [*VI.15*]

[1968b] The equivalence theorem for Cesàro and Riesz summability. Publ. Ramanujan Inst. No. **1**, 107–113. [*I.18*]

Inoue, A., see Bingham

Ishiguro, K(azuo)

[1964] A Tauberian theorem for (J, p_n) summability. Proc. Japan Acad. **40**, 807–812. [*VI.24*]

[1965] Two Tauberian theorems for (J, p_n) summability. Proc. Japan Acad. **41**, 40–45. [*VI.24*]

Ishiguro, K., Meyer-König, W. and Zeller, K.

[1990] Die Turánsche Matrix als Produkt einer Euler–Knopp- und einer Taylor-matrix. Kodai Math. J. **13**, 47–51. [*VI.22*]

Ishiguro, K. and Tietz, H.

[1992] Tauber-Sätze für $J_{p\lambda}$-Verfahren. Math. Japan. **37**, 1201–1204. [*VI.24*]

Ivić, A.

[1985] The Riemann zeta function. Wiley, New York. [*I.26*]

Izumi, S.

[1954] A simple proof of Littlewood's Tauberian theorem. Proc. Japan Acad. **30**, 927–929. [*I.7*]

Jackson, D.

[1930] The theory of approximation. Amer. Math. Soc. Colloq. Publ. vol. 11, Amer. Math. Soc., New York. [*III.6; VII.2*]

Jakimovski, A. (= Amir, A.), see also Baron & Leviatan [1991]

[1954a] On a Tauberian theorem by O. Szász. Proc. Amer. Math. Soc. **5**, 67–70. [*I.7*]

[1954b] Some relations between the methods of summability of Abel, Borel, Cesàro, Hölder and Hausdorff. J. Analyse Math. **3**, 346–381. [*VI.22*]

[1964] Analytic continuation and summability of power series. Michigan Math. J. **11**, 353–356. [*VI.4*]

Jakimovski, A., Meyer-König, W. and Zeller, K.

[1981] Power series methods of summability: positivity and gap perfectness. Trans. Amer. Math. Soc. **266**, 309–317. [*V.21; VI.24*]

[1987] Refined Tauberian gap theorems for power series methods of summability. Rocky Mountain J. Math. **17**, 355–369. [*V.21; VI.24*]

Jakimovski, A. and Tietz, H.

[1980] Regularly varying functions and power series methods. J. Math. Anal. Appl. **73**, 65–84. Errata, ibid. **95**, 597–598. [*VI.24*]

Johansson, B.I.

[1979] A general high indices theorem with an application to a conjecture by Rényi. Ark. Mat. **17**, 279–296. [*I.23*]

[1981] Some distributional Tauberian theorems. Analysis **1**, 197–208. [*I.23*]

[1993] An extremal property for the $L^1(\mathbb{R})$-functions with compactly supported Fourier transform. Preprint, Dept. of Mathematics, Chalmers University of Technology, Gothenburg. [*III.11*]

[1995] A correction to a distributional Tauberian theorem of Ganelius. Indag. Math. (N.S.) **6**, 279–286. [*II.8*]

[1996] A Tauberian vanishing theorem. Anal. Math. **22**, 25–34. [*VII.3*]

Jordan, G.S.

[1974] Regularly varying functions and convolutions with real kernels. Trans. Amer. Math. Soc. **194**, 177–194. [*IV.13*]

[1976] A convolution relation with remainder estimate. J. Reine Angew. Math. **283/284**, 287–293. [*IV.13; VII.18*]

Jukes, K.A.

[1974] Tauberian theorems of Landau–Ingham type. J. London Math. Soc. (2) **8**, 570–576. [*II.18*]

Jurkat, W.B., see also Bojanic

[1956a] Ein funktionentheoretischer Beweis für O-Taubersätze bei Potenzreihen. Arch. Math. **7**, 122-125. [*I.7; III.1*]

[1956b] Ein funktionentheoretischer Beweis für O-Taubersätze bei den Verfahren von Borel und Euler–Knopp. Arch. Math. **7**, 278–283. [*VI.11, 22*]

[1957] Über die Umkehrung des Abelschen Stetigkeitssatzes mit funktionentheoretischen Methoden. Math. Z. **67**, 211–222. [*I.7; III.1*]

Kac, M.

[1965] A remark on Wiener's Tauberian theorem. Proc. Amer. Math. Soc. **16**, 1155–1157. [*II.8*]

Kalton, N.J., see also Bennett

Kalton, N.J. and Wilansky, A.

[1976] Tauberian operators on Banach spaces. Proc. Amer. Math. Soc. **57**, 251–255. [*V.18*]

Kangro, G.

[1970] The weakening of Tauber's conditions (Russian). Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. **19**, 24–33. [*I.24*]

[1974] Theory of summability of sequences and series (Russian). Mathematical Analysis, vol. 12 (Russian), pp 5–70. Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow. (English translation: J. Soviet Math. **5** (1976), 1–45.) [*I.1*]

Kannai, P., see Agmon

Karamata, J., see also Avakumović, Bojanic, Marić [2002], Tomić [2001]

[1930a] Über die Hardy–Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes. Math. Z. **32**, 319–320. [*I.1, 7, 11; III.18; IV.1*]

[1930b] Sur un mode de croissance régulière des fonctions. Mathematica (Cluj) **4**, 38–53. [*I.15; III.17; IV.1, 2, 5*]

[1931] Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche and Stieltjessche Transformation betreffen. J. Reine Angew. Math. **164**, 27–39. [*I.15, 21; III.19; IV.1, 8, 9*]

[1932] Über einen Satz von Vijayaraghavan. Math. Z. **34**, 737–740. [*I.20*]

[1933a] Über die O-Inversionssätze der Limitierungsverfahren. Math. Z. **37**, 582–588. [*I.20*]

[1933b] Sur un mode de croissance régulière. Théorèmes fondamentaux. Bull. Soc. Math. France **61**, 55–62. [*III.17; IV.1, 2*]

[1934] Weiterführung der N. Wienerschen Methode. Math. Z. **38**, 701–708. [*III.7, 10*]

[1936] Über einen Satz von H. Heilbronn und E. Landau. Publ. Inst. Math. Beograd **5**, 28–38. [*III.6*]

[1937a] Allgemeine Umkehrsätze der Limitierungsverfahren. Hamburg. Abhandlungen **12**, 48–63. [*II.1, 3*]

[1937b] Sur les théorèmes inverses des procédés de sommabilité. Act. Sci. et Ind. 450, Hermann, Paris. [*I.1; II.1, 3*]

[1938] Über die B-Limitierbarkeit einer Potenzreihe am Rande. Math. Z. **44**, 156–160. [*I.25; VI.22*]

[1939a] Über einen Tauberschen Satz im Dreikörperproblem. Amer. J. Math. **61**, 769–770. [*I.17*]

[1939b] Über die Indexverschiebung beim Borelschen Limitierungsverfahren. Math. Z. **45**, 635–641. [*VI.2*]

[1952] Review of Freud [1951]. Zentralblatt für Math. **44**, 324. [*VII.10*]

[1958] Sur les procédés de sommation intervenant dans la théorie des nombres. Colloque sur la théorie des suites. Publ. Centre Belge de recherches mathématiques, pp 12–31. [*II.18*]

[1962] Some theorems concerning slowly varying functions. Math. Res. Center Tech. Report 369, Madison, WI. [*IV.2, 5*]

[2004?] Selected Papers – to appear. [*IV.1*]

Karlin, S.

[1955] On the renewal equation. Pacific J. Math. **5**, 229–257. [*V.22*]

Kasahara, Y.

[1978] Tauberian theorems of exponential type. J. Math. Kyoto Univ. **18**, 209–219. [*IV.1, 15, 16*]

Kasahara, Y. and Kosugi, N.

[2000] Large deviation around the origin for sums of nonnegative i.i.d. random variables. Natur. Sci. Rep. Ochanomizu Univ. **51**, 27–31. [*IV.15*]

Katznelson, Y.

[1968/76] An introduction to harmonic analysis. Dover, New York. [*II.8; III.14; V.5*]

Katznelson, Y. and Tzafriri, L.

[1986] On power bounded operators. J. Funct. Anal. **68**, 313–328. [*III.1, 13, 14, 15*]

Keldysh, M.V.

[1951] On a Tauberian theorem (Russian). Trudy Mat. Inst. Steklova **38**, 77–86. English transl. in Amer. Math. Soc. Transl. (2) **102** (1973), 133–143. [*IV.10*]

Kennedy, P.B. and Szüsz, P.

[1966] On a bounded increasing power series. Proc. Amer. Math. Soc. **17**, 580–581. [*I.24*]

Kiesel, R., see also Borwein, D.

[1993a] Power series methods and almost sure convergence. Math. Proc. Cambridge Philos. Soc. **113**, 195–204. [*VI.13*]

[1993b] General Nörlund transforms and power series methods. Math. Z. **214**, 273–286. [*VI.24*]

[1995] On scales of summability methods. Math. Nachr. **176**, 129–138. [*VI.24*]

Kiesel, R. and Stadtmüller, U.

[1991] Tauberian theorems for general power series methods. Math. Proc. Cambridge Philos. Soc. **110**, 483–490. [*VI.12, 24*]

[1994] Tauberian- and convexity theorems for certain (N, p, q)-means. Canad. J. Math. **46**, 982–994. [*VI.24*]

[2000] A large-deviation principle for weighted sums of independent identically distributed random variables. J. Math. Anal. Appl. **251**, 929–939. [*VI.13*]

Kloosterman, H.D.

[1940a] On the convergence of series summable (C, r) and on the magnitude of the derivatives of a function of a real variable. J. London Math. Soc. **15**, 91–96. [*I.6, 17*]

[1940b] Limitierungsumkehrsätze mit Lückenbedingungen für das C-Verfahren. Math. Z. **46**, 375–379. [*I.6*]

[1950] Derivatives and finite differences. Duke Math. J. **17**, 169–186. [*I.17*]

Klüppelberg, C., see Balkema

Knopp, K.

[1922–23] Über das Eulersche Summierungsverfahren. Math. Z. **15**, 226–25; part II, Math. Z. **18**, 125–156. [*VI.1, 20, 22*]

[1954] Tauberian theorem for the C_k and H_k methods. Proc. Amer. Math. Soc. **5**, 572–574. [*I.6*]

[1964] Theorie und Anwendung der unendlichen Reihen. Grundl. math. Wiss. vol. 2, Springer, Berlin. [*VI.20*]

Kogbetliantz, E.

[1931] Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques. Mémorial des Sci. Math. vol. 51. Gauthier–Villars, Paris. [*I.7*]

Kohlbecker, E.E.

[1958] Weak asymptotic properties of partitions. Trans. Amer. Math. Soc. **58**, 346–365. [*IV.1, 15, 16, 23*]

Kolmogorov, A.

[1936] Anfangsgründe der Theorie der Markoffschen Ketten mit unendlich vielen möglichen Zuständen. Mat. Sbornik (N.S.) **1**, 607–610. [*V.22*]

König, H.

[1960] Neuer Beweis eines klassischen Tauber-Satzes. Arch. Math. **11**, 352–365. [*I.7*]

Konjuhovskiĭ, V.S., see Geĭsberg

Koosis, P.

[1988–92] The logarithmic integral I, II. Cambridge Univ. Press. [*V.7*]

Korenblum, B. (= Korenblyum, B.I.), see also Hayman

[1953] A general Tauberian theorem for the ratio of functions (Russian). Dokl. Akad. Nauk SSSR (NS) **88**, 745–748. [*IV.10*]

[1955] On the asymptotic behavior of Laplace integrals near the boundary of a region of convergence (Russian). Dokl. Akad. Nauk SSSR (NS) **104**, 173–176. [*IV.10; VII.20*]

[1956] Generalization of Wiener's Tauberian theorem and the spectrum of rapidly increasing functions (Russian). Dokl. Akad. Nauk SSSR (NS) **111**, 280–282. [*V.9*]

[1957a] On a normed ring of functions with convolution (Russian). Dokl. Akad. Nauk SSSR (NS) **115**, 226-229. [*V.9*]

[1957b] Harmonic analysis of functions of rapid growth (Russian). Uspehi Mat. Nauk (NS) **12** no. 1(73), 201–203. English transl. in Amer. Math. Soc. Transl. (2) **16** (1960), 416–417. [*V.9*]

[1958] A generalization of Wiener's Tauberian theorem and harmonic analysis of rapidly increasing functions (Russian). Trudy Moskov. Mat. Obšč. **7**, 121–148. [*V.9*]

[1960] Methods of the theory of functions of a complex variable in generalized harmonic analysis on the line (Russian). In: Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, pp 526–531. [*V.9*]

Korevaar, J.

[1946] Elementary proof of a Tauberian theorem for Lambert series I, II (Dutch). Simon Stevin **25**, 83–114. [*II.15*]

[1947] The uniform approximation to continuous functions by linear aggregates of functions of a given set. Duke Math. J. **14**, 31–50. [*II.8*]

[1951] An estimate of the error in Tauberian theorems for power series. Duke Math. J. **18**, 723–734. [*III.18; VII.2, 10*]

[1953] Best L^1 approximation and the remainder in Littlewood's theorem. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indag. Math. **15**, 281–293. [*III.18; VII.2*]

[1954a] A very general form of Littlewood's theorem. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 36–45. [*III.18; VII.2, 3, 5, 10*]

[1954b] Another numerical Tauberian theorem for power series. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 46–56. [*I.7; III.1, 12, 16, 17; VII.2, 3, 5, 8*]

[1954c] Numerical Tauberian theorems for Dirichlet and Lambert series. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 152–160. [*VII.20*]

[1954d] Kloosterman's method in Tauberian theorems for C^k summability. Proc. Amer. Math. Soc. **5**, 574–577. [*I.17, 18*]

[1954e] Numerical Tauberian theorems for power series and Dirichlet series. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 432–455. [*I.25; VII.20*]

[1954f] The Riemann hypothesis and numerical Tauberian theorems for Lambert series. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 564–571. [*VII.20*]

[1955] Tauberian theorems (Address Dutch Math. Soc., November 1952). Simon Stevin **30**, 129–139. [*II.3*]

[1956] Notes on differential equations I, Analog of a nonlinear Tauberian theorem of Erdős. Abstract, Bull. Amer. Math. Soc. **62**, 42. [*VII.25*]

[1962/63] Square roots relative to convolution. J. Analyse Math. **10**, 363–379. [*VII.21*]

[1965] Distribution proof of Wiener's Tauberian theorem. Proc. Amer. Math. Soc. **16**, 353–355. [*II.8, 11*]

[1968] Mathematical methods, vol. 1. Academic Press, New York. [*II.3, 4, 7; III.2, 4*]

[1970] Poor approximability and high index Tauberians. Abstract, Notices Amer. Math. Soc. **17**, 182. [*I.23*]

[1979] Commentary on the Tauberian-Theoretic Papers. In: Norbert Wiener, Collected Works, vol. 2. M.I.T. Press, Cambridge, MA, pp 628–635. [*II.1, 8*]

[1982] On Newman's quick way to the prime number theorem. Math. Intelligencer **4**, 108–115. [*III.1, 6, 7*]

[1999] Roots under convolution of sequences. Indag. Math. (N.S.) **10**, 539–548. [*VII.21, 22, 24*]

[2001a] Tauberian theorem of Erdős revisited. Combinatorica **21**, 239–250. [*Pref; VII.21*]

[2001b] Growth of power series with square root gaps. J. Analyse Math. **85**, 177–194. [*VI.1, 19*]

[2001c] Tauberian theory, approximation, and lacunary series of powers. In: Trends in Approximation Theory (K. Kopotun, T. Lyche and M. Neamtu, eds.), Vanderbilt Univ. Press, Nashville, TN, 2001, pp 169–189. [*I.23; VI.1, 19*]

[2002] A century of complex Tauberian theory. Bull. Amer. Math. Soc. (N.S.) **39**, 475–531. [*Pref*]

[2003] A Tauberian theorem for Laplace transforms with pseudofunction boundary behavior. Preprint, KdV Institute for Mathematics, University of Amsterdam. [*III.7, 14*]

Korevaar, J., van Aardenne–Ehrenfest, T. and de Bruijn, N.G.

[1949] A note on slowly oscillating functions. Nieuw Arch. Wisk. (2) **23**, 77–86. [*IV.2, 4*]

Kosambi, D.D.

[1958] Classical Tauberian theorems. J. Indian Soc. Agric. Statist. **10**, 141–149. [*VI.1*]

Kosugi, N., see also Kasahara

[1999a] Tauberian theorem of exponential type and its application to multiple convolution. J. Math. Kyoto Univ. **39**, 331–346. [*IV.15*]

[1999b] Tauberian theorem of exponential type on limits of oscillation. J. Math. Kyoto Univ. **39**, 783–792. [*IV.15*]

Köthe, G.

[1969] Topological vector spaces, vol. I. Grundl. math. Wiss. vol. 159, Springer, New York. (Translation of the second German edition.) [*V.14*]

Kraĭnova, L.I.

[1984] A Tauberian theorem of Fatou type for power series in the form of explicit inequalities (Russian). Dokl. Akad. Nauk Tadzhik. SSR **27**, 550–552. [*III.17*]

Kratz, W., see also Beurer, Borwein, D.

[2000] A Tauberian theorem for the Borel method. Indian J. Math. **42**, 297–307. [*VI.11*]

Kratz, W. and Stadtmüller, U.

[1989a] Tauberian theorems for J_p-summability. J. Math. Anal. Appl. **139**, 362–371. [*VI.24, 25*] 466

[1989b] Tauberian theorems for general J_p-methods and a characterization of dominated variation. J. London Math. Soc. (2) **39**, 145–159. [*VI.24*]

[1990a] O-Tauberian theorems for J_p-methods with rapidly increasing weights. J. London Math. Soc. (2) **41**, 489–502. [*VI.24*]

[1990b] Tauberian theorems for Borel-type methods of summability. Arch. Math. (Basel) **55**, 465–474. [*VI.12, 24, 25*]

Krishnan, V.K.

[1975] Gap Tauberian theorem for generalized Abel summability. Math. Proc. Cambridge Philos. Soc. **75**, 497–500. [*V.21*]

Kuttner, B.

[1977] A Tauberian theorem for Cesàro summability. Compositio Math. **35**, 49–56. [*I.24*]

Kuttner, B. and Parameswaran, M.R.

[1994] A theorem of Mazur–Orlicz type in summability. J. Anal. **2**, 41–44. [*V.19*]

Kwee, B.

[1972] A Tauberian theorem for the (J, p_n) method of summation. J. London Math. Soc. (2) **5**, 139–142. [*VI.24*]

[1983] An improvement on a theorem of Hardy and Littlewood. J. London Math. Soc. (2) **28**, 93–102. [*VI.18, 22*]

[1989] The relation between the Borel and Riesz methods of summation. Bull. London Math. Soc. **21**, 387–393. [*VI.22*]

Landau, E., see also Heilbronn

[1907] Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes. Monatsh. Math. und Phys. **18**, 8–28. [*I.22*]

[1908] Zwei neue Herleitungen für die asymptotische Anzahl der Primzahlen unter einer gegebenen Grenze. Sitz. Ber. Preuss. Akad. Wiss. Berlin, 746–764. [*II.1; III.1, 2*]

[1909] Handbuch der Lehre von der Verteilung der Primzahlen I, II. Teubner, Leipzig. (Second edition with an appendix by P.T. Bateman, Chelsea Publ. Co., New York, 1953.) [*I.4; II.1, 17; III.1, 2, 8*]

[1910] Über die Bedeutung einiger neuerer Grenzwertsätze der Herren Hardy und Axer. Prace mat.- fiz. (Warsaw), **21**, 91–117. [*I.6, 18*]

[1911] Sur les valeurs moyennes de certaines fonctions arithmétiques. Bull. Acad. royale de Belgique (Classe des Sciences), 443–472. [*IV.1*]

[1912] Über einige neuere Grenzwertsätze. Rend. Palermo **34**, 121–131. [*III.6*]

[1913] Über einen Satz des Herrn Littlewood. Rend. Palermo **35**, 265–276. [*I.7, 16*]

[1932a] Über den Wienerschen neuen Weg zum Primzahlsatz. Sitz.-Ber. Preuss. Akad. Wiss., Phys.-Math. Klasse, 1932, 514–521. [*III.4, 6*]

[1932b] Über Dirichletsche Reihen. Nachrichten Göttingen, 525–527. [*III.4, 6*]

[1985–87] Collected works, vols 1–9. Thales-Verlag, Essen.

Landau, E. and Gaier, D.

[1986] Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, third enlarged edition. Springer, Berlin. (First and second edition 1916, 1929 by E. Landau.) [*I.1, 5, 7, 17, 23; III.1, 12; VI.7*]

Lang, S.

[1999] Complex analysis. Grad. Texts in Math., vol. 103, fourth edition. Springer, New York. [*III.6*]

Lavrik, A.F.

[1984] Methods of studying the law of distribution of primes (Russian). Trudy Mat. Inst. Steklov **163**, 118–142. AMS Translation [1985] Proc. Steklov Inst. Math. **163**, 141–167. [*III.1*]

Leviatan, D., see also Baron

[1971] Remarks on some Tauberian theorems of Meyer-König, Tietz and Stieglitz. Proc. Amer. Math. Soc. **29**, 126–132. [*I.24*]

Levin, B.Ja.

[1964] Distribution of zeros of entire functions. Transl. Math. Mono. **5**, Amer. Math. Soc., Providence, RI. (Second edition 1980, original Russian edition 1956.) [*III.19*]

Levinson, N.

[1936] On a class of non-vanishing functions. Proc. London Math. Soc. (2) **41**, 393–407. [*V.7*]

[1938] General gap Tauberian theorems, I. Proc. London Math. Soc. (2) **44**, 289–305. [*I.23; II.15*]

[1940] Gap and density theorems. Amer. Math. Soc. Colloq. Publ. vol. 26, Amer. Math. Soc., New York. [*I.23; II.15; V.7*]

[1964a] Absolute convergence and the general high indices theorem. Duke Math. J. **31**, 241–245. [*I.23; II.15*]

[1964b] The prime number theorem from $\log n!$. Proc. Amer. Math. Soc. **15**, 480–485. [*II.17*]

[1966] Wiener's life. Bull. Amer. Math. Soc. **72**, no. 1, part II, 1–32. [*II.1, 6, 8*]

[1969] A motivated account of an elementary proof of the prime number theorem. Amer. Math. Monthly **76**, 225–245. [*III.1*]

[1973] On the elementary character of Wiener's general Tauberian theorem. J. Math. Anal. Appl. **42**, 381–396. [*II.8*]

[1997] Selected papers of Norman Levinson, vols 1, 2. Birkhäuser, Basel, 1997. [*II.15*]

Levitan, B.M.

[1953] On a special Tauberian theorem. Izv. Akad. Nauk SSSR. Ser. Mat. **17**, 269–284. English transl. in Amer. Math. Soc. Transl. (2) **101** (1973), 222–238. [*III.19*]

[1995] Some special Tauberian theorems for Fourier type integrals and eigenfunction expansions of Sturm–Liouville equations on the whole line. Math. Nachr. **176**, 179–198. [*III.19*]

Levitan, B.M. and Sargsjan, I.S.

[1975] Introduction to spectral theory: selfadjoint ordinary differential equations. (Translation of the Russian edition of 1970.) Amer. Math. Soc. Transl. vol. 39, Providence, RI. [*III.19*]

Lindberg, M.

[1962] On two Tauberian theorems. Pacific J. Math. **12**, 507–515. [*VII.21*]

Linnik, Yu.V., see Ibragimov

Littlewood, J.E., see also Hardy

[1911] The converse of Abel's theorem on power series. Proc. London Math. Soc. (2) **9**, 434–448. [*I.1, 7, 17, 22, 24; II.16; III.1, 18*]

[1953] A mathematician's miscellany. Methuen and Co., London. Pages 66–83 reprinted as 'A mathematical education', Collected Papers, vol. 2. [*I.7*]

[1971] The quickest proof of the prime number theorem. Acta Arith. **18**, 83–86. [*III.1*]

[1982] Collected Papers of J. E. Littlewood, 2 vols. Oxford University Press. [*I.7*]

Loomis, L.H.

[1943] The converse of the Fatou theorem for positive harmonic functions. Trans. Amer. Math. Soc. **53**, 239–250. [*II.19*]

[1953] Abstract harmonic analysis. Van Nostrand, New York. [*II.8; V.5*]

Loomis, L.H. and Widder, D.V.

[1942] The Poisson integral representation of functions which are positive and harmonic in a half-plane. Duke Math. J. **9**, 643–645. [*II.19*]

Lord, R.D.

[1935] On some relations between the Abel, Borel and Cesàro methods of summation. Proc. London Math. Soc. (2) **38**, 241–256. [*VI.8, 22*]

Lorentz, G.G., see also Baron & Leviatan [1997]

[1948] Tauberian theorems and Tauberian conditions. Trans. Amer. Math. Soc. **63**, 226–234. [*I.24; VI.15*]

[1949] Direct theorems on methods of summability. Canad. J. Math. **1**, 305–319. [*I.24; VI.15*]

[1951] Direct theorems on methods of summability II. Canad. J. Math. **3**, 236–256. [*I.24; II.16; VI.15, 18*]

[1997] Mathematics from Leningrad to Austin (Selected works of G.G. Lorentz), vol. 1. Birkhäuser, Boston.

Luh, W., see also Faulstich

[1971] Überkonvergenz bei Euler- und Borel-Verfahren. Mitt. Math. Sem. Giessen No. 92, 37–50. [*VI.4*]

[1974] Über die Summierbarkeit der geometrischen Reihe. Mitt. Math. Sem. Giessen No. 113, 70 pp. [*VI.4*]

Luh, W., Martirosian, V.A. and Müller, J.

[2002] Restricted T-universal functions. J. Approx. Theory **114**, 201–213. [*VI.4*]

Luh, W. and Trautner, R.

[1976] Summierbarkeit der geometrischen Reihe auf vorgeschriebene Mengen. Manuscripta Math. **18**, 317–326. [*VI.4*]

[1978] On the Hausdorff-overconvergence of power series. Arch. Math. (Basel) **30**, 498–507. [*VI.4*]

van de Lune, J.

[1986] An introduction to Tauberian theory: from Tauber to Wiener. Math. Center Syllabi, No. 12, Amsterdam. [*Pref; I.1; II.8; V.3*]

Lyttkens, S.

[1954–56] The remainder in Tauberian theorems. I, Ark. Mat. **2** (1954), 575–588; II, ibid. **3** (1956), 315–349. [*VII.13, 20*]

[1974] General Tauberian remainder theorems. Math. Scand. **35**, 61–104. [*VII.20*]

[1975] An application of a general Tauberian remainder theorem. Ark. Mat. **13**, 237–253. [*VII.20*]

[1978] A Tauberian theorem with rapidly decreasing remainder. In: Fourier analysis and approximation theory (Proc. Colloq., Budapest, 1976), vol. II, pp 513–530. North-Holland, Amsterdam. [*VII.20*]

[1986] General Tauberian theorems in $\mathbb{R}^d$ connected with a theorem of Korenblum. Math. Scand. **59**, 143–176. [*IV.10; VII.20*]

Lyubich, Yu.I. and Vu, Q.P.

[1988] Asymptotic stability of linear differential equations in Banach spaces. Studia Math. **88**, 37–42. [*III.15*]

Maddox, I.J.

[1980] Infinite matrices of operators. Lecture Notes in Math. nr. 786, Springer, Berlin. [*I.7*]

[1989] A Tauberian theorem for statistical convergence. Math. Proc. Cambridge Philos. Soc. **106**, 277–280. [*I.24*]

Malliavin, P.

[1962] Un théorème taubérien avec reste pour la transformation de Stieltjes. C. R. Acad. Sci. Paris **255**, 2351–2352. [*III.1, 16, 19*]

Marčenko, V.A.

[1955] Theorems of Tauberian type in spectral analysis of differential operators (Russian). Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 381–422. English transl. in Amer. Math. Soc. Transl. (2) **102** (1973), 145–190. [*III.19*]

Marić, V.

[2000] Regular variation and differential equations. Lecture Notes in Math. nr. 1726, Springer, Berlin. [*IV.1*]

[2002] Jovan Karamata (1902–1967). Mat. Vesnik **54**, 45–51. [*IV.1*]

Marić, V. and Tomić, M.

[1984] On a method for inverse theorems for $(C, 1)$ and gap $(C, 1)$ summability. Internat. J. Math. Math. Sci. **7**, 469–475. [*I.6; IV.13*]

Markoff (= Markov), A.A.

[1896] Differenzenrechnung (German translation of the Russian edition). Leipzig. [*I.17*]

Markovich, T., see Borwein, D.

Martin, W.T., see also Wiener

Martin, W.T. and Wiener, N.

[1938] Taylor's series of functions of smooth growth in the unit circle. Duke Math. J. **4**, 384–392. [*IV.1, 15, 17, 18*]

Martirosian, V.A., see Luh

Matuszewska, W.

[1961/62] Regularly increasing functions in connection with the theory of $L^{*\phi}$-spaces. Studia Math. **21**, 317–344. [*IV.6*]

[1964/65] A remark on my paper "Regularly increasing functions in connection with the theory of $L^{*\phi}$-spaces". Studia Math. **25**, 265–269. [*IV.2*]

Mazur, S.

[1930] Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren. Studia Math. **2**, 40–50. [*V.14, 16*]

Mazur, S. and Orlicz, W.

[1933] Sur les méthodes linéaires de sommation. C. R. Acad. Sci. Paris **196**, 32–34. [*V.14, 15*]

[1955] On linear methods of summability. Studia Math. **14**, 129–160. [*V.14–16*]

Meir, A., see also Borwein, D.

[1963] Tauberian constants for a family of transformations. Ann. of Math. (2) **78**, 594–599. [*I.5; VI.5, 22*]

Mel'nik, V.I.

[1965] The Tauberian theorem of "large exponents" for the method of Borel (Russian). Mat. Sbornik (NS) **68** (110), 17–25. [*I.9; VI.1, 15*]

[1982] Tauberian theorems with remainder for the Laplace transform on the line (Russian). Ukrain. Mat. Zh. **34**, 462–468, 539. (English translation: Ukrainian Math. J. **34**, 375–380 (1983).) [*IV.15; VII.3*]

Mendès France, M., see Tenenbaum

Mercer, J.

[1907] The limits of real variants. Proc. London Math. Soc. (2) **5**, 206–224. [*IV.13*]

Mertens, F.

[1898] Über eine Eigenschaft der Riemannschen ζ-Funktion. Sitzungsber. Akad. Wiss. Wien, math.-naturw. Cl. **107**, Abt. 2a, 1429–1434. [*I.26*]

Meyer-König, W., see also Ishiguro, Jakimovski

[1939] Limitierungsumkehrsätze mit Lückenbedingungen. I, Math. Z. **45**, 447–478; II, ibid. 479–494. [*I.6; V.21*]

[1949] Untersuchungen über einige verwandte Limitierungsverfahren. Math. Z. **52**, 257–304. [*VI.1, 4, 5, 12, 21, 22*]

[1953] Bemerkung zu einem Lückenumkehrsatz von H.R. Pitt. Math. Z. **57**, 351–352. Cf. comment by Pitt [1958] (p. 92). [*VI.15*]

Meyer-König, W. and Tietz, H.

[1967] On Tauberian conditions of type o. Bull. Amer. Math. Soc. **73**, 926–927. [*I.24*]

[1968] Über die Limitierungsumkehrsätze von Typ o. Studia Math. **31**, 205–216. [*I.24*]

[1969] Über Umkehrbedingungen in der Limitierungstheorie. Arch. Math. (Brno) **5**, 177–186. [*I.24*]

Meyer-König, W. and Zeller, K.

[1956] Lückenumkehrsätze und Lückenperfektheit. Math. Z. **66**, 203–224. [*I.23; V.14, 16–18, 20, 21; VI.1, 15, 23*]

[1958] Funktionalanalytische Behandlung des Taylorschen Summierungsverfahrens. Colloque sur la théorie des suites. Bruxelles 1957, Centre Belge de Recherches Math., Paris–Louvain, pp 32–53. [*VI.23*]

[1960a] Ein high-indices-Theorem für Dirichletsche Reihen. Math. Z. **74**, 221–231. [*I.23; V.21*]

[1960b] On Borel's method of summability. Proc. Amer. Math. Soc. **11**, 307–314. [*VI.15*]

[1962] FK-Räume und Lückenperfektheit. Math. Z. **78**, 143–148. [*V.14, 16, 18–20; VI.23*]

[1978] Abschnittskonvergenz und Umkehrsätze beim Euler-Verfahren. Math. Z. **161**, 147–154. [*V.21*]

[1981] Tauber-Sätze und M-Perfektheit. Math. Z. **177**, 257–266. [*V.21; VI.23*]

Mihalin, G.A.

[1980] Generalization of Tauberian theorems for a class of (J, p_n)-summability (Russian). Izv. Vyssh. Uchebn. Zaved. Mat. **1980**, no. 4, 41–68. Soviet Math. (Iz. VUZ) **24** (1980), 69–76. [*VI.24*]

Milin, I.M.

[1970] Hayman's regularity theorem for the coefficients of univalent functions (Russian). Dokl. Akad. Nauk SSSR **192**, 738–741. English transl. in Soviet Math. Dokl. **11** (1970), 724–728. [*V.1, 24*]

[1971] Univalent functions and orthonormal systems (Russian). Izdat. Nauka, Moscow. English transl. Amer. Math. Soc., Providence, RI (1977). [*V.1, 24*]

Minakshisundaram, S., see Chandrasekharan

Moh, T.T.

[1972] On a general Tauberian theorem. Proc. Amer. Math. Soc. **36**, 167–172. [*IV.11; VI.13*]

Montgomery, H.L.

[1978] The analytic principle of the large sieve. Bull. Amer. Math. Soc. **84**, 547–567. [*III.5*]

[1994] Ten lectures on the interface between analytic number theory and harmonic analysis. CBMS Regional Conf. Series in Math. vol. 84, Amer. Math. Soc., Providence, RI. [*III.5*]

Móricz, F.

[1994] Necessary and sufficient Tauberian conditions, under which convergence follows from summability $(C, 1)$. Bull. London Math. Soc. **26**, 288–294. [*I.24*]

Móricz, F. and Rhoades, B.E.

[1995] Necessary and sufficient Tauberian conditions for certain weighted mean methods of summability. Acta Math. Hungar. **66**, 105–111. [*VI.25*]

Móricz, F. and Stadtmüller, U.

[2001] Necessary and sufficient conditions under which convergence follows from summability by weighted means. Int. J. Math. Math. Sci. **27**, 399–406. [*VI.25*]

Motzer, W. and Stadtmüller, U.

[1994] A Tauberian theorem concerning power series and Cesàro methods of summability. Integral Transform. Spec. Funct. **2**, 131–152. [*VI.18, 24*]

Müller, J., see Luh

Müntz, C.

[1914] Über den Approximationssatz von Weierstrass. Math. Abhandlungen H.A. Schwarz gewidmet, Berlin, pp 303–312. [*VI.17, 23*]

Naimark, M.A.

[1956/72] Normed algebras. Wolters-Noordhoff, Groningen, Netherlands. (Original Russian edition, 'Normed rings', 1956.) [*II.8; V.5*]

Narkiewicz, W.

[2000] The development of prime number theory. Springer Monographs in Math., Springer, Berlin. [*III.1*]

Nathanson, M.B.

[2000] Elementary methods in number theory. Graduate Texts in Math. 195, Springer, New York. [*III.1; VII.21*]

van Neerven, J.M.A.M., see also Batty

[1997] Elementary operator-theoretic proof of Wiener's Tauberian theorem. Rend. Istit. Mat. Univ. Trieste, Suppl. **28**, 281–286. [*II.8*]

Neubrander, F., see Arendt

Neuheuser, H.-G., see Butzer

Nevai, P. (= Nevai, G.P.)
[1972] Einseitige Approximation durch Polynome, mit Anwendungen. Acta Math. Acad. Sci. Hungar. **23**, 495–506. [*VII.7*]
[1986] Géza Freud, orthogonal polynomials and Christoffel functions. A case study. J. Approx. Theory **48**, no. 1, 167 pp. [*VII.2*]

Nevanlinna, R.
[1936/70] Analytic functions. Grundl. math. Wiss. vol. 162, Springer, Berlin. (Original German editions 'Eindeutige analytische Funktionen' 1936, 1953). [*VII.9*]

Newman, D.J.
[1975] A simple proof of Wiener's $1/f$ theorem. Proc. Amer. Math. Soc. **48**, 264–265. [*V.5*]
[1980] Simple analytic proof of the prime number theorem. Amer. Math. Monthly **87**, 693–696. [*III.1, 6, 7*]
[1998] Analytic number theory. Grad. Texts in Math. vol. 177, Springer, New York. [*III.6; IV.23; VII.24*]

Nikolski (= Nikol'skiĭ), N.
[1995] Yngve Domar's forty years in harmonic analysis. Festschrift in honour of L. Carleson and Y. Domar, Uppsala University, pp 45–78. [*V.9*]
[1999] Remarks concerning completeness of translates in function spaces. J. Approx. Theory **98**, 303–315. [*II.8*]

Nikol'skiĭ, S.M.
[1947] On the best linear method of approximation in the mean to differentiable functions by polynomials. Doklady Akad. Nauk SSSR (NS) **58**, 185–188. [*VII.3*]
[1950] On the best approximation of differentiable, nonperiodic functions by means of polynomials. Acta Sci. Math. (Szeged) **12**, part A, 185–197. [*VII.3*]

Noll, D. and Stadler, W.
[1989] Sliding hump technique and spaces with the Wilansky property. Proc. Amer. Math. Soc. **105**, 903–910. [*V.14*]

Northcott, D.G.
[1947] Abstract Tauberian theorems with applications to power series and Hilbert series. Duke Math. J. **14**, 483–502. [*I.7*]

Nyman, B.
[1950] On the one-dimensional translation group and semi-group in certain function spaces. Dissertation, Uppsala. [*V.9*]

Odlyzko, A.M.
[1992] Explicit Tauberian estimates for functions with positive coefficients. J. Comput. Appl. Math. **41**, 187–197. [*IV.23*]

O'Farrell, A.G., see Allan

Okada, Y.
[1925–28] Über die Annäherung analytischer Funktionen. Math. Z. **23**, 62–71; part II, ibid. **27**, 212–217. [*VI.4*]

Olevskiĭ, A., see Atzmon

Omey, E.
[1985] Tauberian theorems with remainder. J. London Math. Soc. (2) **32**, 116–132. [*IV.10*]

Omey, E. and Willekens, E.
[1988] Π-variation with remainder. J. London Math. Soc. (2) **37**, 105–118. [*IV.13*]

Orlicz, W., see Mazur

Ostrogorski, T.
[1998] Regular variation and the Banach–Steinhaus theorem. Nieuw Arch. Wisk. (4) **16**, 27–36. [*IV.2*]

Ostrowski, A.
[1926] On representation of analytic functions by power series. J. London Math. Soc. **1** (1926), 251–263. [*VI.7*]
[1930] Zur Theorie der Überkonvergenz. Math. Ann. **103**, 15–27. [*VI.7*]

Paley, R.E.A.C. and Wiener, N.
[1934] Fourier transforms in the complex domain. Amer. Math. Soc. Colloq. Publ. vol. 19, Amer. Math. Soc., New York. [*III.5; IV.13; V.7*]

Palmer, T.W.
[1994] Banach algebras and the general theory of *-algebras, vol. 1, Algebras and Banach algebras. Cambridge Univ. Press, Cambridge. [*II.8; V.5*]

Parameswaran, M.R., see also Boos, Kuttner
[1961] Partition functions whose logarithms are slowly oscillating. Trans. Amer. Math. Soc. **100**, 217–240. [*IV.15*]
[1975] Some Tauberian theorems for the circle family of summability methods. Math. Z. **143**, 199–201. [*VI.22*]

Patterson, S.J.
[1988] An introduction to the theory of the Riemann zeta-function. Cambridge Univ. Press, Cambridge. [*I.26*]

Pennington, W.B.
[1955] On Ingham summability and summability by Lambert series. Proc. Cambridge Philos. Soc. **51**, 65–80. [*II.18*]

Peterson, G.E.
[1972] Tauberian theorems for integrals. I, J. London Math. Soc. (2) **4**, 493–498; II, ibid. **5**, 182–190. [*IV.11*]

Peyerimhoff, A., see also Bojanic
[1969] Lectures on summability. Lecture Notes in Math. nr. 107, Springer, Berlin. [*I.1, 24; VI.18*]

Phillips, R., see Hille

Pilipović, S. and Stanković, B.
[1993] Wiener Tauberian theorems for distributions. J. London Math. Soc. (2) **47**, 507–515. [*II.8*]

Piranian, G.
[1966] Bounded radial variation and divergence of power series. Math. Scand. **18**, 93–96. [*I.24*]

Pitt, H.R., see also Wiener
[1938a] General Tauberian theorems. Proc. London Math. Soc. **44**, 243–288. [*II.1–3, 5, 8, 9; V.1, 10; VI.1, 15*]
[1938b] A remark on Wiener's general Tauberian theorem. Duke Math. J. **4**, 437–440. [*II.5*]
[1940] General Tauberian theorems II. J. London Math. Soc. **15**, 97–112. [*II.5*]
[1955] A note on Tauberian conditions for Abel and Cesàro summability. Proc. Amer. Math. Soc. **6**, 616–619. [*I.24*]
[1957] An elementary proof of the closure in L of translations of e^{-x^2}, and the Borel Tauberian theorem. Proc. Amer. Math. Soc. **8**, 706–707. [*VI.11*]
[1958] Tauberian theorems. Oxford Univ. Press. [*Pref, I.1, 20; II.5, 8; III.9–11; IV.13; V.1, 10, 11; VI.1, 15*]

Pleijel, Å.
[1963] On a theorem of P. Malliavin. Israel J. Math. **1**, 166–168. [*III.1, 16, 19*]

Poisson, S.D.
[1820] Mémoire sur la manière d'exprimer les fonctions par les séries de quantités périodiques. J. École Polytechn. **11**, 417–489. [*I.3*]
[1823] Suite du Mémoire sur les intégrales définies et sur la sommation des séries. J. École Polytechn. **12**, 404–509. [*I.3*]

Pollard, H., see also Chung, Erdős
[1953] The harmonic analysis of bounded functions. Duke Math. J. **20**, 499–512. [*II.8*]

Pollard, H. and Saari, D.G.
[1970] An elementary Tauberian theorem of nonlinear type. Proc. Amer. Math. Soc. **24**, 593–594. [*I.17*]

Pólya, G.
[1917] Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen. Nachrichten Göttingen, 149–159. [*IV.1*]
[1923] Bemerkungen über unendliche Folgen und ganze Funktionen. Math. Annalen **88**, 169–183, in particular pp 173–177. [*IV.1, 14*]
[1929] Untersuchungen über Lücken und Singularitäten von Potenzreihen. Math. Z. **29**, 549–640. [*VI.19*]

Pólya, G. and Szegő, G.
[1925/78] Problems and theorems in analysis, vol. 1. Grundl. math. Wiss. vol. 193, Springer, Berlin, 1978. (First German ed.: Aufgaben und Lehrsätze aus der Analysis, 1925.) [*I.4; IV.1, 13*]

Postnikov, A.G.
[1951] The remainder term in the Tauberian theorem of Hardy and Littlewood (Russian). Dokl. Akad. Nauk SSSR (NS) **77**, 193–196. [*VII.2*]
[1953] A Tauberian theorem for Dirichlet series (Russian). Dokl. Akad. Nauk SSSR (NS) **92**, 487–490. [*III.1, 16, 18*]
[1980] Tauberian theory and its applications. A translation of Trudy Mat. Inst. Steklov **144** (1979). Proc. Steklov Inst. Math. **1980**, no. 2. [*Pref; I.1; III.1, 12, 16–18; V.23; VII.1*]
[1988] Introduction to analytic number theory. Transl. of math. monographs vol. 68, Amer. Math. Soc., Providence, RI. (Original Russian edition 1971.) [*IV.21, 24*]

Pringsheim, A.
[1904] Über den Divergenz-Charakter gewisser Potenzreihen an der Konvergenzgrenze. Acta Math. **28**, 1–30. [*IV.1*]

Prüss, J., see Arendt

Räbiger, F., see Batty

Raikov, D., see also Gelfand
[1938] Generalization of the Ikehara–Landau theorem (Russian). Mat. Sbornik. Recueil math. Akad. Nauk SSSR (NS) **3**, 559–568. [*III.4*]

Rajagopal, C.T.
[1955] A note on Ingham summability and summability by Lambert series. Proc. Indian Acad. Sci. Sect. A **42**, 41–50. [*II.18*]
[1958] On Tauberian theorems for Abel–Cesàro summability. Proc. Glasgow Math. Assoc. **3**, 176–181. [*I.7*]
[1969] Gap Tauberian theorems on oscillation for the Borel method (B). J. London Math. Soc. **44**, 41–51. [*VI.15*]
[1974] A generalization of Tauber's theorem and some Tauberian constants. IV. Math. Z. **136**, 331–343. Correction, same journal **147** (1976), 301. [*I.5*]

Ramanujan, S., see Hardy

Ramey, W. and Ullrich, D.
[1982] The pointwise Fatou theorem and its converse for positive pluriharmonic functions. Duke Math. J. **49**, 655–675. [*II.19*]

Rangachari, M.S.
[1968/69] On gap Tauberian theorems for Abel, Borel and Lambert summabilities. Publ. Ramanujan Inst. No. **1**, 269–281. [*VI.18*]
[1985] On Lambert and Ingham summability. J. Nat. Acad. Math. India **3**, 145–152. Corrigendum, same journal **5** (1987), 144–145. [*II.18*]

Ransford, T.J., see also Allan
[1988] Some quantitative Tauberian theorems for power series. Bull. London Math. Soc. **20**, 37–44. [*III.13*]

Reid, W.T.
[1954] A Tauberian theorem for power series. Math. Z. **60**, 94–97. [*I.7*]

Reimers, E.
[1984] The problem of equivalence of sequences and Tauberian theorems. II (Russian). Tartu Riikl. Ül. Toimetised No. 661, 64–75. [*VI.25*]

Reiter, H. and Stegeman, J.D.
[2000] Classical harmonic analysis and locally compact groups. Clarendon Press, Oxford. (First edition by Reiter, 1968.) [*II.8, 13; V.9*]

Rényi, A.
[1946] On a Tauberian theorem of O. Szász. Acta Univ. Szeged Sect. Sci. Math. **11**, 119–123. [*I.7*]

Resnick, S.I., see Balkema

Rhoades, B.E., see Móricz

Rickart, C.E.
[1960] General theory of Banach algebras. Van Nostrand, Princeton. [*II.8; V.5*]

Riemann, B.
[1892] Gesammelte mathematische Werke und wissenschaftlicher Nachlass (German). Teubner, Leipzig. (Reprinted by Dover Publ., New York, 1953.) [*III.12*]

Riesz, M., see also Hardy
[1909] Sur les séries de Dirichlet et les séries entières. C. R. Acad. Sci. Paris **149**, 909–912. [*III.1, 12*]
[1911] Über einen Satz des Herrn Fatou. J. Reine Angew. Math. **140**, 89–99. [*III.1, 12*]
[1916] Ein Konvergenzsatz für Dirichletsche Reihen. Acta Math. **40**, 349–361. [*III.1, 12*]
[1923] Sur un théorème de la moyenne et ses applications. Acta Litt. ac Scient. Univ. Hungar. **1**, 114–126. [*I.17*]
[1924] Über die Summierbarkeit durch typische Mittel. Acta Litt. ac Scient. Univ. Hungar. **2**, 18–31. [*III.12*]
[1988] Collected Papers (French). Springer, Berlin.

Roberts, K.L., see Fridy

Robinson, I.J.W. and Borwein, D.
[1975] A Tauberian theorem for Borel-type methods of summability. J. Reine Angew. Math. **273**, 153–164. [*VI.22*]

Rolewicz, S.
[2000] On locally bounded algebras. Algebraic analysis and related topics, pp 191–198, Banach Center Publ. vol. 53, Warsaw. [*V.9*]

Ross, K.A., see Hewitt

Roth, K.F., see Halberstam

Rubel, L.A., see also Ash
[1960] Maximal means and Tauberian theorems. Pacific J. Math. **10**, 997–1007. [*I.7*]

Rudin, W.
[1953/76] Principles of mathematical analysis. McGraw–Hill, New York. [*I.17; II.3*]
[1962/90] Fourier analysis on groups. Interscience, New York. [*II.8; V.3*]
[1966] A converse to the high indices theorem. Proc. Amer. Math. Soc. **17**, 434–435. [*I.24*]
[1966/87] Real and complex analysis. McGraw–Hill, New York. [*II.4, 7; III.2, 4; IV.14; V.2*]
[1973/91] Functional analysis. McGraw–Hill, New York. [*II.8; V.2, 5, 15*]
[1978] Tauberian theorems for positive harmonic functions. Nederl. Akad. Wetensch. Indag. Math. **40**, 376–384. [*II.19*]

Saari, D.G., see also Pollard
[1969] Some large O nonlinear Tauberian theorems. Proc. Amer. Math. Soc. **21**, 459–462. [*I.17*]
[1974] An elementary Tauberian theorem for absolutely continuous functions and series. SIAM J. Math. Anal. Appl. **5**, 649–662. [*I.17*]

Safarov, T.M.
[1974a] Complex Tauberian theorems with a remainder term for general Dirichlet series (Russian). Akad. Nauk Azerbaĭdžan. SSR. Dokl. **30**, 12–15. [*III.17*]

[1974b] Generalization of a Fatou–Korevaar theorem (Russian). Mathematics and the methodology of its teaching, II (Russian), pp 55–61. Tadzhik. Gos. Univ., Dushanbe. [*III.17*]

Safarov, Yu.

[2001] Fourier Tauberian theorems and applications. J. Funct. Anal. **185**, 111-128. [*III.19*]

Sargsjan, I.S., see Levitan

Schieber, E.

[1962] Über das S_α-Verfahren der Limitierungstheorie. Math. Z. **80**, 19–43. [*VI.23*]

Schmaal, A., Stam, A.J. and de Vries, T.

[1976] Tauberian theorems for limitation methods admitting a central limit theorem. Math. Z. **150**, 75–82. [*VI.11*]

Schmeisser, H.-J., see Feichtinger

Schmetterer, L.

[1963] Wahrscheinlichkeitstheoretische Bemerkungen zur Theorie der Reihen. Arch. Math. **14**, 311–316. [*VI.6*]

Schmidt, R.

[1925a] Über divergente Folgen und lineare Mittelbildungen. Math. Z. **22**, 89–152. [*I.7, 16, 21; II.1; IV.1*]

[1925b] Die Umkehrsätze des Borelschen Summierungsverfahren. Schriften Königsberg **1**, 205–256. [*I.9; II.1; VI.1, 12*]

Schoenberg, I.J.

[1959] The integrability of certain functions and related summability methods. Amer. Math. Monthly **66**, 361–375. [*I.24*]

Schwartz, J.T., see Dunford

Schwartz, L.

[1966] Théorie des distributions I, II. Hermann, Paris. (First edition 1950/51.) [*II.11*]

Schwarz, W.

[1968] Schwache asymptotische Eigenschaften von Partitionen. J. Reine Angew. Math. **232**, 1–16. [*IV.15, 23*]

[1969a] Asymptotische Formeln für Partitionen. J. Reine Angew. Math. **234**, 174–178. [*IV.23*]

[1969b] Einführung in Methoden und Ergebnisse der Primzahltheorie. Hochschultaschenbücher 278/278a, Mannheim. [*III.1*]

Segal, I.E.

[1947] The group algebra of a locally compact group. Trans. Amer. Math. Soc. **61**, 69–105. [*V.3*]

Segal, S.L.

[1969] A general Tauberian theorem of Landau–Ingham type. Math. Z. **111**, 159–167. [*II.18*]

[1979] A second note on Ingham's summation method. Canad. Math. Bull. **22**, 117–120. [*II.18*]

Selberg, A.

[1949] An elementary proof of the prime-number theorem. Ann. of Math. (2) **50**, 305–313. [*I.26; III.1; VII.21*]

[1991] Collected papers, vol. 2. Springer, Berlin. [*III.5*]

Seneta, E., see also Bojanic, Drasin

[1976] Regularly varying functions. Lecture Notes in Math. nr. 508, Springer, Berlin. [*IV.2, 3, 6*]

[2002] Karamata's characterization theorem, Feller, and regular variation in probability theory. Publ. Inst. Math. Beograd (N.S.) **71(85)**, 79–89. [*IV.2*]

Shapiro, H.N.

[1959] Tauberian theorems and elementary prime number theory. Comm. Pure Appl. Math. **12**, 579–610. [*VII.21*]

Shapiro, H.S.

[1965] A remark concerning Littlewood's Tauberian theorem. Proc. Amer. Math. Soc. **16**, 258–259. [*I.24*]

[1968] A Tauberian theorem related to approximation theory. Acta Math. **120**, 279–292. [*II.8*]

Shawyer, B. and Watson, B.

[1994] Borel's methods of summability. Oxford Univ. Press, New York. [*I.25; VI.1, 4, 5, 21, 22*]

Shea, D.F., see also Drasin

[1966] On the Valiron deficiencies of meromorphic functions of finite order. Trans. Amer. Math. Soc. **124**, 201–222. [*IV.14*]

[1969] On a complement to Valiron's tauberian theorem for the Stieltjes transform. Proc. Amer. Math. Soc. **21**, 1–9. [*IV.1, 13, 14*]

Shilov, G.E. (= Šilov, G.E.), see also Gelfand

[1940] Sur la théorie des idéaux dans les anneaux normés de fonctions. Dokl. Akad. Nauk SSSR **27**, 900–903. [*V.1, 7*]

[1947] On regular normed rings (Russian). Trudy Mat. Inst. Steklova **21**, 118 pp. [*V.1, 7, 8*]

Siegel, C.L.

[1950] Hand-written letter proposing a simpler final part of the proof in Erdős [1949b]. [*VII.21, 25, 26, 28*]

Simić, S.

[2001] A class of C^∞ slowly varying functions. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **11**, 59–70. [*IV.2*]

Sitaraman, Y. and Swaminathan, V.

[1977] Tauberian theorems for the family $F(a, q)$ of summability methods. Acta Math. Acad. Sci. Hungar. **30**, 45–47. [*VI.12*]

Smet, E., see Borwein, D.

Smith, W.L.

[1954] Asymptotic renewal theorems. Proc. Roy. Soc. Edinburgh. Sect. A. **64**, 9–48. [*V.22*]

Sondow, J.
[1994] Analytic continuation of Riemann's zeta function and values at negative integers via Euler's transformation of series. Proc. Amer. Math. Soc. **120**, 421–424. [*VI.20*]

Sõrmus, T. (= Syrmus, T.)
[1982] One method for weakening Tauberian o-conditions into $\mathcal{O}$-conditions (Russian). Tartu Riikl. Ül. Toimetised No. 596, 75–89. [*I.24*]
[1998] Mercer's theorems for generalized summability methods in Banach spaces. Proc. Estonian Acad. Sci. Phys. Math. **47**, 75–89. [*IV.13*]
[2000] Tauberian theorems for generalized summability methods in Banach spaces. Proc. Estonian Acad. Sci. Phys. Math. **49**, 170–182. [*VI.22*]

Stadler, W., see Noll

Stadtmüller, K., see Faulstich

Stadtmüller, U., see also Balkema, Bingham, Borwein, D., Geluk, de Haan, Kiesel, Kratz, Móricz, Motzer
[1982] Tauberian theorems for Laplace and Stieltjes transforms. J. Math. Anal. Appl. **86**, 146–156. [*IV.10*]
[1993] Tauberian theorems for weighted Laplace transforms. Analysis **13**, 159–183. [*IV.24; VI.24*]
[1995] On a family of summability methods and one-sided Tauberian conditions. J. Math. Anal. Appl. **196**, 99–119. [*VI.18, 24*]

Stadtmüller, U. and Trautner, R.
[1979] Tauberian theorems for Laplace transforms. J. Reine Angew. Math. **311/312**, 283–290. [*IV.10*]
[1985] Ratio Tauberian theorems for Laplace transforms without monotonicity assumptions. Quart. J. Math. Oxford Ser. (2) **36**, 363–381. [*IV.10*]
[1999] Continuity theorems and ratio-Tauberian theorems for integral transforms. Acta. Sci. Math. (Szeged) **65**, 611–633. [*IV.10*]

Stam, A.J., see Schmaal

Stanković, B., see Pilipović

Stef, A. and Tenenbaum, G.
[2001] Inversion de Laplace-effective. Ann. Prob. **29**, 558–575. [*IV.24*]

Stegeman, J.D., see Reiter

Stein, C., see Diaconis

Stein, E.M.
[1970] Singular integrals and differentiability properties of functions. Princeton Univ. Press. [*II.19*]

Štěpánek, F.
[1966] A Tauber's theorem for (J, p_n) summability. Monatsh. Math. **70**, 256–260. [*VI.24*]

Stieglitz, M.
[1969] Über ausgezeichnete Tauber-Matrizen. Arch. Math. (Brno) **5**, 227–233. [*I.24*]
[1971] Die allgemeine Form der O-Tauber-Bedingung für das Euler–Knopp- und Borel-Verfahren. J. Reine Angew. Math. **246**, 172–179. [*VI.18*]

Stirzaker, D., see Grimmett

Stone, Charles
[1965] On characteristic functions and renewal theory. Trans. Amer. Math. Soc. **120**, 327–342. [*V.23*]

Strouse, E., see Esterle

Strube, S.P.M.
[1974] A Tauberian remainder theorem with applications to Lambert summability. Math. Scand. **34**. 137–144. [*VII.20*]
[1975] Tauberian remainder theorems for the Borel methods. J. Reine Angew. Math. **273**, 125–133. [*VII.20*]

Subhankulov, M.A., see also An
[1960] Tauberian theorems with remainder term (Russian). Mat. Sbornik **52(94)**, 823–846; Amer. Math. Soc. Transl. (2) **26** (1963), 311–338. [*III.1, 16, 18*]
[1961a] Some general Tauberian theorems with remainder term (Russian). Trudy Mat. Inst. Steklov **64**, 239–266. [*VII.18*]
[1961b] The remainder term in the Tauberian theorem of Hardy–Littlewood–Carleman (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **25**, 925–934. [*VII.18*]
[1964] On a theorem of Littlewood (Russian). Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk **1**, 22–30. [*III.1, 18*]
[1972] A problem of Lo Kêng Hua (Russian). Tadžik. Gos. Univ. Trudy Meh.-Mat. Fak. Vyp. **2**, 46–49. [*VII.21*]
[1973] Tauberian theorems of Landau–Ikehara type and their connection with the distribution of primes (Russian). Baškir. Gos. Učen. Zap. **74**, 47–69. [*III.4*]
[1976] Tauberian theorems with remainder (Russian). Izdat. Nauka, Moscow. [*Pref; I.1; III.1, 4, 16, 18, 19; VII.1, 18*]

Subhankulov, M.A. and An, F.I.
[1974] Complex Tauberian theorems for the one-sided and two-sided Stieltjes transform (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **38**, 138–169. (English translation: Math. USSR-Izv. **8** (1975), 145–176.) [*III.19*]

Swaminathan, V., see Sitaraman

Swartz, C.
[1996] Infinite matrices and the gliding hump. World Scientific Publishing Co., River Edge, NJ. [*V.14*]

Szász, O.
[1915] Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. Math. Ann. **77**, 482–496. [*VI.17, 23*]
[1928] Verallgemeinerung eines Littlewoodschen Satzes über Potenzreihen. J. London Math. Soc. **3**, 254–262. [*I.7*]
[1929] Verallgemeinerung und neuer Beweis einiger Sätze Tauberscher Art. Sitzungsber. Bayer. Akad. Wiss. München, Math.-Phys. Kl. **59**, 325–340. [*I.15, 21, 22*]
[1935] Generalization of two theorems of Hardy and Littlewood on power series. Duke Math. J. **1**, 105–111. [*I.7*]
[1936] Converse theorems of summability for Dirichlet's series. Trans. Amer. Math. Soc. **39**, 117–130. [*I.22*]
[1948] Introduction to the theory of divergent series. Hafner Publ. Co., New York, 1948. [*I.7*]

[1951] On a Tauberian theorem for Abel summability. Pacific J. Math. **1**, 117–125. [*I.7*]
[1955] Collected mathematical papers (H. D. Lipsich, ed.). Dept. of Math., Univ. of Cincinnati. Hafner Publ. Co., New York.

Szegő, G., see also Pólya
[1939/75] Orthogonal polynomials. Amer. Math. Soc. Colloq. Publ. vol. 23, Amer. Math. Soc., Providence, RI. [*VII.6*]

Sz.-Nagy, B. and Foiaş, C.
[1970] Harmonic analysis of operators on Hilbert space. North-Holland Publ. Co., Amsterdam. [*III.15*]

Szüsz, P., see Kennedy

Tam, L.
[1992] A Tauberian theorem for the general Euler–Borel summability method. Canad. J. Math. **44**, 1100–1120. [*VI.22*]

Tammeraid, I.
[1996] On the comparison of Tauberian remainder theorems. Proc. Estonian Acad. Sci. Phys. Math. **45**, 234–241. [*VII.13*]

Tauber, A., see also Binder [1984], Chatterji [1984]
[1897] Ein Satz aus der Theorie der unendlichen Reihen. Monatsh. Math. u. Phys. **8**, 273–277. [*I.1, 5; III.1*]

Tenenbaum, G., see also Bingham, Hildebrand, A., Stef
[1980] Sur le procédé de sommation de Borel et la répartition des facteurs premiers des entiers. Enseignement math. (2) **26**, 225–245. [*I.9; IV.3; VI.1, 13, 14*]
[1995] Introduction to analytic and probabilistic number theory. Cambridge Univ. Press. [*III.4; IV.24; VII.10, 15*]

Tenenbaum, G. and Mendès France, M.
[2000] The prime numbers and their distribution. Student Math. Library vol. 6, Amer. Math. Soc., Providence, RI. (Translation of French edition of 1997.) [*III.1*]

Teugels, J.L., see Bingham

Thorpe, B., see Badiozzaman

Tietz, H., see also Boos, Ishiguro, Jakimovski, Meyer-König
[1989] Tauberian theorems of $(J_p) \to (M_p)$-type. Math. J. Okayama Univ. **31**, 221–225. [*VI.24*]
[1990] Schmidt'sche Umkehrungen für Potenzreihenverfahren. Acta Sci. Math. (Szeged) **54**, 355–365. [*VI.24*]

Tietz, H. and Trautner, R.
[1988] Tauber-Sätze für Potenzreihenverfahren. Arch. Math. **50**, 164–174. [*VI.24*]

Tietz, H. and Zeller, K.
[1997] Tauber-Sätze für bewichtete Mittel. Arch. Math. (Basel) **68**, 214–220. [*VI.25*]
[1998a] An elementary proof for the $\mathcal{O}_R$-theorem of Hardy and Littlewood. J. Math. Anal. Appl. **218**, 632–635. [*I.7, 23*]
[1998b] A unified approach to some Tauberian theorems of Hardy and Littlewood. Acta Comment. Univ. Tartu. Math. no. 2, 15–18. [*I.7, 12*]

[1998c] Tauber-Bedingungen für Verfahren mit Abschnittskonvergenz. Acta Math. Hungar. **81**, 241–247. [*V.21*]

[1999a] Charakterisierung von O-Tauber-Bedingungen für Teilwirkfelder mit Abschnittskonvergenz. Results Math. **35**, 380–391. [*V.21*]

[1999b] Einseitige O-Tauber-Bedingungen für Teilwirkfelder mit Abschnittskonvergenz. Results Math. **36**, 365–372. [*V.21*]

[1999c] The Tauberian gap theorem for the Abel method. In: Sequence spaces and applications (P.K. Jain and E. Malkowsky, eds.), Narosa, New Delhi, pp 130–134. [*V.21*]

[2000] Tauber-Sätze für Kreisverfahren und der Satz von Müntz. Acta Sci. Math. (Szeged) **66**, 103–113. [*VI.1, 22*]

[2001] Der Satz von Vijayaraghavan und Hardy. Arch. Math. (Basel) **77**, 47–55. [*I.19*]

Titchmarsh, E.C.

[1927] On integral functions with real negative zeros. Proc. London Math. Soc. (2) **26**, 185–200. [*I.21; III.19; IV.9*]

[1939] The theory of functions. Oxford Univ. Press. (Second edition.) [*I.4; III.12; VI.11; VII.12, 18*]

[1937/86] Introduction to the theory of Fourier integrals. Oxford Univ. Press, Chelsea Publ. Co., New York. [*II.7; IV.14*]

[1951/86] The theory of the Riemann zeta-function. Clarendon Press, Oxford. (Second edition edited by D. R. Heath-Brown.) [*I.26*]

[1958] Eigenfunction expansions associated with second-order differential equations, Part II. Clarendon Press, Oxford. [*III.19; IV.9*]

Tomić, M., see also Aljančić, Marić

[2001] Jovan Karamata (1902–1967). Bull. Cl. Sci. Math. Nat. Sci. Math. no. 26, 1–30. [*IV.1*]

Tomilov, Y. and and Zemánek, J.

[2001] A new way of constructing examples in operator ergodic theory. Preprint. (Paper to appear in Math. Proc. Cambridge Philos. Soc.) [*III.13*]

Tomm, L., see Faulstich

Trautner, R., see Luh, Stadtmüller, Tietz

Turán, P.

[1958] A remark concerning the behaviour of a power-series on the periphery of its convergence-circle. Acad. Serbe Sci. Publ. Inst. Math. **12**, 19–26. [*VI.22*]

[1984] On a new method of analysis and its applications. Fourth edition with the assistance of G. Halász and J. Pintz. Wiley and Sons, New York. [*VI.15*]

Tzafriri, L., see Katznelson

Ullrich, David, see Ramey

Vaaler, J.D., see also Graham, Holt

[1985] Some extremal functions in Fourier analysis. Bull. Amer. Math. Soc. (N.S.) **12**, 183–216. [*III.5; IV.24*]

Valiron, G.

[1914] Sur les fonctions entières d'ordre fini et d'ordre nul, et en particulier les fonctions à correspondance régulière. Ann. Toulouse (3) **5**, 117–257. [*I.21; III.19; IV.9, 13*]

[1917] Remarques sur la sommation des séries divergentes par les méthodes de M. Borel. Rend. Palermo **42**, 267–284. [*VI.1, 11*]

de la Vallée Poussin, C.

[1896] Recherches analytiques sur la théorie des nombres premiers, I. Ann. Soc. Sci. Bruxelles **20**:2, 183–256. [*I.10, 26; III.1*]

Vaughan, R.C.

[1983] A remark on Freud's Tauberian theorem. Studies in Pure Mathematics. To the memory of Paul Turán. Edited by Paul Erdős. Birkhäuser, Basel, pp 737–742. [*VII.10*]

Vermes, P.

[1949] Series to series transformations and analytic continuation by matrix methods. Amer. J. Math. **71**, 541–562. [*VI.21*]

[1951] Conservative series to series transformation matrices. Acta Sci. Math. (Szeged) **14**, 23–38. [*VI.21*]

Vijayaraghavan, T.

[1926] A Tauberian theorem. J. London Math. Soc. **1**, 113–120. [*I.19*]

[1928] A theorem concerning the summability of series by Borel's method. Proc. London Math. Soc. (2) **27**, 316–326. [*I.19; VI.1, 8, 12*]

Vladimirov, V.S., Drozhzhinov, Yu.N. and Zav'yalov, B.I.

[1986] Multidimensional Tauberian theorems for generalized functions (Russian). Nauka, Moscow, 1986. (English translation: Tauberian theorems for generalized functions. Kluwer, Dordrecht, 1988.) [*III.1*]

Vretblad, A.

[1973] Spectral analysis in weighted L^1 spaces on $\mathbb{R}$. Ark. Mat. **11**, 109–138. [*V.7, 9*]

de Vries, T., see Schmaal

Vu, Q.P., see Lyubich

Vučković, V.

[1953] Die Stieltjes-Transformation die mit der Geschwindigkeit der Exponentialfunktion unendlich klein wird (Serbo-Croatian; German abstract). Srpska Akad Nauka. Zbornik Radova Mat. Inst. **35(3)**, 255-288. [*VII.2, 18*]

[1954] Quelques théorèmes relatifs à la transformation de Stieltjes. Acad. Serbe Sci. Publ. Inst. Math. **6**, 63–74. [*VII.18*]

Vuilleumier, M.

[1963] On asymptotic behavior of linear transformations of slowly varying sequences and of sequences of regular asymptotic behavior. Math. Res. Center Tech. Report 435, Madison, WI. [*IV.5, 9*]

Wagner, Eberhard

[1966] Taubersche Sätze reeller Art für die Laplace-Transformation. Math. Nachr. **31**. 153–168. [*IV.15*]

[1968] Ein reeller Tauberscher Satz für die Laplace-Transformation. Math. Nachr. **36**, 323–331. [*IV.15*]

Wais, R.
[1935] Das Taylorsche Summierungsverfahren. Dissertation, Tübingen. [*VI.21*]

Waterman, D.
[1950] On some high indices theorems. Trans. Amer. Math. Soc. **69**, 468–478. [*I.23*]
[1999] On some high-indices theorems, II. J. London Math. Soc. (2) **59**, 978–986. [*I.23*]

Watson, B., see Shawyer

Watson, G.N., see Whittaker, E.T.

Weiermann, A.
[2002] Zero-one law characterizations of ε_0. In: Mathematics and Computer Science II: Algorithms, Trees, Combinatorics and Probabilities (B. Chauvin, P. Flajolet, D. Gardy, A. Mokkadem, eds). Birkhäuser, Basel, pp 527–539. [*IV.19*]
[2003] An application of results by Hardy, Ramanujan and Karamata to Ackermannian functions. DMTCS **6**, 133–142. [*IV.8*]

Wermer, J.
[1954] On a class of normed rings. Ark. Mat. **2**, 537–551. [*V.9*]

Whittaker, E.T. and Watson, G.N.
[1927/96] A course of modern analysis, fourth ed. Cambridge Univ. Press, Cambridge. (Reprinted 1996.) [*I.21; II.12, 19; IV.9, 12; VI.5; VII.10, 13*]

Widder, D.V., see also Hirschman, Loomis
[1941] The Laplace transform. Princeton Univ. Press, Princeton. [*I.1, 14, 15; II.8; III.4*]
[1971] An introduction to transform theory. Acad. Press, New York. [*II.3, 6*]

Wielandt, H.
[1952] Zur Umkehrung des Abelschen Stetigkeitssatzes. Math. Z. **56**, 206–207. [*I.7, 12*]

Wiener, N., see also Levinson [1966], Martin, Paley
[1927a] On a theorem of Bochner and Hardy. J. London Math. Soc. **2**, 118–123. [*II.1, 6*]
[1927b] Une méthode nouvelle pour la démonstration des théorèmes de M. Tauber. C. R. Acad. Sci. Paris **184**, 793–795. [*II.1*]
[1928] A new method in Tauberian theorems. J. Math. and Phys. M.I.T. **7**, 161–184. [*I.1, 10; II.1, 8, 12, 14; III.1, 4*]
[1932] Tauberian theorems. Ann. of Math. **33**, 1–100. [*I.1, 10; II.1, 4, 7–10, 12–15; III.1, 4; V.5; VI.1, 9, 12*]
[1933] The Fourier integral and certain of its applications. Cambridge Univ. Press, Cambridge. [*I.1, 5; II.6, 8; III.1, 4*]
[1956] I am a mathematician. M.I.T. Press, Cambridge, MA. [*II.1*]
[1979] Collected works with commentaries, vol. 2. Generalized harmonic analysis and Tauberian theory; classical harmonic and complex analysis. M.I.T. Press, Cambridge, Mass. [*II.1*]
[1997a] The legacy of Norbert Wiener: A centennial symposium. Proc. Symp. in Pure Math vol. 60, Amer. Math. Soc. Providence, RI. [*II.1*]
[1997b] Proceedings of the Norbert Wiener Centenary Congress. Proc. Symp. in Appl. Math. vol. 52, Amer. Math. Soc., Providence, RI. [*II.1*]

Wiener, N. and Martin, W.T.
[1937] Taylor's series of entire functions of smooth growth. Duke Math. J. **3**, 213–223. [*IV.17; VI.13*]

Wiener, N. and Pitt, H.R.
[1939] A generalization of Ikehara's theorem. J. Math. and Phys. M.I.T. **17** (1939), 247–258. [*III.4, 9*]

Wilansky, A., see also Garling, Kalton
[1984] Summability through functional analysis. North-Holland Math. Studies, vol. 85, Elsevier, Amsterdam. [*V.1, 14*]

Wilansky, A. and Zeller, K.
[1955] Summation of bounded divergent sequences, topological methods. Trans. Amer. Math. Soc. **78**, 501–509. [*V.14, 16*]
[1956] Abschnittsbeschränkte Matrixtransformationen; starke Limitierbarkeit. Math. Z. **64**, 258–269. [*V.17*]

Willekens, E., see Omey

Wintner, A.
[1943] Eratosthenian averages. Baltimore. [*II.18*]
[1957] On arithmetical summation processes. Amer. J. Math. **79**, 559–574. [*II.18*]

Włodarski, L.
[1955] Sur les méthodes continues de limitation I, II. Studia Math. **14**, 161–187, 188–199. [*V.21*]

Wolfowitz, J., see Chung

Wright, E.M., see Hardy

Yakymiv, A.L.
[2002] Some applications of Tauberian theorems to probability theory (Russian). Publ. Inst. Math. Beograd (N.S.) **71(85)**, 133–147. [*IV.1*]

Yashchin, E., see Feigin

Yeates, S.B., see Batty

Young, W.H.
[1918] On restricted Fourier series and the convergence of power series. Proc. London Math. Soc. **17**, 353–366. [*III.12*]

Zagier, D.
[1997] Newman's short proof of the prime number theorem. Amer. Math. Monthly **104**, 705–708. [*III.1, 6, 7*]

Zav'yalov, B.I., see Vladimirov

Zeller, K., see also Ishiguro, Jakimovski, Meyer-König, Tietz, Wilansky
[1951a] Allgemeine Eigenschaften von Limitierungsverfahren. Math. Z. **53**, 463–487. [*V.14, 15, 19*]
[1951b] Abschnittskonvergenz in FK-Räumen. Math. Z. **55**, 55–70. [*V.14, 17*]
[1953a] Sur la méthode de sommation d'Abel. C. R. Acad. Sci. Paris **236**, 568–569. [*V.14, 21*]
[1953b] FK-Räume in der Funktionentheorie, I. Math. Z. **58**, 288–305. [*V.14, 18*]
[1953c] Approximation in Wirkfeldern von Summierungsverfahren. Arch. Math. **4**, 425–431. [*V.14, 20; VI.15*]

[1956] Über den perfekten Teil von Wirkfeldern. Math. Z. **64**, 123–130. [*V.14, 16*]

Zeller, K. and Beekmann, W.
[1958/70] Theorie der Limitierungsverfahren. Ergebn. Math. u. Grenzgebiete vol 15, Springer, Berlin. [*Pref; I.1, 5; III.12; IV.13; V.1, 14, 15, 17; VI.1, 4, 12, 21–23*]

Zemánek, J., see Tomilov

Zouakia, F., see Esterle

Zygmund, A.
[1931] On a theorem of Ostrowski. J. London Math. Soc. **6**, 162–163. [*VI.1, 7*]
[1944] On certain integrals. Trans. Amer. Math. Soc. **55**, 170–204. [*I.23*]
[1959] Trigonometric series. Cambridge Univ. Press. (Second edition.) [*I.7, 23; II.19; III.12*]

Index

For an *Author Index*, see the Bibliography.

Grundlehren der mathematischen Wissenschaften

A Series of Comprehensive Studies in Mathematics

A Selection

224. Gilbarg/Trudinger: Elliptic Partial Differential Equations of Second Order
225. Schütte: Proof Theory
226. Karoubi: K-Theory. An Introduction
227. Grauert/Remmert: Theorie der Steinschen Räume
228. Segal/Kunze: Integrals and Operators
229. Hasse: Number Theory
230. Klingenberg: Lectures on Closed Geodesics
231. Lang: Elliptic Curves. Diophantine Analysis
232. Gihman/Skorohod: The Theory of Stochastic Processes III
233. Stroock/Varadhan: Multidimensional Diffusion Processes
234. Aigner: Combinatorial Theory
235. Dynkin/Yushkevich: Controlled Markov Processes
236. Grauert/Remmert: Theory of Stein Spaces
237. Köthe: Topological Vector Spaces II
238. Graham/McGehee: Essays in Commutative Harmonic Analysis
239. Elliott: Probabilistic Number Theory I
240. Elliott: Probabilistic Number Theory II
241. Rudin: Function Theory in the Unit Ball of C^n
242. Huppert/Blackburn: Finite Groups II
243. Huppert/Blackburn: Finite Groups III
244. Kubert/Lang: Modular Units
245. Cornfeld/Fomin/Sinai: Ergodic Theory
246. Naimark/Stern: Theory of Group Representations
247. Suzuki: Group Theory I
248. Suzuki: Group Theory II
249. Chung: Lectures from Markov Processes to Brownian Motion
250. Arnold: Geometrical Methods in the Theory of Ordinary Differential Equations
251. Chow/Hale: Methods of Bifurcation Theory
252. Aubin: Nonlinear Analysis on Manifolds. Monge-Ampère Equations
253. Dwork: Lectures on ρ-adic Differential Equations
254. Freitag: Siegelsche Modulfunktionen
255. Lang: Complex Multiplication
256. Hörmander: The Analysis of Linear Partial Differential Operators I
257. Hörmander: The Analysis of Linear Partial Differential Operators II
258. Smoller: Shock Waves and Reaction-Diffusion Equations
259. Duren: Univalent Functions
260. Freidlin/Wentzell: Random Perturbations of Dynamical Systems
261. Bosch/Güntzer/Remmert: Non Archimedian Analysis – A System Approach to Rigid Analytic Geometry
262. Doob: Classical Potential Theory and Its Probabilistic Counterpart
263. Krasnosel'skiĭ/Zabreĭko: Geometrical Methods of Nonlinear Analysis
264. Aubin/Cellina: Differential Inclusions
265. Grauert/Remmert: Coherent Analytic Sheaves
266. de Rham: Differentiable Manifolds
267. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. I
268. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. II
269. Schapira: Microdifferential Systems in the Complex Domain
270. Scharlau: Quadratic and Hermitian Forms
271. Ellis: Entropy, Large Deviations, and Statistical Mechanics
272. Elliott: Arithmetic Functions and Integer Products
273. Nikol'skiĭ: Treatise on the Shift Operator

274. Hörmander: The Analysis of Linear Partial Differential Operators III
275. Hörmander: The Analysis of Linear Partial Differential Operators IV
276. Liggett: Interacting Particle Systems
277. Fulton/Lang: Riemann-Roch Algebra
278. Barr/Wells: Toposes, Triples and Theories
279. Bishop/Bridges: Constructive Analysis
280. Neukirch: Class Field Theory
281. Chandrasekharan: Elliptic Functions
282. Lelong/Gruman: Entire Functions of Several Complex Variables
283. Kodaira: Complex Manifolds and Deformation of Complex Structures
284. Finn: Equilibrium Capillary Surfaces
285. Burago/Zalgaller: Geometric Inequalities
286. Andrianaov: Quadratic Forms and Hecke Operators
287. Maskit: Kleinian Groups
288. Jacod/Shiryaev: Limit Theorems for Stochastic Processes
289. Manin: Gauge Field Theory and Complex Geometry
290. Conway/Sloane: Sphere Packings, Lattices and Groups
291. Hahn/O'Meara: The Classical Groups and K-Theory
292. Kashiwara/Schapira: Sheaves on Manifolds
293. Revuz/Yor: Continuous Martingales and Brownian Motion
294. Knus: Quadratic and Hermitian Forms over Rings
295. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces I
296. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces II
297. Pastur/Figotin: Spectra of Random and Almost-Periodic Operators
298. Berline/Getzler/Vergne: Heat Kernels and Dirac Operators
299. Pommerenke: Boundary Behaviour of Conformal Maps
300. Orlik/Terao: Arrangements of Hyperplanes
301. Loday: Cyclic Homology
302. Lange/Birkenhake: Complex Abelian Varieties
303. DeVore/Lorentz: Constructive Approximation
304. Lorentz/v. Golitschek/Makovoz: Construcitve Approximation. Advanced Problems
305. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms I. Fundamentals
306. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms II. Advanced Theory and Bundle Methods
307. Schwarz: Quantum Field Theory and Topology
308. Schwarz: Topology for Physicists
309. Adem/Milgram: Cohomology of Finite Groups
310. Giaquinta/Hildebrandt: Calculus of Variations I: The Lagrangian Formalism
311. Giaquinta/Hildebrandt: Calculus of Variations II: The Hamiltonian Formalism
312. Chung/Zhao: From Brownian Motion to Schrödinger's Equation
313. Malliavin: Stochastic Analysis
314. Adams/Hedberg: Function Spaces and Potential Theory
315. Bürgisser/Clausen/Shokrollahi: Algebraic Complexity Theory
316. Saff/Totik: Logarithmic Potentials with External Fields
317. Rockafellar/Wets: Variational Analysis
318. Kobayashi: Hyperbolic Complex Spaces
319. Bridson/Haefliger: Metric Spaces of Non-Positive Curvature
320. Kipnis/Landim: Scaling Limits of Interacting Particle Systems
321. Grimmett: Percolation
322. Neukirch: Algebraic Number Theory
323. Neukirch/Schmidt/Wingberg: Cohomology of Number Fields
324. Liggett: Stochastic Interacting Systems: Contact, Voter and Exclusion Processes
325. Dafermos: Hyperbolic Conservation Laws in Continuum Physics
326. Waldschmidt: Diophantine Approximation on Linear Algebraic Groups
327. Martinet: Perfect Lattices in Euclidean Spaces
328. van der Put/Singer: Galois Theory of Linear Differential Equations
329. Korevaar: Tauberian Theory